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CONSTANT MEAN CURVATURE FOLIATION: SINGULARITY STRUCTURE AND CURVATURE ESTIMATE

RUGANG YE

We study constant mean curvature foliations with isolated center singularities in 3-dimensions. We prove that the leaves become round upon approaching a center. We also derive a priori curvature estimates for constant mean curvature foliations without stability condition. A complete existence, uniqueness and non-existence result for constant mean curvature foliations around a center is derived as a consequence. These results are extended to constant mean curvature foliations on asymptotically flat ends.

0. Introduction.

Consider a foliation by constant mean curvature hypersurfaces in a Riemannian manifold. Using suitable concepts for surfaces such as rectifiable currents, singularities can be allowed in the foliation. A special case is that all leaves have the *same* mean curvature. If the mean curvature is zero, i.e. the leaves are minimal, the calibration argument shows that they are homologically area-minimizing, and hence are smooth if the dimension of the ambient space is no larger than 7. The same holds if all leaves have the same nonzero mean curvature, for under this assumption the leaves still have an area-minimizing property, namely they minimize area under the constant volume constraint. In higher dimensions, singularities generally do occur, and one would like to undersatnd the structure of singularities, mainly the asymptotical behavior of leaves upon approaching singularities. Important results on singularity structures of area-minimizing varieties have been obtained by L.Simon in [**Sm1**], [**Sm2**]. These results apply to the said special case of constant mean curvature foliations and gives e.g. information on the asymptotical behavior of a single leaf upon approaching singularities.

The situation changes dramatically if we allow mean curvature to *vary from leaf to leaf* without assuming any control over its variation. In general, singularities occur, even in the lowest dimension. No nontrivial result about singularity structures has been known before. Besides, we emphasize that the question is not only about a single leaf. The asymptotical behavior of all relevant leaves upon approaching singularities is at the heart of the problem.

In this paper we study a model case: constant mean curvature foliations with center singularities. A center singularity, or simply a center, is an isolated singularity surrounded by closed leaves. In [**Y1**], the concept of "regularity" was introduced. A constant mean curvature foliation is said to be "regular" or "controlled" at a center, or the given center is "controlled", if the rescaled second fundamental form of the surrounding leaves is uniformly bounded. Regularity controls the asymptotical geometry of leaves around center singularities. Indeed, the shape of the leaves becomes round upon approaching a controlled center, in the sense that they converge to the Euclidean sphere after rescaling, see [**Y1**]. The fundamental question is then whether a constant mean curvature foliation is regular at every center, at least in lower dimensions.

For a constant mean curvature foliation on an asymptotically flat manifold or end, the infinity can be thought of as a special kind of singularity. Here, "regular at ∞" is a natural concept concerning structures at infinity. Note that from the point of view of physics, the most significant dimension is 3.

The main results in this paper concerning singularity structures are the following structure theorems in dimension 3. (The 2 dimensional case is rather easy, and should be considered as a trivial case.)

Theorem 1. *Let $\mathcal{F}$ be a C^2 constant mean curvature foliation in a 3-manifold. Then $\mathcal{F}$ is regular at its centers. Consequently, the leaves become round upon approaching a center. (The same holds for a 2-manifold.)*

Theorem 2. *Let $\mathcal{F}$ be a C^2 constant mean curvature foliation with compact leaves on an asymptotically flat end of dimension 3. If it is diameter-pinched at ∞, then it is regular at ∞. Consequently, the leaves become round upon approaching infinity.*

Regularity is a very strong geometric property, and the above results are quite delicate. Note that in a Euclidean space these results become trivial on account of the Alexandrov reflection principle. In the general Riemannian case, since the metric is almost Euclidean near a center, one might hope to apply this principle to show that the leaves become round upon approaching the center. However, the Alexandrov reflection principle depends in a crucial way on the symmetry properties of the Euclidean space, and a quantitative version of it allowing small perturbations does not seem to hold. Hence in general it does not predict the shape of a closed, embedded surface of constant mean curvature in a space which is arbitarily close to being Euclidean. Conceivably such a surface can be dramatically different from round spheres. Indeed, it should be possible to construct such examples by using the classical Delauney surfaces. This issue will be taken up in a subsequent paper.

It is not clear whether Theorem 2 is optimal. Technically, the "diameter-pinched" condition is used to prevent the leaves from falling to the "black hole" singularity after a certain rescaling. On the other hand, there are technical obstacles to extending Theorem 1 and Theorem 2 to higher dimensions. All ingredients in their proof work in arbitary dimensions, except two, which are restricted to dimension 3. The first is the Bernstein type result of Fischer-Colbrie-Schoen [**FS**] and Do Carmo-Peng [**CP**]. Experts believe that it extends to dimensions 4-7, but no proof has become available. The second is an argument for finding "necks", but it seems that an refinement of this argument works in higher dimensions. With such a refinement, a weak version of Theorems 1 and 2 then hold in dimensions 4-7. We replace namely the above Bernstein type result by the result in [**SS**]. An additional condition on rescaled area is needed here. Details will be discussed in another paper.

Besides singularity structures, another important question about constant mean curvature foliations is a priori curvature estimates. For a minimal foliation, the leaves are stable, and hence the curvature estimates in e.g. [**SS**] are applicable. If the mean curvature is a fixed nonzero constant, the leaves are mean stable, and the estimates in [**SS**] apply with some modifications. See also Section 4 of this paper for stronger estimates in dimension 3. No result has been known for the case that the mean curvature varies from leaf to leaf.

In this paper we obtain the following a priori curvature estimates. Note that the regularity results stated above are implied by these estimates, but their seperate formulation helps to make the different aspects of our topic clear. (The 2-dimensional case is again trivial.)

Theorem 3. *Let M be a 3-manifold, Ω a region in M and p a point in Ω. Let $\mathcal{F}$ be a C^2 contant mean curvature foliation of $\bar{\Omega}$ or $\bar{\Omega}\backslash\{p\}$ with closed leaves, which we assume to be simply connected (hence topological spheres). Then there are positive constants $R = R(\Lambda, \delta)$ and $C = C(\Lambda, \delta)$ such that if $\operatorname{diam}\Omega \leq R$, then*

$$\sup_{S\in\mathcal{F}} \left(\sup_S \|A_S\| \cdot \min\{1, \sigma(S), \operatorname{diam} S\} \right) < C. \tag{0.1}$$

Here Λ denotes an absolute bound on Ricci curvatures, δ a positive lower bound on conjugate radius, and the thickness $\sigma(S)$ of $\mathcal{F}$ along S is defined to be the supremum of positive numbers r with the following property. If $q \in S$, then one of the two geodesics which start at q in the direction of one normal of S and have length r is contained in $\bar{\Omega}$.

