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# RATIONAL PONTRYAGIN CLASSES, LOCAL REPRESENTATIONS, AND $K^G$-THEORY

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**Suppose that $X$ and $Y$ are connected, simply connected Spin$^c$-manifolds of the same dimension. Let $G$ be a compact connected Lie group with torsion-free fundamental group which acts upon $X$ and $Y$ such that $X^G$ and $Y^G$ are non-empty and consist entirely of isolated fixed points. Suppose that $f : X \to Y$ is a smooth $G$-map such that the induced map**

$$f^* : K_G^*(Y) \to K_G^*(X)$$

**is an isomorphism. If $X$ and $Y$ are even-dimensional then for each fixed point $x \in X^G$, the local representations of $G$ at $x$ and at $f(x)$ are equivalent. If $f : X \to Y$ is an equivalence then**

$$f^* : H^*(Y;\mathbb{Q}) \to H^*(X;\mathbb{Q})$$

**preserves Pontryagin classes.**

## 1. Introduction.

Suppose that $X$ and $Y$ are compact smooth manifolds and $f : X \to Y$ is a smooth (homotopy) equivalence. In general, the map $f$ does not preserve rational Pontryagin classes, which depend a priori upon the smooth structures on $X$ and $Y$, unless $f$ happens to be a diffeomorphism. S. P. Novikov proved in 1965 [**N1**, **N2**] that if $f$ is a *homeomorphism* then rational Pontryagin classes are indeed preserved, and this remains the best general positive result on the subject. In 1981 Sullivan and Teleman jointly provided a proof of Novikov's result using differential geometric and analytic techniques, and recently Shmuel Weinberger gave a "short and conceptually simple analytic proof" of Novikov's theorem drawing upon new ideas in index theory for non-compact complete Riemannian manifolds. Baum and Connes [**BC**] have studied the foliated version of the problem and have positive results in the presence of negatively curved leaves.

In the early 1970's Ted Petrie developed a connection between the problem of preservation of Pontryagin classes and another classical problem. If $G$ is a compact Lie group, $X$ and $Y$ are smooth $G$-manifolds, and $f : X \to Y$

is a smooth $G$-map which restricts to a diffeomorphism on fixed point sets then what are the relations among the local representations[1] at the various fixed points $x$, $f(x)$ for $x \in X^G$? The classic theorem of this genre is due to Atiyah, Bott, and Milnor:

**Theorem** [**AB**, Theorem 7.27]. *Let $G$ be a compact Lie group acting differentiably on a homology sphere $X$. Assume that $G$ has just two fixed points $p$ and $q$ and that elsewhere the action is free. Then the local representations of $G$ at $p$ and $q$ are equivalent.*

This is a very sensitive result. There are counterexamples to the theorem (due to Petrie [**P 5, 6**] and to Cappell-Shaneson [**CS**]) if the action is not assumed to be free off the fixed point set. For instance, Cappell-Shaneson construct an action of the group $\mathbb{Z}/8$ on a homotopy 9-sphere with exactly two fixed points $p$ and $q$ where the local representation at $p$ is not equivalent to the local representation at $q$.

Petrie [**P 1, P 2**] considered a very special situation. He took $G = S^1$ and he assumed that the fixed point sets consist of isolated points which correspond under $f$. In that situation, with some further assumptions on $X$ and $Y$, he showed that if

$$f^* : K^*_{S^1}(Y) \to K^*_{S^1}(X)$$

is an isomorphism then rational Pontryagin classes are indeed preserved and the induced local representations at $x$ and $f(x)$ are equivalent for each $x \in X^{S^1}$.

In the present paper we shall focus attention upon the more general case where $G$ is a compact connected Lie group with $\pi_1(G)$ torsionfree. Our most precise results still require that fixed point sets consist of isolated points. [2] In that situation we are able to obtain results which are analogous to those of Petrie.

**Note:** Throughout this paper a "manifold" is understood to be smooth, compact (with the obvious exceptions of vector spaces) and without boundary. Following [**RS**] we shall call a compact connected Lie group with torsionfree fundamental group a "Hodgkin group" to recognize Luke Hodgkin's fundamental contribution [**H**]. An action of a compact Lie group $G$ upon a

[1] If $x \in X^G$ is a fixed point then differentiation produces a linear action of $G$ on the tangent space $\tau X_x$; this action is called the **local representation** of $G$ at $x$. Two local representations are said to be **equivalent** if they are linearly equivalent as $G$-representations.

[2] If $X^G$ is allowed to be more complicated then Cappell and Weinberger have constructed counterexamples to the following Theorem.

manifold is understood to be a smooth action, and we may take this to be an action by isometries without loss of generality by averaging the metric. We shall assume that this has been done wherever necessary. If $X$ is a $G$-manifold then its underlying manifold is denoted $uX$, and if $f : X \to Y$ is a $G$-map then $uf : uX \to uY$ is the same map with all $G$-structures forgotten.

Here is the main theorem.

**Theorem 10.1.** *Suppose that $X$ and $Y$ are connected, simply connected* $\mathrm{Spin}^c$*-manifolds of the same dimension. Let $G$ be a Hodgkin group which acts upon $X$ and $Y$ such that $X^G$ and $Y^G$ are non-empty and consist entirely of isolated fixed points. Suppose that $f : X \to Y$ is a smooth $G$-map such that the induced map $f^* : K_G^*(Y) \to K_G^*(X)$ is an isomorphism. Then:*

(1) *If $X$ and $Y$ are of the same even dimension then for each fixed point $x \in X^G$, the local representations of $G$ at $x$ and at $f(x)$ are equivalent.*

(2) *If $uf : uX \to uY$ is an equivalence then $uf^* : H^*(uY;\mathbb{Q}) \to H^*(uX;\mathbb{Q})$ preserves Pontryagin classes.*

Although the statement of the main results of this paper would seem to be within the realm of classical algebraic topology, our proofs rely ultimately on the Universal Coefficient Spectral Sequence [**RS**] which is a spectral sequence which converges to the equivariant Kasparov group $KK_*^G(A,B)$ for suitable $G$-$C^*$-algebras $A$ and $B$ [**K 2**]. (In the present application $A = C(X)$ and $B = \mathbb{C}$.) It may be possible to construct the special case of the spectral sequence needed in this paper or, more likely, a spectral sequence converging to $KK_*^G(C(X), C(Y))$ by classical methods as suggested, for instance, in the Seattle Notes of J.F. Adams [**Ad**]. This might be a worthwhile project in order to eliminate the torsionfree hypothesis on $\pi_1(G)$, or as part of a larger project of studying bivariant theories (equivariant or not) of the form

$$EE^*(X,Y) = [X, E \wedge Y]_*$$

for some ring spectrum $E$.

It is possible that a version of this Theorem holds for arbitrary compact connected Lie groups. The Hodgkin condition comes into play in the proof of Theorem 10.1 in two completely different parts of the proof. First, it is used in an essential manner in the proof of the Universal Coefficient Spectral Sequence. The version of that spectral sequence which is needed here (4.1) is very weak, and it is quite plausible that (4.1) generalizes to arbitrary compact connected Lie groups. See Remark 4.14. Second, the Hodgkin condition is used in §9 to show that certain non-equivariant $K$-orientations give rise to $K^G$-orientations. If one were given the $K^G$- orientations a priori (for instance, by assuming that $X$ and $Y$ were complex $G$-manifolds) then here too it would suffice to assume that $G$ is a compact connected Lie group.

The remainder of the paper is organized as follows:

**§2. The relative $\hat{A}$-class and $K^G$-orientations.**

We briefly recall information on the Pontryagin classes and their relationship to the $\hat{A}$-class. Then we introduce the notion of $K^G$-orientations and fundamental classes and demonstrate their utility in our context.

**§3. The Theta invariant.**

In this section the $K^G$- Theta invariant $\Theta^G(f) \in K^0_G(Y)$ of a $G$-map $f : X \to Y$ is defined and its relation to the relative $\hat{A}$-class $\hat{A}(\nu_f)$ is established. If $f$ preserves $H$-orientation, then

$$\hat{A}(\nu_f) = 1 \Longleftrightarrow \Theta(f) \equiv 1 \qquad \text{mod } \mathbb{Z}\text{-torsion.}$$

We switch attention to $K^G$-homology and demonstrate that

$$\Theta^G(f) \cap [Y] = f_*[X]$$

so that $\Theta^G(f)$ measures the truth of the hypothetical equality

$$[Y] \stackrel{?}{=} f_*[X] \in K^G_*(Y)$$

of $K^G$-fundamental classes.

**§4. Spectral sequences: moving from cohomology to homology.**

We demonstrate under suitable hypotheses (4.9) that if

$$f^* : K^*_G(Y) \longrightarrow K^*_G(X)$$

is an isomorphism then the associated map on homology

$$f_* : K^G_*(X) \longrightarrow K^G_*(Y)$$

and the Gysin map

$$f_! : K^*_G(X) \longrightarrow K^*_G(Y)$$

are also isomorphisms. Our proof requires $G$ to be a Hodgkin group, though we believe that the result should be true for arbitrary compact connected Lie groups.

**§5. Localizations and fixed point sets.**

In this section attention turns to the fixed point sets of the $G$-actions. Under very general circumstances the Atiyah-Segal localization theorem allows us to restrict to fixed points, provided that the theory itself is suitably localized. It turns out that if $G$ is a Hodgkin group and $K^*(X^G)$ is $\mathbb{Z}$-torsionfree then the map

$$\imath_* : K_*^G(X^G) \longrightarrow K_*^G(X)$$

is a monomorphism which induces an isomorphism on fraction fields. These results help us to relate $K^G$-orientations on $X$ to those on $X^G$ and hence to localize the data to fixed point sets.

## §6. Isolated fixed points.

This section is devoted to a detailed analysis of the additional structure that is forced by the assumption that the fixed point set $X^G$ consist of isolated points. For example, under the usual hypotheses, if $X^G$ consists of isolated fixed points then the induced map

$$\imath_*^X : K_*^G(X^G) \longrightarrow K_*^G(X)$$

is a *canonically split* monomorphism.

## §7. The main theorem for $K^G$-oriented Manifolds.

In this section we shall establish the main theorem on $K^G$- oriented manifolds, showing that Pontryagin classes are preserved and control is maintained at the fixed point sets. These results involve assuming that $X$ and $Y$ have $K^G$-orientations.

## §8. Bilinear pairings.

In this section we pause to address two questions raised by Petrie in Part II, §3 of [**P 1**]. Suppose that $G$ is a compact connected Lie group. Define

$$\mathfrak{K}_G^*(X) = K_G^*(X)/(R(G)\text{-torsion}).$$

Suppose that $X$ is a $K^G$-oriented $G$-manifold. Then there is a natural $R(G)$-valued bilinear form on $\mathfrak{K}_G^*(X)$ determined by Poincaré duality and the Kronecker pairing. Petrie [**P 1**, page 144] asks the following two questions:

**Question 1:** *When is the bilinear form on $\mathfrak{K}_G^*(X)$ nondegenerate?*

Answer: If $G$ is a Hodgkin group and $\mathfrak{K}_G^*(X)$ is $R(G)$-free, then the form is always nondegenerate.