The remarks above about higher dimensions also apply here. We point out that the condition on the diameter of $\mathcal{F}$ is solely for the purpose of having $\mathcal{F}$ contained in a (geometrically controlled) coordinate chart. Thus an alternative version of Theorem 3 is the following result.

Theorem 4. *Let $\mathbb{R}^3$ be equipped with a Riemannian metric g. Assme that the C^2-norm (or a $C^{1,\alpha}$-norm) of g is bounded by a constant K and that $g \geq \lambda g_0$ for a positive constant λ, where g_0 denotes the Euclidean metric. Let $\mathcal{F}$ be a constant mean curvature foliation of $\bar{\Omega}\backslash\{p\}$ or $\bar{\Omega}$ for a bounded region Ω in $\mathbb{R}^3$ with $p \in \Omega$, where mean curvature is measured in g. Assume that its leaves are all topolgical spheres. Then there is a positive constant $C = C(K, D, \lambda)$ such that*

$$\sup_{S\in\mathcal{F}}\left(\sup_{S}\|A_S\| \cdot \min\{1, \sigma(S), \operatorname{diam} S\}\right) < C, \tag{0.2}$$

where D denotes an upper bound for the maximum distance of points in $\mathcal{F}$ from the origin. In the estimates, all geometric quantities can be measured in g as well as in the Euclidean metric.

A parallel version for foliations on asymptotically flat ends holds, in which the number 1 is *removed* from (0.2), but the constant C depends in addition on a measure of the asymptotical flatness and a measure of the diameter-pinching property. Moreover, the foliation is required to be contained in an asymptotical chart. We leave it to the reader to formulate the precise statements. An open question regarding the above curvature estimates is whether the condition of simple connectedness is indispensible. Note that it is quite unusual that global topological structure plays a role in pointwise curvature estimates. On the other hand, it would be very nice if the condition on the size of the foliation can be dropped.

The initial motivation for the research undertaken in this paper came from the following context: exsitence and uniqueness of foliations by constant mean curvature spheres around a point and on an aymptotically flat end. In [**Y1**], we showed that around a nondegenerate critical point of the scalar curvature function in a Riemannina manifold, there exists a regular foliation by constant mean curvature spheres. Uniqueness was obtained there under the regularity assumption. The arguments in [**Y1**] extend to produce so-called "balanced" foliations by constant mean curvature spheres on an asymptotically flat end. Uniqueness in this context is more subtle, but a crucial ingredient is still regularity. For a detailed discussion see [**Y2**].

As a consequence of Theorem 1 and [**Y1**], we obtain the following complete existence (non-existence) and uniqueness results for constant mean curvature foliations around a point in dimensions 3 and 2:

Theorem 5. *Let M be a 3-manifold or a 2-manifold and $p \in M$. If there is a C^2 foliation by constant mean curvature spheres around p, then p must* *be a critical point of the scalar curvature function. If p is a non-degenerate critical point of the scalar curvature function, then there is a neighborhood of p which contains a unique C^2 foliation by constant mean curvature spheres around p, which is actually smooth.*

Part of the above results can be formulated in the following way:

Theorem 5′. *In dimensions 2 and 3, the center singularities of a constant mean curvature foliation must be critical points of the scalar curvature function.*

Similarly, Theorem 2 and the Main Theorem in [**Y2**] imply the following

Theorem 6. *On a 3-dimensional asymptotically flat end of non-zero mass, there is a unique diameter-pinched C^2 foliation by constant mean curvature spheres, which is actually smooth.*

We would like to mention that G.Huisken and S.T.Yau [**HY**] obtained independently existence of foliations by constant mean curvature spheres on asymptotically flat ends of positive mass, and showed uniqueness in the class of foliations with (mean) stable leaves in this case.

Now some words about the proof of Theorem 1. Since the proof is rather technical, we try to describe its main points here. We start with a rescaling procedure. Assuming namely that there is a non-controlled center, we rescale the leaves around it suitably to make the maximum value of the norm of their second fundamental form become 1, and then obtain limits. The limits more or less constitute a foliation of some Euclidean region by constant mean curvature surfaces. If these surfaces are actually minimal, then we can apply the Bernstein type theorem mentioned before to derive a contradiction. Hence the key is to show the minimality of the limit leaves. This is rather intricate and quite involved. The basic idea is this. There is a known height estimate for compact graphs of constant mean curvature, say 1, which is obtained by applying the maximum principle to a natural function which we call the *geometric height function.* If our limit leaves are not minimal, we can rescale them to achieve mean curvature 1. On the other hand, they are noncompact and come from compact surfaces, and hence we hope to find portions in them which are compact, tall graphs, thereby producing a contradiction to the said height estimate.

One might want to try to apply the height estimate directly to the rescaled leaves before taking limit. One trouble here is that for this purpose we need the leaves to be uniformly close to having Euclidean constant mean curvature 1, which may not happen because of the noncompactness of their limits. Another more important point is that this estimate only applies to graphs.

On the other hand, it is not clear how to find tall graphs in the rescaled leaves without utilizing limits. A basic method for proving graph property of constant mean curvature surfaces is the Alexandrov reflection principle, which does not extend to an almost Euclidean situation in general (c.f. a relevant discussion before). Now how can one find compact, tall graphs in the limit leaves? The limit leaves come from compact surfaces, so one may naively think that they are easy to come by. But it is by no means clear that they can always be found. The whole process of finding useful graphs turns out to be rather involved, and the fact that the original leaves are simply connected is used in a crucial way. First we observe that we can utilize the entire rescaled leaves, namely we can use limits of them other than the chosen ones.