**Question 2:** *When the preceeding question has an affirmative answer, can one relate the algebraic invariants of the bilinear form to the representations of $G$ on the fibers normal to the fixed points sets?*

Answer: Yes, and in a very concrete fashion when each fixed point is isolated.

### §9. Invariant Spin$^c$ orientations.

In this section we recall the relationship between $K$-orientations and Spin$^c$-structures. Then we generalize these results and demonstrate that $K^G$-orientations are fairly common as indicated by the following theorem.

**Theorem 9.12.** *Suppose that $G$ is a Hodgkin group. Let $X$ be a connected, simply connected $G$-manifold. Suppose that $w_2(X)$ is the reduction of an integral class, or equivalently that $\tau X$ has a (non-equivariant)* Spin$^c$ *structure. Then $\tau X$ has a $G$-invariant reduction to* Spin$^c$ *which is (non-equivariantly) equivalent to the given* Spin$^c$*-structure, and thus $X$ has a $K^G$-orientation.*

### §10. Conclusion.

In this short section we pull together the results of the previous sections in order to establish the main theorem stated in the Introduction.

It is a pleasure to thank Sylvain Cappell, Peter May, Jonathan Rosenberg, Julius Shaneson, and Shmuel Weinberger for their generous assistance.

## 2. The Relative $\hat{A}$-class and $K^G$-Orientations.

In this section we briefly recall information on the Pontryagin classes and their relationship to the $\hat{A}$-class. Then we introduce the notion of $K^G$-orientations and demonstrate their utility in our context. The question of existence of such orientations is a classical problem in bundle theory. In order not to interrupt the flow of the argument, we put off the question of existence of such structures until Section 9.

**Definition 2.1.** Suppose that $X$ and $Y$ are manifolds and that $f : X \to Y$ is an equivalence. Denote the stable normal bundle (the formal difference in $KO$-theory) by

$$\nu_f = f^*(\tau Y) - \tau X.$$

The **relative Â-class** $\hat{A}(\nu_f) \in H^{**}(X; \mathbb{Q})$ is defined by [3]

$$\hat{A}(\nu_f) = \hat{A}(f^*\tau Y)/\hat{A}(\tau X).$$

The relative $\hat{A}$-class measures the extent to which $f$ fails to respect Pontryagin classes. The following proposition is well-known:

[3] We let $H^{**}(X)$ denote the $\mathbb{Z}_2$-graded cohomology theory $\left(\sum_{j\,even} H^j(X)\right) \oplus \left(\sum_{j\,odd} H^j(X)\right)$.

**Proposition 2.2.** *Suppose that $X$ and $Y$ are manifolds and $f : X \to Y$ is an equivalence. Then $\hat{A}(\nu_f) = 1$ if and only if the map*

$$f^* : H^{**}(Y; \mathbb{Q}) \to H^{**}(X; \mathbb{Q})$$

*preserves Pontryagin classes.*

Next we recall the basic structure of $K^G$-orientations and Poincaré duality in a context suitable for applications. In the non-equivariant setting this material is presented in the seminal paper of Atiyah, Bott, and Shapiro [**ABS**]. The equivariant case is presented in the context of generalized equivariant cohomology theories in [**LMS**]; we follow their treatment but specialize immediately to equivariant $K$-theory. Following [**LMS**] we index the theory by elements $\alpha \in R(G)$ with the understanding that in applications it suffices by the equivariant Thom isomorphism theorem to focus upon the cases $\alpha = 0, 1$. Let $G$ be a compact Lie group and let $\epsilon : R(G) \to \mathbb{Z}$ be the augmentation map. If $\xi$ is a vector bundle over a space $Y$ then its Thom space will be denoted $T\xi$. If $W$ is a finite-dimensional $G$-representation then $S^W$ denotes the one-point compactification of $W$ with its associated $G$-space structure. If $\jmath : G/H \to Y$ is the inclusion of an orbit then $\jmath^*\xi$ is of the form $G \times_H W \to G/H$ where $W$ is the fibre $H$-representation at $\jmath H$ and thus

$$T(\jmath^*\xi) \cong G^+ \wedge_H S^W.$$

**Definition 2.3.** A **$\mathbf{K^G}$-orientation** of $\xi$ is an element $\alpha \in R(G)$ such that $\epsilon(\alpha)$ is the fibre dimension of $\xi$ together with a class $\mu = \mu(\xi) \in \tilde{K}^\alpha_G(T\xi)$ such that the restriction of $\mu$ to $\tilde{K}^*_G(T(\jmath^*\xi)) \cong \tilde{K}^*_H(S^W)$ is a $\pi^H_*(K^G)$ generator for each orbit inclusion $\jmath : G/H \to Y$ with fibre representation $W$.

This makes sense, since $\tilde{K}^*_H(S^W)$, regarded as graded over $R(G)$, is a free $\pi^H_*(K^G)$-module on one generator, where $\pi^H_*(K^G)$ is also understood in the $R(H)$-graded sense. [4] In the present paper we may take $\mu \in \tilde{K}^d_G(T\xi)$ via the equivariant Thom isomorphism, where $d = 0$ or $1$ is the mod 2 reduction of $\epsilon(\alpha)$.

**Definition 2.4.** A $G$-manifold $X$ is said to be **$\mathbf{K^G}$-orientable** if its tangent bundle $\tau X$ is $K^G$-orientable. An orientation $\mu$ of $\tau$ is also called an **orientation** of $X$.

A $K^G$ orientation $\mu$ of a compact $G$-manifold $X$ determines a Poincaré duality isomorphism as follows. Let $\mu \in \tilde{K}^\alpha_G(T(\tau X))$ be an orientation of $X$

[4] Note that $\pi^H_0(K^G) = R(H)$ and $\pi^H_1(K^G) = 0$.

where $\epsilon(\alpha) = \dim X$. Embed $X$ in a $G$-representation $V$ and let $\nu$ denote the normal bundle of the embedding. Let

$$\lambda \in \tilde{K}_G^{v-\alpha}(T(\nu))$$

be the unique orientation such that $\lambda \oplus \mu$ is the canonical orientation in $K_G^v(\Sigma^v X^+)$. Multiplication by $\lambda$ determines the equivariant Thom isomorphism

$$K_G^\beta(X) \to \tilde{K}_G^{v-\alpha+\beta}(T(\nu)).$$

Since $T(\nu)$ is stably equivalent to the equivariant Spanier-Whitehead dual of $X$ by the result of Atiyah [**At 1**], there is a natural duality isomorphism

$$\tilde{K}_G^*(T(\nu)) \to K_*^G(X).$$

**Definition 2.5.** If $X$ is a $K^G$-oriented $G$-manifold, then the composite

$$D = D_X \, : \, K_G^\beta(X) \to \tilde{K}_G^{v-\alpha+\beta}(T(\nu)) \to K_{\alpha-\beta}^G(X)$$

of the Thom isomorphism and the Spanier-Whitehead-Atiyah duality isomorphism is the **Poincaré duality isomorphism**. The element

$$[X] = D(1_X) \in K_\alpha^G(X)$$

is called the **fundamental class** associated to the orientation.

The map $D_X$ depends upon the class of the orientation $\mu$ but is independent of other choices made in the construction. The map is of the same mod 2 degree as $X$; that is, it preserves mod 2 degree if the dimension of $X$ is even and switches mod 2 degrees if the dimension of $X$ is odd. It is important to note that the map $D$ above is given by the usual cap product as

$$D(x) = x \cap [X]. \tag{2.6}$$

The fundamental class of a $K^G$-oriented $G$-manifold admits a local description, as in the nonequivariant case.

**Definition 2.7.** Let $X$ be a $G$-manifold. For $x \in X$, let

$$t_x : X^+ \to G^+ \wedge_H S^Z$$

be a local Thom map at $x$ where $H$ is the isotropy group of $x$ and $Z$ is the fibre at $x$ of the normal bundle of $Gx \subseteq X$. A **$\mathbf{K}^G$-fundamental class** of $X$ is an element $\alpha \in R(G)$ such that $\epsilon(\alpha) = \dim X$ and an element $[X] \in K_\alpha^G(X^+)$ such that the image of $[X]$ under the composite

$$K_\alpha^G(X^+) \xrightarrow{t_{x*}} \tilde{K}_\alpha^G\,(G^+ \wedge_H S^Z) \cong \tilde{K}_\alpha^H\,(S^{L+Z})$$

is a $\pi_*^H(K^G)$-generator of $\tilde{K}_\alpha^H(S^{L+Z})$ for each $x \in X^+$.

**Proposition 2.8** [**LMS**, p. 159]. *Let $X$ be a $G$-manifold smoothly embedded in a $G$-representation $V$. Then the Spanier-Whitehead-Atiyah duality isomorphism*

$$K_G^{v-\alpha}(T(\nu)) \xrightarrow{\cong} K_\alpha^G(X^+)$$

*restricts to a bijective correspondence between $K^G$-orientations of $\tau^\perp$ (and thus of $\tau$ and $X$) and $K^G$-fundamental classes of $X$.*

**Proposition 2.9.** *Let $X$ be a $K^G$-oriented $G$-manifold with associated $K^G$-fundamental class $[X] \in K_\alpha^G(X)$.*

(1) *Suppose that $w \in K_G^0(X)$. Then $w \cap [X] \in K_\alpha^G(X)$ is also a $K^G$ fundamental class if and only if $w$ is a unit.*

(2) *Suppose that $W \in K_\alpha^G(X)$ and the map*

$$\cap W\; :\; K_G^\beta(X) \longrightarrow K_{\alpha-\beta}^G(X)$$

*is an isomorphism. Then $W$ is a $K^G$-fundamental class.*

*Proof.* (1) Suppose first that $w$ is a unit. Let $t_x : X^+ \to G^+ \wedge_H S^Z$ be a local Thom map. Then the class

$$t_{x*}[X] \in \tilde{K}_\alpha^H\,(S^Z) \cong R(H)$$

is a generator of $R(H)$ since $[X]$ is a fundamental class. Since

$$t_{x*}(w \cap [X]) = t_x^*(w) \cap t_{x*}[X]$$

and $t_x^*(w)$ is a unit, the class $t_{x*}(w \cap [X])$ is also a generator of $R(H)$. This is true for each local Thom map, which implies that $w \cap [X]$ is a fundamental class.

Conversely, suppose that $w \cap [X]$ is a $K^G$-fundamental class. Define

$$E : K_G^*(X) \to K_*^G(X)$$

by

$$E(x) = x \cap (w \cap [X]).$$

Then $E$ is an isomorphism by assumption. Let $v \in K^*_G(X)$ be the unique element with $E(v) = [X]$. Then

$$v \cap (w \cap [X]) = [X]$$

and hence

$$\begin{aligned} 1_X \cap [X] &= [X] \\ &= v \cap (w \cap [X]) \\ &= (vw) \cap [X] \qquad \text{since in general} \quad (rs) \cap t = r \cap (s \cap t). \end{aligned}$$

Cap product with $[X]$ is an isomorphism, and so $vw = 1_X$ and $w$ is a unit as required.