To construct useful graphs in limits of the rescaled leaves, we design a key cutting scheme: *trimming*. Using it we isolate pieces which are suitably confined and sufficiently tall. It is here that we need to use the fact that the original leaves are simply connected. After enough trimming, we pass to a limit. The limit surface contains a point where the geometric height function assumes a maximum. To show that it is a graph, we try to apply the Alexandrov reflection principle as mentioned before. But there is an obstacle here: the limit surface may not be compact. It also bring with it another difficulty, namely the tallness may be lost in the limit. The trick for overcoming this obstacle is to build the reflection argument into the limit process. An ingredient here is an argument for finding *necks*, which is a device for utilizing the trimming property. Eventually we arrive at a situation in which we have a tall graph and the said height estimate holds, and we obtain the desired contradiction.

The proof of the curvature estimates is essentially the same as the above arguments. We find it convenient to also include some results on a priori estimates for mean stable surfaces, see Section 4. These are quite easy, but useful. A relevant open question here is this: when are the leaves in the constant mean curvature foliations constructed in [**Y1**] stable or unstable?

We acknowledge interesting discussions with G. Huisken concerning foliations on asymptotically flat manifolds.

1. Regularity and Curvature Estimates.

Let M be a Riemannian manifold of dimension $n+1$ with $n \geq 1$. Let $\mathcal{F}$ be a codimension one foliation on a domain Ω of M. We shall assume that $\mathcal{F}$ has C^2 leaves. We say that $\mathcal{F}$ is of class C^k, if the local coordinates for $\mathcal{F}$ are of class C^k.

Definition 1. A point $p \in M \backslash \Omega$ is called a *center* or *center singularity* of

$\mathcal{F}$, if there is a neighborhood U of p such that $\overline{U}\setminus\{p\} \subset \Omega$ and the restriction of $\mathcal{F}$ to $\overline{U}\setminus\{p\}$ has closed leaves. U is called a *center region* of $\mathcal{F}$. If we have moreover

$$\sup_{S\in\mathcal{F}'}\left(\sup_{S}\|A_S\|\operatorname{diam}S\right)<\infty,$$

where $\mathcal{F}'$ is the restriction of $\mathcal{F}$ to $\overline{U}\setminus\{p\}$ and A_S denotes the second fundamental form of S, then we say that $\mathcal{F}$ is *regular* or *controlled* at p, and that p is a *controlled* center.

Lemma 1. *Let U be a center region of a foliation $\mathcal{F}$ of class C^k and p the corresponding center. Let $\mathcal{F}'$ denote the restriction of $\mathcal{F}$ to $\overline{U}\setminus\{p\}$. Then the leaves of $\mathcal{F}'$ can be parametrized as a C^k-family S_t, $0<t\leq 1$ with $S_t\neq S_{t'}$ if $t\neq t'$ and $\lim_{t\to 0}\operatorname{diam}S_t=0$. Moreover, these leaves are simply connected if $n\geq 2$.*

Proof. The proof of Lemma 2.1 in [**Y1**] can be quoted word by word to produce the desired parametrization. Now assume $n\geq 2$. Consider the closed region V_t bounded by S_1 and S_t, which is homotopy equivalent to S_1 on account of the parametrization. If t is sufficiently small, then all loops on S_t can be homotoped to a point in a closed region which is enclosed in S_1 and does not contain the center. Hence they can be homotoped to a point in the region $V_{t'}$ for a smaller t'. By the said homotopy equivalence we deduce that all the leaves are simply connected. □

Definition 2. A *constant mean curvature foliation* in M is a codimension one foliation of class C^2 on a domain of M whose leaves have (generally varying) constant mean curvature.

Proof of Theorem 1. Let $\mathcal{F}$ be a constant mean curvature foliation in a 3-manifold M and p a center of $\mathcal{F}$. Consider an associated center region U. Let S_t, $0<t\leq 1$ be the parametrization of the restriction $\mathcal{F}'$ of $\mathcal{F}$ to $\overline{U}\setminus\{p\}$ as given by Lemma 1. We assume $\lim_{t\to 0}\ \sup_{S_t}\|A_{S_t}\|\operatorname{diam}S_t=\infty$ and are going to derive a contradiction. Choose $t_k\to 0$, $T_k\to 0$ with $T_k>t_k$ such that,

i) $\operatorname{diam} S_{T_k}\to 0$,
ii) $\sup_{S_{t_k}}\|A_{S_{t_k}}\|$ diam $S_{t_k}=\sup_{t_k\leq t\leq 1}(\sup_{S_t}\|A_{S_t}\|$ diam $S_t)$,
iii) $\sup_{S_{t_k}}\|A_{S_{t_k}}\|$ diam $S_{t_k}\to\infty$,
iv) $\operatorname{dist}(S_{t_k},\ S_{T_k})\geq 10$ diam S_{t_k} and
v) $\operatorname{dist}(S_{t_k},\ S_{T_k})\geq 1/\sup_{S_{t_k}}\|A_{S_{t_k}}\|$.

It's not hard to see that such a choice is possible. Note that ii) implies

$$\sup_{S_{t_k}}\|A_{S_{t_k}}\|=\sup_{t_k\leq t\leq 1}\left(\sup_{S_t}\|A_{S_t}\|\right), \tag{1.1}$$

because the diameter is an increasing function. We denote by $\mathcal{F}_k$ the foliation $\mathcal{F}'$ restricted to the parameter range $t_k \le t \le T_k$ and by Ω_k the domain bounded by S_{T_k}. Via the exponential map at p we can consider the Ω_k's as domains in $\mathbb{R}^3$. For each fixed k, we choose $p_k \in S_{t_k}$ with $\|A_{S_{t_k}}(p_k)\| = \sup_{S_{t_k}} \|A_{S_{t_k}}\|$ and translate Ω_k so that p_k becomes the origin. Then we dilate Ω_k by the factor $a_k = \|A_{S_{t_k}}(p_k)\|$ to get a new domain Ω_k^*. The dilation of $\mathcal{F}_k$ yields a foliation $\mathcal{F}_k^*$ whose leaves have constant mean curvature with respect to the dilated metric g_k. Moreover,

$$\sup_{t_k \le t \le T_k} \left(\sup_{S_t^{(k)}} \left\| A_{S_t^{(k)}} \right\| \right) = 1 = \left\| A_{S_{t_k}^{(k)}}(0) \right\|, \tag{1.2}$$

where $S_t^{(k)}$ denotes the image of S_t under the dilation and the second fundamental form is measured in g_k. We also have

$$\min_{t_k \le t \le T_k} \left(\operatorname{diam} S_t^{(k)} \right) \to \infty \text{ as } k \to \infty, \tag{1.3}$$

where the diameter of $S_t^{(k)}$ is measured in g_k. Notice that the domains Ω_k^* approach $\mathbb{R}^3$ and g_k approach the euclidean metric in the smooth topology everywhere uniformly in Ω_k^* (the uniform convergence follows from the property i) above).