(2) Define $F : K^*_G(X) \to K^G_*(X)$ by

$$F(x) = x \cap W.$$

Let

$$v = D_X^{-1} F(1_X) = D_X^{-1}(1_X \cap W) = D_X^{-1} W.$$

Then $v$ is a unit with inverse $F^{-1}D_X(1_X)$ and hence the class $v \cap [X]$ is a fundamental class by (1). However,

$$v \cap [X] = D_X(v) = D_X(D_X)^{-1} W = W$$

and hence $W$ is a fundamental class. □

With strong control of orientations and fundamental classes, it is then possible to define a Gysin map associated to any continuous map $f : X \to Y$.

**Definition 2.10.** Let $G$ be a compact Lie group. Suppose that $X$ and $Y$ are $K^G$-oriented $G$-manifolds. Let $f : X \to Y$ be a $G$-map. Then the **Gysin map**

$$f_! : K^*_G(X) \to K^*_G(Y)$$

is defined to be the composite

$$\begin{array}{ccc} K^*_G(X) & \xrightarrow{f_!} & K^*_G(Y) \\ {\scriptstyle D_X}\downarrow & & \uparrow{\scriptstyle D_Y^{-1}} \\ K^G_*(X) & \xrightarrow{f_*} & K^G_*(Y). \end{array}$$

Note that the degree of the Gysin map is given by dim $X$ - dim $Y$ mod 2.

If $X$ is a compact $G$-space and $\xi$ is a real $G$-bundle over $X$ with a compatible $\mathrm{Spin}^c(n)$-structure, [5] then the **equivariant Euler class**

$$\chi^G(\xi) \in K_G^{\dim X - \dim \xi}(X)$$

is defined to be the restriction of the Thom class of $\xi$ to the zero section $X \to E(\xi)$. If $\xi$ is a complex $G$-bundle then

$$\chi^G(\xi) = \lambda_{-1}(\xi) \equiv \Sigma(-1)^i\lambda^i(\xi) \tag{2.11}$$

(complex exterior powers). If $f : X \to Y$ is a $G$-map then

$$\chi^G(f^*(\xi)) = f^*(\chi^G(\xi)).$$

It is easy to show that $\chi^G(\xi \oplus \xi') = \chi^G(\xi)\chi^G(\xi')$.

**Proposition 2.12.** *The Gysin map satisfies the following properties:*

(1) *The Gysin map $f_! : K_G^*(X) \to K_G^*(Y)$ is $R(G)$-linear.*

(2) *If $X \xrightarrow{f} Y \xrightarrow{g} Z$ in the $K^G$-oriented setting then $(gf)_! = g_! f_!$.*

(3) *For $x \in K_G^*(X)$, $y \in K_G^*(Y)$,*

$$f_!(xf^*(y)) = f_!(x)y.$$

(4) *If $X \to Y$ is an equivariant embedding with normal bundle $\nu$, then*

$$f^*f_!(x) = xf^*f_!(1) \equiv x\chi^G(\nu).$$

*Proof.* This follows exactly as in the non-equivariant case; c.f. [**Kar**, p. 233]. □

**Remark 2.13**. The analogous results for ordinary cohomology are well-known. We shall have occasion to use the theories $K_G^*$, $K^*$, and $H^*$ in this paper. Each theory has duality maps, Gysin maps, etc. When the theory is not clear from context we shall append the superscript $G$, $K$ and $H$ respectively, so, e.g.,

$$D_X^K : K^*(X) \to K_*(X).$$

Similarly, fundamental classes in the three theories will be denoted $[X]_G$, $[X]_K$ and $[X]_H$ respectively.

[5] This is discussed in detail in §9, starting with Definition 9.4.

## 3. The Theta Invariant.

In this section the $K^G$- Theta invariant $\Theta^G(f)$ of a $G$-map $f : X \to Y$ is defined and its relation to the relative $\hat{A}$-class $\hat{A}(\nu_f)$ is established. The basic facts are:

(1) (3.2) Under the map $K^0_G(Y) \to K^0(Y)$, $\quad \Theta^G(f) \mapsto \Theta(f)$.

(2) (3.6) If $f$ preserves $H$-orientation, then

$$\hat{A}(\nu_f) = 1 \iff \Theta(f) \equiv 1 \qquad \text{mod } \mathbb{Z}\text{-torsion}.$$

Then we switch attention to $K^G$-homology and demonstrate that

$$\Theta^G(f) \cap [Y] = f_*[X] \tag{3.9}$$

so that $\Theta^G(f)$ measures the extent to which the hypothetical equality

$$[Y] \stackrel{?}{=} f_*[X] \in K^G_*(Y)$$

holds.

**Definition 3.1.** Let $G$ be a compact Lie group. Suppose that $X$ and $Y$ are $K^G$-oriented $G$-manifolds and $f : X \to Y$ is a $G$-map. Define the **$\mathbf{K^G}$-Theta invariant** of the map $f$ by

$$\Theta^G(f) = f_!(1_X) \in K^0_G(Y).$$

If $G = \{e\}$ then write $\Theta(f) = \Theta^{\{e\}}(f)$.

Petrie calls the class $\Theta^{S^1}(f) \in K^0_{S^1}(Y)$ the "torsion invariant" thereby stressing the parallel with Whitehead torsion, but we prefer a more neutral term, especially since in cases of interest the class $\Theta^G(f)$ is a unit!

**Lemma 3.2.** *Let $G$ be a compact Lie group. Suppose that $X$ and $Y$ are $K^G$-oriented $G$-manifolds and $f : X \to Y$ is a $G$-map. Let $J$ be some*[6] *closed subgroup of $G$ and let*

$$r : K^*_G(Y) \to K^*_J(Y)$$

*be the natural map. Then $r(\Theta^G(f)) = \Theta^J(f)$.*

*Proof.* It is clearly enough to show that restriction to closed subgroups commutes with Gysin maps, and for this it suffices to show that restriction to closed subgroups commutes with the duality maps $D_Y$. This is the case, since the basic duality is given by the equivariant Thom isomorphism and by Atiyah duality, both of which restrict to closed subgroups. $\square$

[6] The case $J = \{e\}$ is of particular interest.

The Theta invariant $\Theta^G(f)$ is defined by means of the $K^G$-orientations of $X$ and $Y$ and hence a priori would be expected to depend strongly upon the smooth $G$-structures on $X$ and $Y$. The Theta invariant controls the behavior of Pontryagin classes as follows.

**Proposition 3.3.** *Suppose that $X$ and $Y$ are $K$-oriented manifolds and $f : X \to Y$ is a smooth equivalence. Then*

$$Ch(\Theta(f)) = f_!^H\left(\hat{A}\left(\nu_f\right)\right) \tag{3.4}$$

*and more generally*

$$Ch(\Theta(f)) \cap [Y]_H = f_!^H\left(\hat{A}(\nu_f) \cap [X]_H\right). \tag{3.5}$$

*If in addition $f_*[X]_H = [Y]_H$ then*

$$\hat{A}\left(\nu_f\right) = 1 \quad \Longleftrightarrow \quad \Theta(f) \equiv 1 \qquad \text{mod } \mathbb{Z}\textit{-torsion.} \tag{3.6}$$

*Proof.* The Atiyah - Hirzebruch version of the Riemann-Roch theorem [**AH 1**] asserts that

$$Ch\left(f_!^K(x)\right) = f_!^H\left(e^{c_1(f)}\hat{A}\left(\nu_f\right) Ch(x)\right)$$

for $x \in K^*(X)$.[7] Since $f$ is an equivalence, $c_1(f) = 0$. Take $x = 1_X \in K^0(X)$; then the formula becomes

$$Ch(\Theta(f)) = f_!^H\left(\hat{A}(\nu_f)\right)$$

which establishes (3.4). Write

$$Ch = \left(D_Y^H\right)^{-1} Ch_* D_X^K$$

where $Ch_* : K_*(X) \to H_{**}(X;\mathbb{Q})$ and recall that

$$f_!^H = \left(D_Y^H\right)^{-1} f_*^H D_X^H.$$

Then the Riemann-Roch formula becomes

$$\left(D_Y^H\right)^{-1} Ch_* D_Y^K \Theta(f) = \left(D_Y^H\right)^{-1} f_*^H D_X^H\left(\hat{A}\left(\nu_f\right)\right)$$

[7] If the normal bundle $\nu_f$ has a complex structure (and hence a stable Spin$^c$-structure), then $e^{c_1(f)}\hat{A}\left(\nu_f\right)$ is the Todd class of $\nu_f$ and hence (3.5) reduces to the familiar formula

$$Ch\left(f_!^K(x)\right) = f_!^H\left(Todd\left(\nu_f\right) Ch(x)\right).$$

which implies that

$$Ch_* D_Y^K \Theta(f) = f_*^H D_X^H \left(\hat{A}\left(\nu_f\right)\right) = f_* \left(\hat{A}\left(\nu_f\right) \cap [X]_H\right).$$

Now

$$\begin{aligned} Ch_* D_Y^K \Theta(f) &= D_Y^H Ch \left(D_Y^K\right)^{-1} D_Y^K \Theta(f) \\ &= D_Y^H Ch \Theta(f) \\ &= Ch(\Theta(f)) \cap [Y]_H \end{aligned}$$

and hence

$$Ch(\Theta(f)) \cap [Y]_H = f_!^H \left(\hat{A}\left(\nu_f\right) \cap [X]_H\right)$$

which proves (3.5).

Suppose that $\hat{A}(\nu_f) = 1$ and $f_*$ preserves $H$-orientations. Then

$$Ch(\Theta(f)) \cap [Y]_H = f_*^H [X]_H = [Y]_H.$$

Since cap product with the fundamental class $[Y]_H$ is an isomorphism in cohomology, this implies that $Ch(\Theta(f)) = 1$. As $Ch$ is a monomorphism modulo $\mathbb{Z}$-torsion, we have $\Theta(f) \equiv 1$ mod $\mathbb{Z}$-torsion as required. Conversely, suppose that $\Theta(f) \equiv 1$ mod $\mathbb{Z}$-torsion. Then

$$[Y]_H = 1_Y \cap [Y]_H = f_*^H \left(\hat{A}(\nu_f) \cap [X]_H\right).$$

We also know that $[Y]_H = f_*[X]_H$. The map $f$ is an equivalence, hence $f_*^H$ is an isomorphism, and so

$$\hat{A}(\nu_f) \cap [X]_H = [X_H]$$

which implies that $\hat{A}(\nu_f) = 1$. □

It is clear from this proposition that the class $\Theta(f) \in K^0(Y)$ provides the link between Pontryagin classes and local behavior. In order to make this matter more transparent, we prefer to work in $K$-homology which is more natural for this problem. So let us consider again the definition (3.1) of the class $\Theta^G(f)$ via the commutative diagram

$$\begin{array}{ccc} K_G^*(X) & \xrightarrow{f_!} & K_G^*(Y) \\ {\scriptstyle D_X}\downarrow & & \uparrow{\scriptstyle D_Y^{-1}} \\ K_*^G(X) & \xrightarrow{f_*} & K_*^G(Y) \end{array}.$$

In this diagram the class $\Theta^G(f) = f_!(1_X)$ arises as

$$\begin{array}{ccc} 1_X & \xrightarrow{f_!} & \Theta^G(f) \\ {\scriptstyle D_X}\downarrow & & \uparrow{\scriptstyle D_Y^{-1}} \\ [X] & \xrightarrow{f_*} & f_*[X] \end{array} .$$

Since $D_Y$ is an isomorphism, study of $\Theta^G(f)$ is thus equivalent to the study of $f_*[X] = D_Y\Theta^G(f)$. This implies the following proposition.

**Proposition 3.8.** *Let $G$ be a compact Lie group. Suppose that $X$ and $Y$ are $K^G$-oriented $G$-manifolds and $f : X \to Y$ is a $G$-map. Then*

$$\Theta^G(f) \cap [Y] = f_*[X]. \tag{3.9}$$

Thus our attention turns to the question of how the map

$$f_* : K_*^G(X) \longrightarrow K_*^G(Y)$$

treats $K^G$-fundamental classes. This is the subject of the following sections. Before we leave this section, however, note the case of Proposition 3.8 specialized to the case where $G$ is trivial.

**Corollary 3.10.** *Suppose that $X$ and $Y$ are $K$-oriented manifolds, and $f : X \to Y$ is a smooth equivalence with $f_*[X]_H = [Y]_H$. Then the following are equivalent:*

(1) *The map $f$ preserves Pontryagin classes.*

(2) $\Theta(f) \equiv 1_Y \in K^0(Y)$ mod $\mathbb{Z}$*-torsion.*

(3) $f_*[X]_K \equiv [Y]_K \in K_0(Y)$ mod $\mathbb{Z}$*-torsion.*

## 4. Spectral sequences; moving from cohomology to homology.

We begin by recalling some of the consequences of the Universal Coefficient spectral sequence and the Hodgkin spectral sequence. We demonstrate under suitable hypotheses (4.9) that if

$$f^* : K_G^*(Y) \longrightarrow K_G^*(X)$$

is an isomorphism then so too is the associated map on homology

$$f_* : K_*^G(X) \longrightarrow K_*^G(Y)$$

and the Gysin map

$$f_! : K_G^*(X) \longrightarrow K_G^*(Y).$$

The Universal Coefficient spectral sequence used to prove these results is constructed [**RS**] using operator-algebra techniques and machinery. The spectral sequence cited below (4.1) is a very special case of the spectral sequence of [**RS**] which for appropriate classes of $G$-$C^*$-algebras $A$ and $B$ converges to the equivariant Kasparov group $KK_G^*(A, B)$ and has

$$E_2^{s,*} \cong \operatorname{Ext}^{s,*}_{R(G)}(K_*^G(A), K_*^G(B)).$$

The spectral sequence of Theorem 4.1 is obtained by setting $A = C(X)$ and $B = \mathbb{C}$.

**Theorem 4.1. Universal Coefficient Spectral Sequence [RS].** *Let $G$ be a Hodgkin group and let $X$ be a compact $G$-space. Then there is a strongly convergent spectral sequence of $R(G)$-modules which converges to $K_*^G(X)$ with*

$$E_2^{s,*} \cong \operatorname{Ext}^{s,*}_{R(G)}(K_G^*(X), R(G)).$$

*The edge homomorphism*

$$K_*^G(X) \to E_\infty^{0,*} \hookrightarrow E_2^{0,*} \cong \operatorname{Hom}_{R(G)}(K_G^*(X), R(G)) \tag{4.2}$$

*is the Kronecker pairing. The spectral sequence is natural with respect to $G$-maps $X \to X'$.*

**Theorem 4.3.** *Suppose that $G$ is a Hodgkin group, $X$ and $Y$ are compact $G$-$ENR$ spaces*[8] *and $f : X \to Y$ is a $G$-map which induces an isomorphism*

$$f^* : K_G^*(Y) \to K_G^*(X).$$

*Then the map*

$$f_* : K_*^G(X) \to K_*^G(Y) \tag{4.4}$$

*is an isomorphism.*

*Proof.* The map $f$ induces a morphism of spectral sequences of type (4.1) which is an isomorphism at the $E_2$ level. This implies that the two spectral sequences are isomorphic for each $E_n$ for $n > 1$ which implies that $f_* : K_*^G(X) \to K_*^G(Y)$ is an isomorphism as required. □

[8] A $G$-space is a $G$-$ENR$ (Euclidean neighborhood retract) if it can be embedded as a retract of an open subset of some $G$-representation. For example, each compact locally linear topological $G$-manifold is a $G$-$ENR$.

**Remark 4.5.** It is possible that the theorem above holds for every compact connected Lie group, but this would seem to be a deep fact. The lowest dimensional compact connected Lie group for which the conclusion of the theorem is unknown is $SO(3)$. This group is a quotient of $SU(2)$ which is Hodgkin, and $\mathrm{Ker}(SU(2) \to SO(3)) \cong \mathbb{Z}_2$. The problem may revolve about characterizing $K^*_{SO(3)}(X)$ as an appropriate functor of $K^*_{SU(2)}(X)$.

**Theorem 4.6.** *Let $G$ be a Hodgkin group, let $X$ and $Y$ be $K^G$-oriented $G$-manifolds with $K^G$-fundamental classes $[X]$ and $[Y]$ respectively, both of degree $\alpha$, and suppose given a $G$-map $f : X \to Y$ which induces an isomorphism*

$$f^* : K^*_G(Y) \to K^*_G(X).$$

*Then the class $f_*[X] \in K^G_\alpha(Y)$ is a $K^G$-fundamental class for $Y$.*

*Proof.* By (2.9)(1), it suffices to show that cap product induces an isomorphism

$$(-) \cap f_*[X] \,:\, K^\beta_G(Y) \longrightarrow K^G_{\alpha-\beta}(Y).$$

This may be verified directly. The cap product satisfies the identity

$$y \cap f_*[X] \;=\; f_*(f^*(y) \cap [X]) \tag{4.7}$$

for $y \in K^*_G(Y)$. The map $f^*$ is an isomorphism by assumption and the map $f_*$ is an isomorphism by (4.4). This completes the proof. □

**Theorem 4.8.** *Let $G$ be a Hodgkin group, let $X$ and $Y$ be $K^G$-oriented $G$-manifolds with $K^G$-fundamental classes $[X]$ and $[Y]$ respectively, both of degree $\alpha$, and suppose given a $G$-map $f : X \to Y$ which induces an isomorphism*

$$f^* : K^*_G(Y) \to K^*_G(X).$$

*Then $\Theta^G(f)$ is a unit in $K^0_G(Y)$.*

*Proof.* We know that $f_*[X]$ is a $K^G$-fundamental class, by (4.6), and hence

$$f_*[X] = w \cap [Y]$$

for some unique unit $w \in K^0_G(Y)$ by (2.9)(1). On the other hand, the map $f_*[X]$ satisfies

$$f_*[X] = \Theta^G(f) \cap [Y]$$

by (3.9) and thus $\Theta^G(f) = w$ is a unit. □

Theorem 4.8 is the key to a refined understanding of $\hat{A}(f)$ as will become evident in Theorem 7.1.

Next we introduce the Hodgkin spectral sequence.

**Theorem 4.10. Hodgkin spectral sequence [H, RS].** *Let $G$ be a Hodgkin group acting upon a compact space $X$, and let $H$ be a closed subgroup of $G$. Then there is a strongly convergent spectral sequence of $R(G)$-modules which converges to $K^*_H(X)$ with*

$$E^2_{s,*} = \mathrm{Tor}^{R(G)}_*(R(H), K^*_G(X)).$$

*The spectral sequence is natural with respect to $G$-maps $X \to X'$.*

**Corollary 4.11.** *Let $G$ be a Hodgkin group. If $f : X \to Y$ is a $G$-map of compact $G$-spaces which induces an isomorphism*

$$f^* : K^*_G(Y) \xrightarrow{\cong} K^*_G(X)$$

*then for any closed subgroup $J \subseteq G$ the map*

$$f^* : K^*_J(Y) \longrightarrow K^*_J(X)$$

*is an isomorphism. In particular (with $J = \{e\}$) the map*

$$uf^* : K^*(uY) \longrightarrow K^*(uX)$$

*is an isomorphism and hence the Chern character induces an isomorphism*

$$uf^* : H^{**}(uY; \mathbb{Q}) \xrightarrow{\cong} H^{**}(uX; \mathbb{Q}).$$

**Proposition 4.12.** *Let $G$ be a Hodgkin group. Suppose that $X$ and $Y$ are $K^G$-oriented $G$-manifolds. Let $f : X \to Y$ be a continuous $G$-map which induces an isomorphism*

$$f^* : K^*_G(Y) \to K^*_G(X).$$

*Then the Gysin map*

$$f_! : K^*_G(X) \to K^*_G(Y)$$

*is an isomorphism.*

*Proof.* By the definition of $f_!$ (2.11) it suffices to prove that the map

$$K^G_*(X) \xrightarrow{f_*} K^G_*(Y)$$

is an isomorphism. This follows from Theorem 4.4. □

**Corollary 4.13.** *Let $G$ be a Hodgkin group. Suppose that $X$ and $Y$ are $G$-manifolds and that each is $K^G$-oriented. Let $f : X \to Y$ be a continuous $G$-map such that the induced map*

$$f^* : K_G^*(Y) \to K_G^*(X)$$

*is an isomorphism. Then*

(1) *for any [9] closed subgroup $J$ of $G$ the maps*

$$f_* : K_*^J(X) \to K_*^J(Y)$$

*and*

$$f_! : K_J^*(X) \to K_J^*(Y)$$

*are isomorphisms.*

(2) *The induced map*

$$uf^* : H^*(uY; \mathbb{Q}) \longrightarrow H^*(uX; \mathbb{Q})$$

*is an isomorphism.*

(3) *The induced maps*

$$uf_! : H^*(uX; \mathbb{Q}) \longrightarrow H^*(uY; \mathbb{Q})$$

*and*

$$uf_* : H_*(uX; \mathbb{Q}) \longrightarrow H_*(uY; \mathbb{Q})$$

*are isomorphisms.*

*Proof.* The proof of a) is immediate from Corollary 4.4 and Proposition 4.12. To prove b), take $J = \{e\}$ in part a) and apply the Chern character. Part c) follows from b) and the fact that duality holds in homology as well. □

**Remark 4.14.** Theorem 4.1 also holds for $G$ finite, by Bökstedt [**Bok**]. It seems likely that Theorem 4.1 holds for arbitrary compact connected Lie groups, and not just for Hodgkin groups. In contrast, the Hodgkin spectral sequence would not be expected to generalize beyond Hodgkin groups.

[9]The case $J = \{e\}$ is permitted and is interesting.

## 5. Localizations and fixed point sets.

In this section attention turns to the fixed point sets of the $G$-actions. Under very general circumstances the Atiyah-Segal localization theorem allows us to restrict to fixed points, provided that the theory itself is suitably localized. It turns out that if $G$ is a Hodgkin group and $K^*(X)$ is $\mathbb{Z}$-torsionfree then the map

$$\imath_* : K_*^G(X^G) \longrightarrow K_*^G(X)$$

is a monomorphism which induces an isomorphism on fraction fields. These results help us to relate $K^G$-orientations on $X$ to those on $X^G$ and hence to localize the data to fixed point sets.