Our goal is to obtain suitable limits from the foliations $\mathcal{F}_k^*$ in order to derive a contradiction. We start with the "innermost" leaves $S_{t_k}^{(k)}$. By the curvature bound (1.2), the constancy of mean curvature and the smooth convergence of g_k it is easy to see that a subsequence of $S_{t_k}^{(k)}$ converges locally smoothly. This subsequence will still be denoted by $S_{t_k}^{(k)}$. (Throughout the following we shall adopt this convention of retaining the notation for a sequence when passing to a subsequence. Often we do not explicitly mention passing to a subsequence.) The limit of $S_{t_k}^{(k)}$ consists of countably many complete, noncompact and connected immersed surfaces of constant mean curvature. Here the noncompactness follows from (1.3) and the fact that the $S_{t_k}^{(k)}$'s are embedded (and connected). One of these surfaces passes through the origin and its second fundamental form has length 1 there. We denote this surface by S^∞. Note that self-intersection of S^∞ can only occur in the way that one (embedded) piece of S^∞ meets another from one side. Such self-intersection will be called "one-sided".

We need the following crucial lemma.

Lemma 2. *S^∞ has zero mean curvature.*

Proof. **Part 1. Initial Step.** Assume the contrary. By rescaling and choosing orientation we may assume that the mean curvature of S^∞ is 1. To derive a contradiction, we shall utilize the entire leaves $S_{t_k}^{(k)}$. It is convenient to first perturb them so that the coordinate function x^3 becomes a Morse function on each of them. We can choose the perturbations so small that the limit S^∞ is unaffected. We assume that these leaves have been so perturbed and keep their notations.

Let S stand for $S_{t_k}^{(k)}$. Consider horizontal (with respect to x^3) planes P_i, $i = 1, 2, ..., m$ for some m, which lie one above another successively at a distance between 10 and 11 and intersect S transversally, such that the highest point of S lies above P_1 at a distance between 10 and 11, and the lowest point of S lies below P_m at a distance between 10 and 21. We can assume that S is tall enough such that these hyperplanes exist.

Part 2. Trimming. Consider the part of S lying above P_1. Let S' be one of its components. A component of $S \backslash S'$ is called a *leg* of S *growing from* S'. A leg of height at least 40 is called a *long leg*.

Claim 1. A leg has only one boundary component.

Consider a component S' and the legs growing from it, one of which S'' has at least 2 boundary components L_1 and L_2. Note that S' and its legs are topologically plane domains, and S is obtained by gluing S' to its legs along their common boundary circles. Using these facts, it is easy to see that the genus of S must be at least 1, contradicting the simple connectedness assumption. (This argument can be extended to all dimensions.) Thus the claim is proved.

Next we expand the concept of legs. First, if S'' denotes a component of the part of S lying *below* P_1, then following the terminology before, a component of $S \backslash S''$ is called a leg of S growing from S''. These new legs can be called “upward legs”, whereas the old legs can be called “downward legs”. The words “upward” and “downward” indicate how a leg grows out its boundary; both kinds of legs reach low in the end. Second, for the purpose of defining legs, so far we have only used the plane P_1 to cut S; from now on we may use any of the $P_i' s$. With the expanded definition, Claim 1 still holds. Now we choose a smallest long leg $S(1)$, i.e. a long leg which does not contain properly another long leg.

Consider the components of the part of $S(1)$ above P_m. One of them connects to the boundary of $S(1)$. Denote it by $S(2)$. Let $S(3)$ be another component. We reach its highest level, and choose the P_i which lies right below it. Consider an upward leg $S(4)$ contained in $S(3)$ which is defined with respect to P_{i+1} and containes a highest point of $S(3)$. In particular, $S(4)$ is no shorter than 10, measured from its boundary level P_{i+1}. Since $S(1)$ is a smallest leg, the height of $S(4)$ is less than 40. Notice that by Claim 1, $S(4)$ has only one boundary component. If no $S(3)$ exists, then we consider the closure of the part of $S(1)$ lying below P_m. Denote it by $S(4)'$. It has only one boundary component, which is contained in P_m. Moreover, it lies between P_m and the horizontal level 23 below P_m and is no shorter than 10, measured downward from its boundary level P_m.

We let $S(k)$ denote the one among $S(4)$ and $S(4)'$ which occurs. After suitable translation and rotation, we can achieve the following: the boundary of $S(k)$ is contained in $P = \{x^3 = 0\}$, the highest point of $S(k)$ occurs in the open upper half space $V = \{x^3 > 0\}$ at height at least 10, and all points of $S(k)$ are higher than -30. We say that $S(k)$ is *trimmed* in the x^3-direction.

Part 3. Further Trimming. Consider a direction $\mathbf{e}$ orthogonal to x^3. Consider a plane P' orthogonal to $\mathbf{e}$ for which there is a subsequence of $S(k)$ with the following property. We perform an arbitarily small perturbation of each $S(k)$ to make them intersect P' transversally away from their boundary (the boundary is unperturbed). Legs of $S(k)$ with respect to P' can be defined as before, with the additional requirement that a leg's boundary does not intersect the boundary of $S(k)$. We assume that each $S(k)$ (in the chosen subsequence) has a long leg. We replace $S(k)$ by one such leg. This process is called *separation.* If no P' with the said property exists, then we say that $S(k)$ is *inseparable in the direction of* $\mathbf{e}$, otherwise *separable in the direction of* $\mathbf{e}$. Now if $S(k)$ is separable in a direction $\mathbf{e}$ orthogonal to x^3, we perform the said separation process and trimm it in the $\mathbf{e}$ direction. Then we change coordinates, letting $\mathbf{e}$ be the new x^3, and change the old x^3 to x^2. Now we check whether the new sequence $S(k)$ is separable in some direction orthogonal to both x^2 and x^3. If it is, we perform the separation and trimming process in that direction, and then change coordinates. It is easy to see that we then arrive at a sequence $S(k)$ which is trimmed in the direction of x^3 and is inseparable in all directions orthogonal to x^3. We say that it is *well-trimmed.*