**Theorem 5.1 Localization Theorem.** [**Se 1, ASe**]. *Suppose that $G$ is a compact Lie group which acts upon a locally compact space $X$. Let $\gamma$ be a conjugacy class in $G$, and denote by $\mathfrak{p}$ the prime ideal in $R(G)$ of characters which vanish on $\gamma$. Define*

$$X^\gamma = \bigcup_{g\in\gamma} X^g.$$

*Then the inclusion $\iota_\gamma : X^\gamma \to X$ induces an isomorphism*

$$\iota_\gamma^* : K_G^*(X)_\mathfrak{p} \longrightarrow K_G^*(X^\gamma)_\mathfrak{p}.$$

Note that this theorem holds even when $X^\gamma$ is empty, in which case it asserts that $K_G^*(X)_\mathfrak{p} = 0$.

Recall that if $G$ is a compact connected Lie group then Segal [**Se 1**] shows that $R(G)$ is a Noetherian domain. Let $F(G)$ denote its field of fractions. If $M$ is an $R(G)$-module then write its associated $F(G)$-localization by

$$M_{F(G)} = M \otimes_{R(G)} F(G).$$

**Corollary 5.2.** *Let $G$ be a compact connected Lie group and let $X$ be a compact $G$-space. Then the inclusion $\imath : X^G \to X$ induces an isomorphism*

$$\imath^* : K_G^*(X)_{F(G)} \longrightarrow K_G^*(X^G)_{F(G)}.$$

For example, if $X^G$ is empty then $K_G^*(X)_{F(G)} = 0$ and hence $K_G^*(X)$ is an $R(G)$-torsion module.

*Proof.* Since $G$ is connected, $F(G) \cong R(G)_{\{0\}}$ The ideal $\mathfrak{p} = \{0\}$ corresponds to $X^G$ and so the result is immediate from the Localization Theorem (5.1) $\square$

The spectral sequences of Section Four must be slightly modified in order to be of optimal use. (A similar modification has been made by Rosenberg and Weinberger [**RW**].) As these modifications are purely algebraic, we state the requisite results in that context.

**Proposition 5.4.** *Let $R$ be a commutative ring. Suppose that $\{E_r, d^r\}$ is a spectral sequence of $R$-modules which strongly converges to some $R$-module $M$ and suppose that $S$ is a set of prime ideals of $R$ with associated localization $R_S$. Then $\{E_{rS}, d^r_S\}$ is a spectral sequence of $R_S$-modules which strongly converges to $M_S$.*

*Proof.* Localization is an exact functor, and thus it commutes with taking the homology of a differential module. □

**Proposition 5.5.** *Suppose that $R$ is a commutative ring and $M$ and $N$ are $R$-modules. Let $S$ be a collection of prime ideals in $R$. Then for each integer $s$ there is a natural isomorphism*

$$\mathrm{Tor}_s^R(M,N)_S \cong \mathrm{Tor}_s^{R_S}(M_S,N_S).$$

*If $R$ is a Noetherian ring and $M$ is finitely generated then for each $s$ there is a natural isomorphism of $R_S$-modules*

$$\mathrm{Ext}^s_R(M,N)_S \cong \mathrm{Ext}^s_{R_S}(M_S,N_S).$$

*Proof.* The proof of the property for Tor is very easy given the fact that $R_S$ is a flat $R$-module. The Ext result is implied by [**CE**] Ch. VI, Exercise 11 and Ch. VII, Exercise 10, but we insert a proof for completeness. It is easy to show that if $P$ is a finitely generated projective $R$-module then there is a natural isomorphism

$$\mathrm{Hom}_R(P,N)_S \cong \mathrm{Hom}_{R_S}(P_S,N_S).$$

Since $M$ is finitely generated and $R$ is Noetherian, $M$ has a projective $R$-resolution $P_\bullet$ of the form

$$0 \to P_k \to P_{k-1} \to \cdots \to P_0 \to M \to 0$$

with each $P_j$ finitely generated. Then

$$\mathrm{Hom}_R(P_j,N)_S \cong \mathrm{Hom}_{R_S}(P_{jS},N_S)$$

for each $j$. This implies the result, since there are natural isomorphisms

$$\begin{aligned}\operatorname{Ext}^s_R(M,N)_S &\cong H_s(\operatorname{Hom}_R(P_\bullet, N))_S \\ &\cong H_s(\operatorname{Hom}_R(P_\bullet, N)_S) \\ &\cong H_s(\operatorname{Hom}_{R_S}(P_{\bullet S}, N_S)) \\ &\cong \operatorname{Ext}^s_{R_S}(M_S, N_S).\end{aligned}$$

□

These techniques allow a refinement of the Localization Theorem as follows:

**Theorem 5.6.** *Suppose that $G$ is a Hodgkin group and $X$ is a $G$-manifold. Let*

$$\imath^X : X^G \longrightarrow X$$

*be the inclusion. Then the map*

$$(\imath^X_*)^F : K^G_*(X^G)_{F(G)} \longrightarrow K^G_*(X)_{F(G)} \tag{5.7}$$

*is an isomorphism. Similarly, there is an isomorphism of $R(G)$-modules*

$$(\imath^{X*})^F : K^*_G(X)_{F(G)} \longrightarrow K^*_G(X^G)_{F(G)} \tag{5.8}$$

*and hence the kernel and the cokernel of the map*

$$(\imath^{X*})^F : K^*_G(X) \longrightarrow K^*_G(X^G) \tag{5.9}$$

*are $R(G)$-torsion modules. Suppose in addition that $K^*(X^G)$ is a $\mathbb{Z}$-torsion-free module.*[10] *Then $\imath$ induces a monomorphism*

$$\imath^X_* : K^G_*(X^G) \longrightarrow K^G_*(X)$$

*and* $\operatorname{Cok}(\imath^X_*)$ *is an $R(G)$-torsion module.*

*Proof.* The map

$$(\imath^{X*})^F : K^*_G(X)_{F(G)} \longrightarrow K^*_G(X^G)_{F(G)} \tag{5.9}$$

[10] Since $X^G$ is a compact manifold, this is equivalent to the assumption that $K_*(X^G)$ is $\mathbb{Z}$-torsionfree, and under this assumption the Kronecker map

$$K_*(X^G) \to \operatorname{Hom}_{R(G)}(K^*(X^G), \mathbb{Z})$$

is an isomorphism by the Universal Coefficient Theorem.

is an isomorphism by (5.1). To proceed further in the proof we record a preliminary lemma which may be of independent interest.

**Lemma 5.10.** *Suppose that $G$ is a Hodgkin group, $X$ and $Y$ are compact $G$-$ENR$-spaces, and $\mathfrak{p}$ is some prime ideal in $R(G)$. Suppose that the map*

$$f^* : K_G^*(Y)_{\mathfrak{p}} \longrightarrow K_G^*(X)_{\mathfrak{p}}$$

*is an isomorphism. Then the map*

$$f_* : K_*^G(X)_{\mathfrak{p}} \longrightarrow K_*^G(Y)_{\mathfrak{p}}$$

*is an isomorphism.*

*Proof.* This lemma is an obvious generalization of Theorem 4.4. It is proved by localizing the proof of that theorem, using the fact that by (5.4) and (5.5) there is a spectral sequence which converges to $K_*^G(X)_{\mathfrak{p}}$ with

$$E_2^{p,*} \cong \mathrm{Ext}^p_{R(G)_{\mathfrak{p}}}(K_G^*(X)_{\mathfrak{p}}, R(G)_{\mathfrak{p}}).$$

$\square$

*Continuation of Proof of* 5.6. Statements (5.8) and (5.9) are restatements of Corollary 5.2. Taking $\mathfrak{p} = \{0\}$ in the Lemma yields a spectral sequence which allows to to deduce (5.7) directly. In fact the situation is very simple, since $E_2^{p,*} = 0$ for $p > 2$. In order to prove that $\imath_*^X$ is mono, consider the commutative diagram

$$\begin{array}{ccc}
K_*^G(X^G) & \xrightarrow{\imath_*^X} & K_*^G(X) \\
{\scriptstyle\kappa}\downarrow & & \downarrow \\
K_*^G(X^G)_{F(G)} & \xrightarrow{(\imath_*^X)^F} & K_*^G(X)_{F(G)}
\end{array}$$

The map $(\imath_*^X)^F$ is an isomorphism by (5.7) and the map $\kappa$ is a monomorphism since $K_*^G(X^G) \cong K_*(X^G) \otimes R(G)$ is a free $R(G)$-module. Thus the map $\imath_*^X$ is a monomorphism. Its cokernel is an $R(G)$-torsion module by (5.7). $\square$

In the following proposition a dimension condition appears for the first time, namely that the difference of the dimensions of $X$ and $X^G$ is even. Eventually we shall assume that $X^G$ consists entirely of isolated points and so $X$ must be of even dimension to satisfy this hypothesis. Note that if $X$ is of odd dimension and $X^G$ consists entirely of isolated points then $\chi^G(\nu) \in$

$K^1_G(X^G) = 0$ so that there is no hope for arguing as in the proof of the proposition.

**Proposition 5.11.** *Let $G$ be a Hodgkin group. Let $X$ be a $K^G$-oriented $G$-manifold with $X^G$ nonempty. Suppose that the inclusion map $\imath^X : X^G \to X$ is $K^G$-oriented with (real) even-dimensional normal bundle $\nu$ and that $\chi^G(\nu) = 1$. [This will be the case, e.g., when $\nu$ is a trivial bundle, and hence this condition holds whenever $X^G$ consists of isolated points.] Let $\tau X^G$ have the unique $K^G$-orientation compatible with the isomorphism*

$$\tau X^G \oplus \nu \cong \imath^{X*}\tau X.$$

*Let $[X^G] \in K^G_\alpha(X^G)$ denote the associated $K^G$ fundamental class*[11]*. Then*

$$\imath^X_*[X^G] \equiv [X] \qquad \text{mod} \quad R(G)\text{-}torsion \qquad \in K^G_\alpha(X) \tag{5.12}$$

*and*

$$\Theta^G(1_{X^G}) \equiv 1_X \qquad \text{mod} \quad R(G)\text{-}torsion \qquad \in K^0_G(X).$$

Note that if $X^G$ is empty then $K^G_*(X)$ and $K^*_G(X)$ are $R(G)$-torsion modules and hence (5.11) holds in a trivial sense.

*Proof.* By (2.12) we have

$$\imath^{X*}\imath^X_!(x) = \chi^G(\nu)x = x \qquad \text{since} \quad \chi^G(\nu) = 1$$

for any $x \in K^*_G(X^G)$. Taking $x = 1_{X^G}$ and writing $\imath^X_! = D_X^{-1}\imath_* D_{X^G}$, this equation becomes

$$\imath^{X*}D_X^{-1}\imath_*[X^G] = 1_{X^G} = \imath^{X*}1_X.$$

Thus

$$D_X^{-1}\imath_*[X^G] - 1_X \in \mathrm{Ker}(\imath^{X*})$$

and $\mathrm{Ker}(\imath^{X*})$ is an $R(G)$-torsion module by (5.6). Since $D_X$ is an isomorphism,

$$\imath_*[X^G] \equiv [X] \qquad \text{mod} \quad R(G)\text{-torsion}.$$

The final statement follows by checking definitions. □

[11]Since $\nu$ is of even dimension we may take $[X]$ to lie in this degree by the equivariant Bott periodicity theorem.

## 6. Isolated fixed points.

This section is devoted to a detailed analysis of the additional structure that is forced by the assumption that the fixed point set $X^G$ consist of isolated points. Theorem 6.5 improves upon Theorem 5.6: under the usual hypotheses, if $X^G$ consists of isolated fixed points then the induced map

$$\imath_*^X : K_*^G(X^G) \longrightarrow K_*^G(X)$$

is a *canonically split* monomorphism. The work starting with Theorem 6.7 is used to prove the main theorems of the paper.

**Proposition 6.1.** *Suppose that $G$ is a compact Lie group and $X$ is a compact $G$-space. The natural map*

$$r : K_G^*(X) \longrightarrow K^*(X)$$

*factors as*

$$K_G^*(X) \xrightarrow{p} \mathbb{Z} \otimes_{R(G)} K_G^*(X) \xrightarrow{\bar{r}} K^*(X).$$

*The short exact sequence*

$$0 \longrightarrow I_G \longrightarrow R(G) \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

*yields the exact sequence*

(6.2)

$$\mathrm{Tor}_1^{R(G)}(\mathbb{Z}, K_G^*(X)) \longrightarrow I_G \otimes_{R(G)} K_G^*(X) \xrightarrow{q} K_G^*(X) \xrightarrow{p} \mathbb{Z} \otimes_{R(G)} K_G^*(X) \to 0.$$

*Assume that $G$ is a Hodgkin group. Then the map $\bar{r}$ is the edge homomorphism in the Hodgkin spectral sequence*

$$(6.3) \qquad E_{t,*}^2 \cong \mathrm{Tor}_t^{R(G)}(\mathbb{Z}, K_G^*(X)) \qquad \Longrightarrow \qquad K^*(X).$$

*Thus if $K_G^*(X)$ is the direct sum of $R(G)$-torsion and $R(G)$-flat modules then $\bar{r}$ is an isomorphism mod $R(G)$-torsion. If $K_G^*(X)$ is an $R(G)$-torsion module then $\bar{r}$ is an isomorphism, $\mathrm{Tor}_1^{R(G)}(\mathbb{Z}, K_G^*(X)) = 0$ and hence there is a short exact sequence of $R(G)$-modules*

$$0 \to I_G \otimes_{R(G)} K_G^*(X) \longrightarrow K_G^*(X) \xrightarrow{r} K^*(X) \to 0.$$

*Finally, if $K_G^*(X) \cong R(G)^s$ then there is a short exact sequence of $R(G)$-modules*

$$0 \to I_G \otimes \mathbb{Z}^s \longrightarrow K_G^*(X) \xrightarrow{r} K^*(X) \to 0.$$

*Proof.* The short exact sequence (6.2) is immediate from the exactness properties of tensoring. When $G$ is Hodgkin so that the Hodgkin spectral sequence (6.3) is available then use the fact that $\mathrm{Tor}_t^{R(G)}(\mathbb{Z}, M)$ vanishes for $t > 0$ when $M$ is $R(G)$-flat (by definition) and is an $R(G)$-torsion module when $M$ is itself an $R(G)$-torsion module (by (5.5)) to conclude that $E^2_{t,*}$ is an $R(G)$-torsion module for $t > 0$. This implies that $E^\infty_{t,*}$ is also an $R(G)$-torsion module for $t > 0$ and hence $\bar{r}$ is an $R(G)$-isomorphism mod $R(G)$-torsion. If $K_G^*(X)$ is an $R(G)$-flat module then $E^2_{t,*} = 0$ for each $t > 0$ which implies that $\bar{r}$ is an isomorphism. The rest of the proposition is immediate. □

Suppose that $G$ is a compact Lie group and $X$ is a $G$-manifold. Then $X^G$ is a $G$-submanifold of $X$ and $\imath^X : X^G \to X$ is a smooth embedding. If $X$ is $K^G$-oriented and the map $\imath^X$ is $K^G$-oriented then this gives $X^G$ a $K^G$-orientation by (2.10). For example, if the normal bundle of the embedding is trivial (if $X^G$ consists of isolated points then this will be the case) then the map $\imath^X$ has a canonical $K^G$-orientation and hence $X^G$ has a canonical $K^G$-orientation.

The presence of only isolated fixed points allows the following improvement of Theorem 5.6.

**Theorem 6.5.** *Suppose that $G$ is a Hodgkin group. Let $X$ be an even-dimensional $K^G$-oriented $G$-manifold such that the fixed point set $X^G$ and the inclusion map $\imath : X^G \to X$ are compatibly $K^G$-oriented. Assume further that $X^G$ consists of isolated points. Then the induced map*

$$\imath_*^X : K_*^G(X^G) \longrightarrow K_*^G(X)$$

*is a canonically split monomorphism.*

*Proof.* The map is a monomorphism by (5.6). Define

$$\ell : K_*^G(X) \to K_*^G(X^G)$$

to be the composite

$$K_*^G(X) \xrightarrow{(D_X)^{-1}} K_G^*(X) \xrightarrow{\imath^{X*}} K_G^*(X^G) \xrightarrow{D_{X^G}} K_*^G(X^G).$$

Then $\ell$ is $R(G)$-linear. We claim that $\ell\imath_*^X[X^G] = [X^G]$. This is the case since (writing "$\equiv$" for "mod $R(G)$-torsion") we have

$$\begin{aligned} \ell\imath_*^X[X^G] &\equiv \ell[X] \qquad \text{by} \quad (5.11) \\ &= D_{X^G}\imath^{X*}(1_X) \\ &= D_{X^G}(1_{X^G}) \\ &= [X^G] \end{aligned}$$

so that $\ell\imath_*^X[X^G] \equiv [X^G]$ in the $R(G)$-free module $K_*^G(X^G)$, and hence $\ell\imath_*^X[X^G] = [X^G]$. Since $X^G$ consists of isolated fixed points $\{x_1, \ldots, x_s\}$, the class $[X^G]$ decomposes as $[X^G] = \Sigma[x_i]$ for $[x_i] \in K_0^G(\{x_i\}) \cong R(G)$. (Note that each $[x_i]$ must be a free generator of $R(G)$.) Hence

$$\Sigma\, \ell\imath_*^X[x_i] = \Sigma\, [x_i].$$

The $[x_i]$ are a free basis, and hence

$$\ell\imath_*^X[x_i] = [x_i] \tag{6.6}$$

for each $i$. Thus $\ell\imath_*^X(x) = x$ for all $x$, and $\imath_*$ is a canonically split monomorphism. $\square$

Henceforth when dealing with two even-dimensional manifolds we shall assume that their $K^G$-fundamental classes have been chosen to lie in the same grading; for definiteness we take $\alpha = 0$ below.

**Theorem 6.7.** *Let $G$ be a Hodgkin group. Let $X$ and $Y$ be $K^G$-oriented even-dimensional $G$-manifolds with $X^G$ and $Y^G$ consisting of $s$ isolated points each. Let $X^G$ and $Y^G$ have the induced $K^G$-orientations. Suppose that $f : X \to Y$ is a $G$-map which is an isomorphism on fixed point sets. Then the Theta class pulls back to $\imath^{Y*}\Theta^G(f) \in K_G^0(Y^G)$ and satisfies*

$$\left(\imath^{Y*}\Theta^G(f)\right) \cap [Y^G] = f_*\,[X^G] \in K_0^G(Y^G)\,. \tag{6.8}$$

*Proof.* By (3.8) we know that

$$\Theta^G(f) \cap [Y] = f_*\,[X]\,.$$

Proposition 5.11 implies that $\imath_*^Y\,[Y^G] \equiv [Y]$ modulo $R(G)$-torsion and similarly for $X$ and hence

$$\Theta^G(f) \cap \imath_*^Y\,[Y^G] \equiv f_*\imath_*^X\,[X^G] \equiv \imath_*^Y f_*^G\,[X^G]\,.$$

But

$$\Theta^G(f) \cap \imath_*^Y\,[Y^G] = \imath_*^Y\left(\imath^{Y*}\Theta^G(f) \cap [Y^G]\right)$$

and hence

$$\imath_*^Y\left(\imath^{Y*}\Theta^G(f) \cap [Y^G]\right) \equiv \imath_*^Y f_*^G\,[X^G]\,.$$

The map $\imath_*^Y$ is a monomorphism by (5.6), and hence

$$\imath^{Y*}\Theta^G(f) \cap [Y^G] \equiv f_*^G\,[X^G]\,.$$

There is no $R(G)$-torsion in $K_*^G(Y^G)$ and so the equality holds. $\square$

**Theorem 6.9.** *Suppose that $G$ is a Hodgkin group and suppose that $X$ and $Y$ are even-dimensional $K^G$-oriented manifolds. Let $f : X \to Y$ be a $G$-map which induces an isomorphism of finite sets $X^G \cong Y^G$ so that*

$$\imath^{Y*}\Theta^G(f) \cap [Y^G] = f_*[X^G]$$

*by Theorem* 6.7. *Then with respect to the decomposition*

$$K_0^G(Y^G) \cong \sum_{1=1}^{s} K_0^G(\{y_i\}),$$

*we have*

$$f_*[X^G] = \sum_{1=1}^{s} \lambda_{-1}\left(\tau Y_{f(x_i)} - \tau X_{x_i}\right)[y_i]. \tag{6.10}$$