Part 4. Controlling Graph Height. We choose a point $\tilde{q}_k$ on $S(k)$ such that the function $x^3 - \nu^3$ achieves its maximum at $\tilde{q}_k$, where ν denotes the "upward" unit normal of $S(k)$ (it points upward at the highest points of $S(k)$). By a translation we can arrange that $\tilde{q}_k$ lie on the x^3-axis. Then we obtain from $\tilde{q}_k$ a limit point $\tilde{q}$ and from $S(k)$ a maximal connected limit surface S which contains $\tilde{q}$. Clearly the function $x^3 - \nu^3$ on S achieves its maximum at $\tilde{q}$. Set $W = \{x^3 > 6\}$. We are going to show that $S \cap W$ is a graph over P. We first estimate from below the height of the portion of $S(k)$ which is a graph.

A horizontal plane P' intersecting $S(k)$ is called a "critical plane" for $S(k)$, if the following is true. Let V' be the open half space above P'. Then $S(k) \cap V'$ is a graph over P and its reflection across P' lies on one side of $S(k) \cap (\mathbb{R}^3 \backslash V')$. Moreover, either its reflection meets $S(k) \cap (\mathbb{R}^3 \backslash V')$ at a point $p \notin P'$ or there is a point $q \in P'$ where $S(k)$ is vertical, i.e. its normal is parallel to P'. We call p a "first touching point" on $S(k)$ and q a "first vertical point". Since $S(k)$ are compact, the Alexandrov reflection procedure implies that $S(k) \cap W$ is a graph over P if W does not contain critical planes of $S(k)$. (Note that no maximum principle is needed for this assertion.) Let δ_k be the maximal height of the critical planes of S_k with the convention that $\delta_k = 0$ if S_k has no critical planes. We claim $\limsup_{k\to\infty} \ \delta_k \leq 6$.

Assume $\limsup_{k\to\infty} \ \delta_k > 6$. We are going to derive a contradiction. Passing to a subsequence we can assume $\delta_k > 6$ for all k. For each $S(k)$ choose a critical plane P_k of maximal height and a first touching or vertical point p_k associated with P_k. Passing to a subsequence we may achieve that either all p_k are first touching points or they are all first vertical points.

Case 1. All p_k are first touching points.

Let V_k denote the open half space above P_k and $q_k \in V_k$ the reflection of p_k across P_k. Let $S(k)^+$ be the component of $S(k) \cap V_k$ containing q_k and $S(k)^-$ the interior of the component of $S(k) \cap (\mathbb{R}^3 \setminus V_k)$ containing p_k. Then the reflection of $S(k)^+$ across P_k lies on one side of $S(k)^-$ and meets $S(k)^-$ at p_k. We claim that the maximal height h_k of $S(k)^+$ measured from P_k is uniformly bounded away from zero. To see this we first note that because of the curvature bound (1.2) there is a number $r > 0$ independent of k with the following property. If $q_k^* \in S(k)^+$ is a point where h_k is achieved, then the component $\mathbb{B}^k$ of $S(k) \cap \mathbb{B}_r(q_k^*)$ containing q_k^* is a graph over P.

Now if h_k approaches zero, we would be able to find a geodesic ball $B^k = B_\varepsilon(q_k')$ on $S(k)$ with $\varepsilon > 0$ independent of k such that B^k is disjoint from and lies below $\mathbb{B}^k$ and $dist(q_k', q_k^*) \to 0$. After suitable translations and passing to subsequencies we then obtain limits $q = \lim q_k^* = \lim q_k'$, $\mathbb{B}^\infty = \lim \mathbb{B}^k$ and $B^\infty = \lim B^k$. Since $\mathbb{B}^\infty$ is concave (i.e. upward convex) at q, it follows that B^∞ is also concave at q. From the normal ν on $S(k)$ we obtain limit normals ν^+ on $\mathbb{B}^\infty$ and ν^- on $\mathbb{B}$. Since we assumed that S^∞ has mean curvature 1, $\mathbb{B}^\infty$ and B^∞ also have mean curvature 1 with respect to ν^+ and ν^- respectively. We conclude that $\nu^+(q) = \nu^-(q)$. This, however, is impossible because $\mathbb{B}^k$ and B^k belong to the embedded and closed hypersurface $S_{t_k}^{(k)}$, which bounds a domain and does not permit its inside (outside) to approach its outside (inside). Thus the uniform positivity of h_k is verified.

Now we translate $S(k)$ so that p_k, q_k are moved to the x^{n+1}-axis. Then we obtain a limit plane P_∞ from P_k, a limit point p_∞ from p_k and a limit point q_∞ from q_k. We deal with two possible cases separately.

Subcase 1. $p_\infty \neq q_\infty$.

We obtain from $S(k)$ a maximal connected limit surface $S(\infty)$ which contains p_∞ and a maximal connected limit surface $S'(\infty)$ which contains q_∞. (It will be shown that these two limits are actually the same.) Let $S'(\infty)^+$ be the component of $S'(\infty)\cap V_\infty$ which contains q_∞, where V_∞ denotes the open half space above P_∞. (Note that by the maximum principle the closure $\overline{S'(\infty)^+}$ can meet P_∞ only along its boundary.) The interior of $S(\infty)\cap(\mathbb{R}^3\backslash V_\infty)$ has one component which contains p_∞. We call it $S(\infty)^-$. We can choose $S'(\infty)$, $S(\infty)$ and $S(\infty)^-$ in such a way that the reflection $P_\infty(S'(\infty)^+)$ of $S'(\infty)^+$ across P_∞ lies on one side of $S(\infty)^-$ and meets $S(\infty)^-$ at p_∞. Then the maximum principle implies that $P_\infty(S'(\infty)^+)$ coincides with $S(\infty)^-$.

Now if $\overline{S'(\infty)^+}$ is disjoint from P_∞, then it is a complete surface which covers the entire P_∞. We take a sphere of mean curvature $1/2$ lying below $P(\infty)$, and move it upward until it touches $S'(\infty)^+$ for the first time. Since $S'(\infty)^+$ is a limit of graphs, the sphere has to meet it from its downward side. This violates the maximum principle. Thus the closure of $S'(\infty)^+$ must intersect P_∞, and it extends across the intersection into $S'(\infty)$. Next consider an arbitary point p in this intersection, which is the same as the intersection of the closure of $S'(\infty)^+$ with the closure of $S(\infty)^-$. At p, the two surfaces must be tangent to each other, because they are limits from the same sequence of embedded surfaces. One sees readily that their commom tangent plane is either horizontal or vertical. The former can be ruled out. Indeed, if it happens, then $S'(\infty)$ and $S(\infty)$ are different. They must meet each other from one side at p. But this violates the maximum principle on acount of their orientation. Note that one consequence of vertical tangent planes is that the commom boundary L of the two surfaces is a smooth submanifold in P_∞. Next we claim that $S'(\infty)$ and $S(\infty)$ are indeed the same limit. In other words, $S'(\infty)^+$ and $S(\infty)^-$ fit together along their common boundary to make a maximal limit. There are various ways to see this. One way is this. If these two surfaces do not fit together at some point $p\in L$, then near it one piece of $S(k)$ approaches another from the wrong side, in a way similar to the one discussed earlier in the argument about the uniform positivity of the numbers h_k.

To proceed, let Ω denote the domain in P_∞ which lies underneath $S(\infty)$. Thus $L=\partial\Omega$. We first observe that Ω must be unbounded, for otherwise $S(\infty)$ would be a closed surface, which is impossible because $S(k)$ have nonempty boundary. Consequently we can find a straight line σ in P_∞ which divides Ω into components, such that one is unbounded and at least one on the other side of σ is 50 tall (seen in the direction orthogonal to σ). We can choose σ such that the the intersection of the vertical plane P_σ containing it with $S(\infty)$ is transversal. If the intersection of σ with the closure of Ω is compact, then we choose a component σ_0 of it, such that one unbounded component of $\Omega\backslash\sigma$ lies on one side of it, while another component of at least 50 tall lies on the other side. The corresponding component of the intersection of P_σ with $S(\infty)$ which lies above σ_0 is then a closed curve, which we call a *neck* of $S(\infty)$. Using this neck it is easy to find long legs in $S(k)$ for k large, contradicting their well-trimmed property. It follows that $\sigma \cap \Omega$ is unbounded. We choose one unbounded component σ_1. Consider a round disk in P_∞ whose radius is 4 and whose center p_0 lies on σ_1, such that both components of $\sigma_1 \backslash \{p_0\}$ are longer than 60. It is divided by σ_1 into two half disks. Both must meet L. Othewise, using the moving process introduced before, we can find a sphere of mean curvature $1/2$ touching the interior of $S(\infty)$ from below, violating the maximum principle. Now it is easy to find a neck of $S(\infty)$ whose projection in P_∞ is a line segment connecting L from one half disk to another. One then readily finds a long leg in $S(k)$ for k large, contradicting the well-trimmed property.

We have arrived at contradictions in all possiblities of Subcase 1.

Subcase 2. $p_\infty = q_\infty \in P_\infty$.

We can again obtain two maximal connected limit surfaces $S(\infty)$ and $S'(\infty)$, both passing through p_∞ this time. Their choice is as follows. Take a local piece of $S(k)$ containing p_k for each k and obtain from them a limit containing p_∞. Extend it maximally we obtain $S(\infty)$. $S'(\infty)$ is obtained similarly by considering q_k. By the arguments in Subcase 1, $S(\infty)^-$ and $S'(\infty)^+$ must meet at p_∞ vertically. Applying the Hopf boundary point lemma to $S(\infty)^-$ and the reflection $P_\infty(S'(\infty)^+)$ we deduce that they coincide with each other. From now on we can apply the arguments in Subcase 1 to reach condradictions.

Case 2. All p_k are first vertical points.

This is clearly similar to Subcase 2 of Case 1.

We have proven that $\limsup_{k\to\infty} \delta_k \le 6$. Applying the Alexandrov moving plane process we then deduce that for each plane P' parallel to P whose height lies in the open interval $(6, 10)$ the following holds. For large k the intersection $S(k) \cap V'$ (V' is the open half space above P') is a graph over P and its reflection across P' lies on one side of $S(k) \cap (\mathbb{R}^3 \backslash V')$. Taking limits we conclude that the reflection of $S \cap V'$ across P' lies on one side of $S \cap (\mathbb{R}^3 \backslash V')$ or $S \cap V'$ is empty. Now if $S \cap W$ is not a graph, then we would be able to find one such P' along with a point $p' \in P' \cap S$ such that S is vertical at p'. But then the above arguments for proving the inequality $\limsup_{k\to\infty} \delta_k \le 6$ can be applied to yield contradictions.

We have established that $S \cap W$ is a graph. By the moving sphere argument used before, we then deduce that the closure of $S \cap W$ must intersect the horizontal plane ∂W at height 6. Now we can argue as e.g. in [**M**]. We have

$$\Delta(x^3 - \nu^3) = (-2 + |A|^2)\nu^3 \ge 0$$

on the domain Ω in P which lies below $S \cap W$. Hence the maximum principle implies that $x^3 - \nu^3$ is a constant on $S \cap W$. Since $\overline{S \cap W} \cap (\partial W) \neq \phi$, it follows that $x^3 - \nu^3 \leq 6$ on S, contradicting the obvious fact that the maximum of $x^3 - \nu^3$ on S is at least 9.