*Proof.* By additivity, it suffices to consider only one fixed point $\{x\}$. Identify $\{f(x)\}$ with $\{x\}$ via $f$ and let $V = \tau X_x$ and $W = \tau Y_{f(x)}$. By a standard argument (c.f. [**ASi**, p. 498]), we may assume that $V$ is a subrepresentation of $W$. Let $\jmath : V \to W$ be the inclusion. Then we must consider the diagram

$$\begin{tikzcd}
K_G^0(\{x\}) \arrow[r, "\cong"] \arrow[d, "\varphi_V"'] & K_G^0(\{x\}) \arrow[d, "\varphi_W"'] \\
K_G^0(V) \arrow[r, shift right, "\jmath_!"'] \arrow[d, "\psi_V"'] & K_G^0(W) \arrow[l, shift right, "\jmath^*"'] \arrow[d, "\psi_W"'] \\
K_0^G(\{x\}) \arrow[r, "\cong"] & K_0^G(\{x\})
\end{tikzcd}$$

where $\varphi_V$ and $\varphi_W$ are Thom isomorphisms, $\psi_V$ and $\psi_W$ are Atiyah duality maps, and we recall that

$$D_V(1_X) = \psi_V\varphi_V(1_X) = \psi_V(\lambda_{-1}(V)) = [x_V]$$

and

$$D_W(1_X) = \psi_W\varphi_W(1_X) = \psi_W(\lambda_{-1}(W)) = [x_W].$$

We compute:

$$\lambda_{-1}(W - V)[x_V] = \lambda_{-1}(W - V)\psi_V\varphi_V(1_X) = \psi_V(\lambda_{-1}(W - V)\varphi_V(1_X))$$

since $\psi_V$ is $R(G)$-linear,

$$= \psi_V \jmath^* \jmath_! \varphi_V(1_X)$$

since $\jmath^* \jmath_!(m) = \lambda_{-1}(W-V)m$

$$= \psi_W \varphi_W(1_X)$$

since $\psi_V \jmath^* = \psi_W$ and $\jmath_! \varphi_V = t_W$

$$= [x_W].$$

Thus

$$\lambda_{-1}(W-V)[x_V] = [x_W] \tag{6.11}$$

which implies the theorem. □

**Remark 6.12** Suppose that $G$ is a compact connected Lie group and suppose that $X$ and $Y$ are $K^G$-oriented manifolds such that $X^G$ and $Y^G$ consist of isolated fixed points. Let $f : X \to Y$ be a $G$-map which induces an isomorphism on $K_G^*(-)$. Clearly $f(X^G) \subset Y^G$. The Localization Theorem (5.1) implies that

$$\begin{aligned} K_G^0(X^G)_{F(G)} &\cong K_G^0(X)_{F(G)} \\ &\cong K_G^0(Y)_{F(G)} \\ &\cong K_G^0(Y^G)_{F(G)} \end{aligned}$$

and if $W$ is a set of $s$ points with trivial $G$-action then $K_G^0(W)_{F(G)}$ is a vector space over $F(G)$ of dimension $s$. Thus $X^G$ and $Y^G$ have the same cardinality. The only possible maps $f : X^G \to Y^G$ which induces an isomorphism on $K_G^0(-)$ are bijections. Thus $f : X^G \to Y^G$ is a bijection of finite sets.

## 7. The Main Theorem for $K^G$-oriented Manifolds.

In this section we shall establish the main theorem on $K^G$- oriented manifolds, showing that Pontryagin classes are preserved and control is maintained at the fixed point sets. These results involve assuming that $X$ and $Y$ have $K^G$-orientations. In Section 9 we shall demonstrate that this condition is frequently satisfied. The results in Section 9 and this section will be combined in Section 10 to demonstrate the results stated in the Introduction.

**Theorem 7.1.** *Let $G$ be a Hodgkin group and let $X$ and $Y$ be $K^G$−oriented connected $G$-manifolds of the same dimension such that $X^G$ and $Y^G$ are nonempty and consist of isolated fixed points. Let $f : X \to Y$ be a $G$-map* *which induces an isomorphism on fixed point sets. Suppose that $\Theta^G(f) \in K^0_G(Y)$ is a unit. Then $\Theta(f) \equiv 1$ mod $\mathbb{Z}$-torsion. If in addition the map $uf : uX \to uY$ is an $H$-orientation-preserving equivalence then $uf : uX \to uY$ preserves Pontryagin classes.*

*Proof.* Since $\Theta^G(f)$ is a unit in $K^*_G(Y)$ and restriction is a ring map, the class

$$\imath^{Y*}\Theta^G(f)_i \in K^*_G(y_i) \cong R(G)$$

is a unit. Thus for any copy of $S^1 \subset G$ the class

$$\imath^{Y*}\Theta^{S^1}(f)_i \in K^*_{S^1}(y_i) \cong R(S^1) \cong \mathbb{Z}[t,t^{-1}]$$

is a unit. Using a characteristic class argument which takes advantage of the specific formula for $\Theta^{S^1}$, Petrie [**P 2**, Lemma 6.3] shows that

$$\imath^{Y*}\Theta^{S^1}(f)_i = \pm t^{n(i,S^1)}$$

for some integer $n(i,S^1)$ and hence

$$t^{-n(i,S^1)}\imath^{Y*}\Theta^G(f)_i$$

is a unit of order 2 in $K^*_{S^1}(y_i)$. As $i$ runs over the (finite) fixed point set and (for $\Theta^G(f)$ given) it suffices to check for a finite number of circles, there is some sequence of integers $\mathbf{n} = (n_1, \dots, n_k)$ such that

$$t^{-\mathbf{n}}\imath^{Y*}\Theta^T(f)$$

is a unit of order 2 in $K^*_T(Y^T)$ where $t^{\mathbf{n}} = t_1^{n_1}\dots t_k^{n_k}$ and $T$ is a maximal torus for $G$.

Let $Ch^G_W$ denote the composite

$$K^*_G(W) \xrightarrow{r} K^*(W) \xrightarrow{Ch} H^{**}(W;\mathbb{Q}). \tag{7.2}$$

Then

$$Ch^T_{Y^T}\left(t^{-\mathbf{n}}\imath^{Y*}\Theta^T(f)\right) \in H^{**}\left(Y^T;\mathbb{Q}\right)$$

is a unit of order 2 in $H^{**}(Y^T;\mathbb{Q})$. The only units of order 2 in this ring are $\pm 1$ and it is easy to see that the degree zero term of $Ch^T_{Y^T}(t^{-\mathbf{n}}\Theta^T(f))$ is 1, so it must be the case that

$$Ch^T_{Y^T}\left(t^{-\mathbf{n}}\imath^{Y*}\Theta^T(f)\right) = 1 \in H^{**}\left(Y^T;\mathbb{Q}\right).$$

Since $Ch^T_{Y^T}$ is a ring map and

$$Ch^T_{Y^T}(t_i) = Ch(\epsilon(t_i)) = 1$$

it follows that

$$(7.3) \qquad Ch^G_{Y^G}\left(\imath^{Y*}\Theta^G(f)\right) = Ch^T_{Y^T}\left(\imath^{Y*}\Theta^T(f)\right) = 1 \in H^{**}\left(Y^T;\mathbb{Q}\right).$$

Consider the following commutative diagram:

$$\begin{array}{ccccc} K^*_G(Y) & \xrightarrow{r} & K^*(uY) & \xrightarrow{Ch} & H^{**}(uY;\mathbb{Q}) \\ \big\downarrow \imath^{Y*} & & \big\downarrow \imath^{Y*} & & \big\downarrow \imath^{Y*} \\ K^*_G(Y^G) & \xrightarrow{r} & K^*(uY^G) & \xrightarrow{Ch} & H^{**}(uY^G;\mathbb{Q}) \end{array}.$$

The composite of the maps in the upper row is $Ch^G_Y$ by (7.2), and similarly the composite of the maps in the lower row is $Ch^G_{Y^G}$. Thus

$$Ch^G_{Y^G}\imath^{Y*} = \imath^{Y*}Ch^G_Y = \imath^{Y*}Ch\, r$$

and since $\Theta(f) = r\Theta^G(f) \in K^*(uY)$ by Lemma 3.2, we see that

$$\imath^{Y*}Ch(\Theta(f)) = 1.$$

The map

$$\imath^{Y*} : H^0(uY;\mathbb{Q}) \longrightarrow H^0\left(uY^G;\mathbb{Q}\right)$$

is a monomorphism provided that $Y^G$ is nonempty and $Y$ is connected, and hence

$$Ch(\Theta(f)) = 1.$$

The Chern character is a ring monomorphism mod $\mathbb{Z}$-torsion, so $\Theta(f) \equiv 1$ mod $\mathbb{Z}$-torsion. The final statement of the Theorem follows from Corollary 3.10. □

**Proposition (7.4) (Petrie).** *Suppose that $M$ and $N$ are real representations of a compact connected Lie group $G$ of the same dimension, and let $\bar{M}$ and $\bar{N}$ be their complexifications. Suppose that $\lambda_{-1}(\bar{M})$ and $\lambda_{-1}(\bar{N})$ are non-zero and $\lambda_{-1}(\bar{M} - \bar{N})$ is a unit in $R(G)$. Then $M$ and $N$ are equivalent representations.*

*Proof.* The complexification map $RO(G) \to R(G)$ is an inclusion, so it suffices to show that $\bar{M}$ and $\bar{N}$ are equivalent representations. Representations are determined by characters and the obvious argument implies that it suffices to prove the theorem for the case $G = S^1$. This is the case which is established by Petrie [**P 2**, p. 365]. The key fact is that both $\lambda_{-1}(\bar{M})$ and $\lambda_{-1}(\bar{N})$ are products of cyclotomic polynomials so that classical algebraic number theory in the Dedekind domain $R(S^1)$ may be applied. □

It is now possible to prove the main technical theorem.

**Theorem 7.5.** *Suppose that $G$ is a Hodgkin group and $X$ and $Y$ are $K^G$-oriented connected $G$-manifolds of the same dimension such that $X^G$ and $Y^G$ are non-empty and consist of isolated fixed points. Let $f : X \to Y$ be a smooth $G$-map such that*

$$f^* : K_G^*(Y) \to K_G^*(X)$$

*is an isomorphism. Then:*

(1) *If $X$ and $Y$ are of the same even dimension then for each fixed point $x \in X^G$ the local representations of $G$ at $x$ and at $f(x)$ are equivalent.*

(2) *If the map $uf : uX \to uY$ is an $H$-orientation-preserving equivalence of the underlying manifolds then the map*

$$uf^* : H^*(uY; \mathbb{Q}) \to H^*(uX; \mathbb{Q})$$

*preserves Pontryagin classes.*

*Proof.* Note first that the map $f$ induces a bijection $f : X^G \to Y^G$ by Remark 6.12, so that the statement of the theorem makes sense. Further, the Theta class $\Theta^G(f) \in K_G^*(Y)$ is a unit by Theorem 4.8, since $f^*$ is an isomorphism.

To complete the proof of (1) we argue as follows. The class $\imath^{Y*}\Theta^G(f) \in K_G^*(Y^G)$ is a unit, since $\Theta^G(f)$ is a unit and $\imath^{Y*}$ is a ring map, and similarly $\imath^{Y*}\Theta^G(f)_i \in R(G)$ is a unit for each fixed point $x_i$. However,

$$\imath^{Y*}\Theta^G(f)_i = \lambda_{-1}(\tau Y_{f(x_i)} - \tau X_{x_i})$$

for each $i$ by Theorem 6.9 (which uses the fact that $X$ and $Y$ are of the same even dimension) and hence the class $\lambda_{-1}(\tau Y_{f(x)_i} - \tau X_{x_i})$ is a unit in $R(G)$. Then Proposition 7.4 implies that

$$\tau X_{x_i} \cong \tau Y_{f(x_i)}$$

for each fixed point $x_i \in X^G$.

To complete the proof of (2) we note that since $uf$ is an $H$-orientation-preserving equivalence, then by Theorem 7.1 the map $uf$ preserves Pontryagin classes. $\square$

**Remark 7.6.** Theorem 7.5 holds for the class of groups which satisfy the conclusion of Theorem 4.1. In Remark 4.14 we note that this is expected to be the case for all compact connected Lie groups, in which case Theorem 7.5 would also hold in that generality.