The proof of Lemma 2 is now finished. □

Now we proceed with the proof of Theorem 1. We are going to show that S^∞ is stable. It is possible to construct a minimal foliation which contains S^∞ as a leaf. Then it will follow that S^∞ is area-minimizing in the support region of the foliation. But we shall follow a somewhat different strategy. Since S^∞ is minimal, it follows that S^∞ is embedded. Now we choose for a fixed k and each $\sigma \in (0,1)$ a leaf $S^k(\sigma)$ in the foliation $\mathcal{F}_k$ along with a point q^k_σ such that $\operatorname{dist}(q^k_\sigma, p_k) = \sigma/(\sup_{S_{t_k}} \|A_{S_{t_k}}\|)$. (Note that the points p_k were introduced before the proof of Lemma 2.) Translating p_k to the origin and then dilating $S^k(\sigma)$ by the factor a_k we obtain surfaces $S^k(\sigma)^*$. Note that $S^k(\sigma)^*$ contains a point $\overline{q}^k_\sigma$ such that $\operatorname{dist}(\overline{q}^k_\sigma, o) = \sigma$. Now we choose a sequence $\sigma_m \to 0$. Fixing m and letting $k \to \infty$ we obtain from $\overline{q}^k_{\sigma_m}$ a limit point q_{σ_m} and from $S^k(\sigma_m)^*$ a limit surface $S(\sigma_m)$ containing q_{σ_m}. The proof of Lemma 2 implies that $S(\sigma_m)$ is minimal. Note that the $S(\sigma_m)$'s are disjoint from S^∞. Since $\operatorname{dist}(q_{\sigma_m}, o) = \sigma_m$, the surfaces $S(\sigma_m)$ approach S^∞ locally smoothly as $m \to \infty$. For each large m we can represent a suitable geodesic ball in $S(\sigma_m)$ with center q_{σ_m} as a normal graph over a domain Ω_m in S^∞, such that Ω_m approaches S^∞ and the positive defining function φ_m for the graph approaches zero locally smoothly as $m \to \infty$. We put $\overline{\varphi}_m = \varphi_m/\varphi_m(o)$. Since φ_m satisfies the minimal surface equation, the Harnack inequality implies that a subsequence of $\overline{\varphi}_m$ converges locally smoothly to a positive function φ on S^∞. Let L be the Jacobi operator on S^∞. Then $L\varphi = 0$. It follows that S^∞ is stable. Indeed, let u be a positive eigenfunction on a compact domain Ω of S^∞. Multiplying the equation $L\varphi = 0$ by u and integrating over Ω show that the first eigenvalue is positive (see [**FS**] for the simple details).

Now the Berstein type theorem of Fischer-Colberie-Schoen [**FS**] and Do Carmo-Peng [**CP**] implies that S^∞ is a plane. This contradicts the fact that the maximal length of the second fundamental form of S^∞ is 1. □

Proof of Theorem 3. We apply the proof for Theorem 1 with minor modifications. Basically, we need to consider all possible metrics which satisfy a given $C^{1,\alpha}$-norm bound. Thus we deal with sequences of foliations associated with sequences of metrics. Scaling is slightly more complicated, but scaling limits will always consist of surfaces of constant Euclidean mean curvature. We omit the details. □

Proof of Theorem 4. By passing to a local covering, we can replace the conjugate radius bound by an injectivity radius bound. It is well-known that controlled harmonic coordinates exist under a Ricci curvature bound and an injectivity radius bound. In these coordinates $C^{1,\alpha}$-bounds on the metric tensor hold. Now we work in harmonic coordinates, and the situation is similar to that in Theorem 4. □

Proof of Theorem 5. This is a direct consequence of Theorem 1 and the main result in [**Y1**]. □

2. Asymptotically Flat Manifolds.

Let M be a complete Riemannian manifold of dimension $n+1$ with $n \geq 2$. Let g be the metric of M. A closed domain Ω of M is called an asymptotically flat end of order $\sigma > 0$ if there is a coordinate map (i.e. a diffeomorphism) from Ω to $\mathbb{R}^{n+1} \backslash \overset{o}{\mathbb{B}}_R(o)$ for some $R > 0$ such that on this coordinate chart the metric g satisfies

$$g_{ij} = \delta_{ij} + O(r^{-\sigma}),\ \partial_k g_{ij} = O(r^{-\sigma-1}),\ \partial_k \partial_\ell g_{ij} = O(r^{-\sigma-2}),$$

as $r = |x| \to \infty$. (c.f. [**LP**]) The most important asymptotically flat ends arising in general relativity have the following more special property in place of the above asymptotical formulas

$$g_{ij}(x) = \left(1 + \frac{m}{r^{n-1}}\right) \delta_{ij} + h_{ij}(x)$$

with

$$\begin{aligned} &h_{ij} = O(r^{-n}),\ \partial_k h_{ij} = O(r^{-n-1}),\ \partial_k \partial_\ell h_{ij} = O(r^{-n-2}), \\ &\partial_k \partial_\ell \partial_{\ell'} h_{ij} = O(r^{-n-3}),\ \partial_k \partial_\ell \partial_{k'} \partial_{\ell'} h_{ij} = O(r^{-n-4}), \end{aligned}$$

where m is a constant called "mass" or "energy" (this differs from the conventional definition of mass by a dimensional factor). We shall call them "*standard asymptotically flat ends*". For sake of convenience, requirements on asymptotical flat ends were not spelled out in Theorem 2 and Theorem 5. The specific requirements are as follows: in Theorem 2 we allow general asymptotically flat ends of an arbitrary order $\sigma > 0$; in Theorem 5 we only allow standard asymptotically flat ends.

Definition 3. Let $\mathcal{F}$ be a C^2-foliation on an asymptotically flat end with compact leaves. We say that $\mathcal{F}$ is *regular* at ∞, if

$$\sup_{S \in \mathcal{F}} \left(\sup_S \|A_S\| \operatorname{diam} S \right) < \infty.$$

Definition 4. Let $\mathcal{F}$ be a codimension one foliation on an asymptotically flat end Ω with compact leaves. We say that $\mathcal{F}$ is *diameter-pinched* at ∞ or diameter pinched, if

$$\limsup \frac{\operatorname{diam}(p_o, S)}{\operatorname{dist}(p_o, S)} < \infty \text{ for } S \in \mathcal{F} \text{ as } \operatorname{diam}(p_o, S) \to \infty,$$

where p_o is a fixed point in Ω and $\operatorname{diam}(p_o, S) = \max_{q \in S} \operatorname{dist}(p_o, q)$.

Proof of Theorem 2. Let Ω be an asymptotically flat end of dimension 3 with metric g and $\mathcal{F}$ a constant mean curvature foliation on Ω whose leaves are all closed. Lemma 1 readily extends to the situation here and we have a C^2 parametrization S_t, $0 < t \leq 1$ of the leaves of $\mathcal{F}$ such that $S_t \neq S_{t'}$ if $t \neq t'$ and $\lim_{t \to 0} \operatorname{diam} S_t = \infty$. Moreover, the leaves are topological spheres. Applying the arguments in the proof of Theorem 1 one readily shows that $\limsup_{t \to 0} (\sup_{S_t} \|A_{S_t}\|) = 0$. Now assume that $\mathcal{F}$ is not regular at ∞. Then we can find $t_k \to 0$ such that

i) $\sup_{S_{t_k}} \left\|A_{S_{t_k}}\right\| = \sup_{t_k \geq t > 0} \left(\sup_{S_t} \|A_{S_t}\|\right)$,

ii) $\sup_{S_{t_k}} \left\|A_{S_{t_k}}\right\| \operatorname{diam} S_{t_k} \to \infty$.

We denote by $\mathcal{F}_k$ the foliation $\{S_t : t_k \geq t > 0\}$. Dilating $\mathcal{F}_k$ by the factor $a_k = \sup_{S_{t_k}} \left\|A_{S_{t_k}}\right\|$ yields a new foliation $\mathcal{F}_k^*$ which has constant mean curvature leaves in the dilated metric g_k. Let $S_t^{(k)}$ denote the image of S_t under the dilation. Then

$$\sup_{t_k \geq t > 0} \left(\sup_{S_t^{(k)}} \left\|A_{S_t^{(k)}}\right\|\right) = 1 = \sup_{S_{t_k}^{(k)}} \left\|A_{S_{t_k}^{(k)}}\right\|,$$

where the second fundamental form is measured in g_k. Moreover,

$$\operatorname{diam}\ S_{t_k}^{(k)} \to \infty \text{ as } k \to \infty.$$

Because $\mathcal{F}$ is diameter-pinched at ∞, we have $\operatorname{dist}_o(o, S_{t_k}^{(k)}) \to \infty$ as $k \to \infty$, where dist_o denotes Euclidean distance. This ensures that g_k converges smoothly to the Euclidean metric on the domain of $\mathcal{F}_k^*$. (We only need $\operatorname{dist}_o(o, S_{t_k}^{(k)})$ to be uniformly bounded away from zero.) From here on we can apply the arguments in the proof of Theorem 1 to derive a contradiction. □

Proof of Theorem 6. It is easy to see that the condition "diameter-pinched" implies the weak balance condition in [**Y2**]. Hence Theorem 6 follows from Theorem 2 and the Main Theorem in [**Y2**]. □

Finally we note that there is another natural geometric condition which implies regularity at ∞.

Definition 6. Let $\mathcal{F}$ be a C^2-foliation on an asymptotically flat end Ω with compact leaves. We say that $\mathcal{F}$ is *non-degenerate* at ∞ or non-degenerate, if

$$\liminf_{\operatorname{dist}(p_o,S)\to\infty} \operatorname{dist}(p_o,S)\sup_S\|A_S\| > 0 \text{ for } S\in\mathcal{F},$$

where p_o is a fixed point in Ω. We say that $\mathcal{F}$ is *weakly non-degenerate* (at ∞), if

$$\liminf_{\operatorname{dist}(p_o,S)\to\infty} \max\left\{\operatorname{dist}(p_o,S)\sup_S\|A_S\|,\ \frac{1}{\sup_S\|A_S\|\operatorname{diam} S}\right\} > 0 \text{ for } S\in\mathcal{F}.$$

It is easy to see that one can replace the condition "diameter-pinched" in Theorem 2, Theorem 3 and Theorem 5 by "weakly non-degenerate".

3. Stable Surfaces.

Consider a Riemannian manifold M of dimension 3. Let S be a closed, immersed surface in M which has constant mean curvature and is (mean) stable, i.e.

$$\int_S (\|\nabla\varphi\|^2 - \|A\|^2\varphi^2 - Rc(\nu)\varphi^2)\,d\text{vol} \geq 0$$

for all C^1 functions φ of compact support with $\int_S \varphi d\text{vol} = 0$. Here ν denotes a unit normal of S and Rc the Ricci curvature. Our goal is to derive a priori curvature estimates for S which are independent of the mean curvature of S. Assume that there is a sequence of such surfaces S_k so that $\sup_{S_k}\|A_{S_k}\|\operatorname{diam} S_k \to \infty$ as $k\to\infty$. Choose a point $p_k\in S_k$ such that $\sup\|A_{S_k}\|$ is achieved at p_k. Let r be the injectivity radius of M. Applying suitable coordinate maps we can consider the intersection $S_k\cap B_r(p_k)$ of S_k with the geodesic ball $B_r(p_k)$ as lying in the Euclidean ball $\mathbb{B}_r(o)$. As in the last sections, after suitable dilation we can obtain from S_k a limit S such that S is immersed, complete and noncompact and that $\sup\|A_S\| = \|A_S(o)\| = 1$. Moreover, S is stable. The Bernstein type theorem of B. Palmer [**P**] and A. Silveira [**Sv**] then implies that S is a plane, contradicting the fact that $\|A_S(o)\| = 1$. We conclude that the sequence S_k cannot exist. Hence we have proved the estimate stated in the following theorem, but with a constant C depending on the manifold. To achieve the claimed explicit dependence, we apply the arguments in the proof of Theorem 3.

Theorem 7. *Let M be a 3-manifold. Then there is a constant $C>0$ depending on an absolute bound for Ricci curvatures and a positive lower* *bound for injectivity radius, such that*

$$\sup_S \|A_S\| \operatorname{diam} S \leq C$$

for every stable closed immersed surface S of constant mean curvature in M.

Corollary . *Let M be a compact manifold of dimension 3. Then there is a constant $r > 0$ with the following property. If $p \in M$ is not a critical point of the scalar curvature function, then $B_r(p)$ contains no stable closed immersed surface of constant mean curvature.*

This corollary follows rather easily from Theorem 7 and the arguments in [**Y1**]. Note however that the Alexandrov's reflection principle used in [**Y1**] should be replaced by the theorem of L. Barbosa and M. do Carmo in [**BC**] (see also Wente's proof of this theorem [**W**]), which says that stable closed immersed hypersurfaces of constant mean curvature in Euclidean spaces are round spheres.

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BOCHUM UNIVERSITY
44780 BOCHUM, GERMANY
E-mail address: ye@dgeo.ruhr-uni-bochum.de

UNIVERSITY OF CALIFORNIA
SANTA BARBARA, CA 93106
E-mail address: yer@math.ucsb.edu