## 8. Bilinear Pairings.

In this section we pause to address two questions raised by Petrie in Part II, §3 of [**P 1**]. First some notation. Suppose that $G$ is a compact connected Lie group. Define

$$\mathfrak{K}_G^*(X) = K_G^*(X)/(R(G) - \text{torsion}).$$

Suppose that $X$ is a $K^G$-oriented $G$-manifold. Then there is a duality isomorphism

$$D : K_G^*(X) \to K_*^G(X)$$

and hence natural maps

$$K_G^*(X) \xrightarrow{D} K_*^G(X) \xrightarrow{e} \mathrm{Hom}_{R(G)}(K_G^*(X), R(G))$$

where $e$ is the Kronecker pairing. This induces a bilinear form

$$K_G^*(X) \times K_G^*(X) \xrightarrow{\langle\,\rangle} R(G)$$

on $K_G^*(X)$ with values in $R(G)$ via the formula

$$\langle a, b\rangle \;=\; (eD(a))(b)$$

which passes to a bilinear form on $\mathfrak{K}_G^*(X)$. Petrie [**P 1**, p. 144] asks the following two questions:

**Question 1:** When is the bilinear form on $\mathfrak{K}_G^*(X)$ nondegenerate?

**Question 2:** When the preceeding question has an affirmative answer, can one relate the algebraic invariants of the bilinear form to the representations of $G$ on the fibers normal to the fixed points sets?

He comments further: "One hopes that the bilinear form ... is nondegenerate when $\mathfrak{K}_G^*(X)$ is free over $R(G)$." He then works out some very specific situations where this is the case.

With the tools developed this paper, we can offer answers to these questions, under the assumption that the group $G$ has torsion free fundamental group. Theorem 8.1 gives a strong affirmative answer to the first question.

**Theorem 8.1.** *Suppose that $G$ is a Hodgkin group. Let $X$ be a $K^G$-oriented $G$-manifold such that $\mathfrak{K}_G^*(X)$ is a free $R(G)$-module. Then the associated bilinear form induced on $\mathfrak{K}_G^*(X)$ by duality and the Kronecker pairing ~~is~~ non-degenerate.*

*Proof.* It suffices to prove that the map

$$(8.2) \qquad e : K_*^G(X) \to \mathrm{Hom}_{R(G)}(K_G^*(X), R(G))$$

is an isomorphism modulo $R(G)$-torsion. This map is the edge homomorphism in the Universal Coefficient spectral sequence

$$E_2^{s,*} \cong \operatorname{Ext}^{s,*}_{R(G)}(K_G^*(X), R(G)) \quad \Longrightarrow \quad K_*^G(X).$$

Since $\mathfrak{K}_G^*(X)$ is a free $R(G)$-module, there is an unnatural isomorphism

$$K_G^*(X) \cong M \oplus \mathfrak{K}_G^*(X)$$

where $M$ is the $R(G)$-torsion submodule of $K_G^*(X)$. Thus for $s > 0$,

$$E_2^{s,*} \cong \operatorname{Ext}^{s,*}_{R(G)}(M, R(G))$$

which is again an $R(G)$-torsion module. [This fact follows from (5.5) and the fact that $M$ must be finitely generated. It would be false in general if $\mathfrak{K}_G^*(X)$ were only torsion free.] This implies that $E_\infty^{s,*}$ is also an $R(G)$-torsion module for $s > 0$, and hence the map $e$ in (8.2) is an isomorphism modulo $R(G)$-torsion. $\square$

Our response to Petrie's second question is not so precise unless there is some condition on the fixed point set. Write

$$X^G = X_1^G \sqcup \cdots \sqcup X_s^G$$

as a disjoint union of connected components. Each component $X_j^G$ is a manifold and clearly the bilinear form decomposes into the sum of bilinear forms on each component. So we may analyze one component at a time.

**Proposition 8.3.** *Suppose that $G$ is a Hodgkin group . Let $X$ be a $K^G$-oriented $G$-manifold. Let $\imath : X^G \to X$ be the inclusion of the fixed point set. Then the associated bilinear forms on $\mathfrak{K}_G^*(X)$ and on $\mathfrak{K}_G^*(X^G)$ are related as follows:*

(1)

$$\langle \imath_! a, \imath_! b\rangle_X = \langle a, \chi(\nu) b\rangle_{X^G}$$

*for $a,\ b \in \mathfrak{K}_G^*(X^G)$.*

(2) *If $X$ is even-dimensional and if $X^G$ consists of isolated fixed points with $\sigma$ the canonical splitting of $\imath_!$ described below, then*

$$\langle a, b\rangle_X = \langle \sigma(a), \chi^G(\nu)\sigma(b)\rangle_{X^G}$$

*for $a,\ b \in \mathfrak{K}_G^*(X)$.*

*Proof.* Suppose that $a \in \mathfrak{K}^*_G(X^G)$, $c \in \mathfrak{K}^*_G(X)$ are represented by $a \in K^*_G(X^G)$, $c \in K^*_G(X)$ respectively. Then

$$\begin{aligned}\langle \imath_! a, c\rangle_X &= (e_X D_X \imath_! a)(c)\\ &= (e_X \imath_* D_{X^G} a)(c)\\ &= (\imath^* e_{X^G} D_{X^G} a)(c)\\ &= (e_{X^G} D_{X^G} a)(\imath^* c)\\ &= \langle a, \imath^* c\rangle_{X^G}.\end{aligned}$$

Set $c = \imath_! b$ for some $b \in K^*_G(X^G)$. Then

$$\langle \imath_! a, \imath_! b\rangle_X \;=\; \langle a, \imath^* \imath_! b\rangle_{X^G} \;=\; \langle a, \chi^G(\nu) b\rangle_{X^G}$$

since $\imath^* \imath_!(y) \;=\; \chi^G(\nu) y$ by (2.12)(4). This proves (1). If $X^G$ consists of isolated points then

$$\imath_* : K^G_*(X^G) \longrightarrow K^G_*(X)$$

is a canonically split monomorphism with $\mathrm{Cok}(\imath_*)$ an $R(G)$-torsion module by Theorem 6.5, and by the definition of $\imath_!$ the same is true for the map

$$\imath_! : K^*_G(X^G) \longrightarrow K^*_G(X).$$

Let $\sigma : K^*_G(X) \to K^*_G(X^G)$ denote the splitting. Then $\mathrm{Ker}(\sigma)$ is an $R(G)$-torsion module. This implies that if $y \in K^*_G(X)$ is any element then $y$ may be written uniquely as

$$y = \imath_! \sigma(y) \;+\; (y - \imath_! \sigma(y))$$

where the second term is in the kernel of $\sigma$, hence $R(G)$-torsion. The pairing $\langle\, ,\, \rangle$ vanishes whenever either entry is $R(G)$-torsion, and hence

$$\begin{aligned}\langle a, b\rangle_X &= \langle \imath_! \sigma(a), \imath_! \sigma(b)\rangle_X\\ &= \langle \sigma(a), \chi^G(\nu)\sigma(b)\rangle_{X^G}\end{aligned}$$

by part (1), and this proves (2). □

Thus the study of the bilinear form reduces down completely to the study of non-equivariant bilinear forms on components of the fixed point set when $X^G$ consists of isolated points. The $K_G$-theory of a space with trivial $G$-action is well-understood:

$$K^*_G(X^G_j) \;\cong\; K^*\left(X^G_j\right) \otimes_{\mathbb{Z}} R(G).$$

A $K^G$-orientation of $X_j^G$ corresponds to a $K$-orientation of $X_j^G$ via this isomorphism. If $X^G$ consists of $s$ isolated points then the local fixed point data may be read off from the bilinear pairing and hence Petrie's question is resolved. If $X_j^G$ is a connected manifold of dimension greater than zero then $K^*(X_j^G)$ may well be non-trivial and hence the situation is much more complex.

## 9. Spin$^c$ orientations.

In this section we recall the relationship between $K$-orientations and Spin$^c$-structures. Then we generalize these results and demonstrate that $K^G$-orientations are fairly common, at least in the context of interest in this work when, for instance $X$ is simply connected and has a non-equivariant Spin$^c$-structure.

First we recall the non-equivariant situation. Suppose that $\xi$ is an oriented $n$-dimensional vector bundle with total space $E(\xi)$ over some compact space $X$. Then it has an associated principal right $SO(n)$-bundle

$$SO(n) \longrightarrow Q \longrightarrow X \tag{9.1}$$

with

$$E(\xi) \cong Q \times_{SO(n)} V$$

where SO(n) acts on $Q$ on the right, on a vector space $V \cong \mathbb{R}^n$ on the left, and

$$(qs, v) = (q, sv) \qquad \text{for} \quad q \in Q,\ s \in SO(n), \text{ and } \quad v \in V.$$

The bundle (9.1) is classified by a map $f_\xi : X \to BSO(n)$ and determines a Stiefel-Whitney class

$$w_2(\xi) = f_\xi^*(w_2) \in H^2(X; \mathbb{Z}/2).$$

There is a natural group extension

$$S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n)$$

and hence a principal $S^1$-bundle

$$S^1 \longrightarrow P \xrightarrow{\pi} Q \tag{9.2}$$

where $P$ is a right Spin$^c(n)$-space and the map $\pi$ is equivariant in the obvious sense. The bundle $\xi$ is said to have a reduction to a Spin$^c$-structure if there is an isomorphism of $SO(n)$-bundles

$$E(\xi) \cong P \times_{\mathrm{Spin}^c(n)} V.$$

Such a structure is classified by a class $c \in H^2(X;\mathbb{Z})$ which reduces to the class $w_2(\xi)$.

The matter may be expressed entirely in terms of classifying spaces. The exact sequence of groups

$$1 \longrightarrow \mathbb{Z}/2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n) \times U(1) \longrightarrow 1$$

gives rise to a fibration sequence

$$B\,\mathrm{Spin}^c \longrightarrow BSO \times K(\mathbb{Z},2) \xrightarrow{\kappa} K(\mathbb{Z}/2,2) \tag{9.3}$$

with

$$\kappa^*(\imath_2) = w_2 \otimes 1 + 1 \otimes \imath.$$

An oriented real vector bundle $\xi$ over $X$ is classified by a map $f_\xi : X \to BSO$ and any class $c \in H^2(X;\mathbb{Z})$ corresponds to a map $X \to K(\mathbb{Z},2)$. Thus the pair $(\xi, c)$ corresponds to a map

$$\varphi : X \to BSO \times K(\mathbb{Z},2).$$

The bundle $\xi$ reduces to a $\mathrm{Spin}^c$-bundle if and only if the map $\kappa\varphi$ is null-homotopic, which (by the formula for $\kappa$) is true if and only if $w_2$ is the reduction of $c$ for some choice of $c$. For example, if $H^3(X;\mathbb{Z})$ has no 2-torsion then exactness of the sequence

$$H^2(X;\mathbb{Z}) \xrightarrow{\times 2} H^2(X;\mathbb{Z}) \longrightarrow H^2(X;\mathbb{Z}_2) \xrightarrow{Sq^1} H^3(X;\mathbb{Z}) \xrightarrow{\times 2} H^3(X;\mathbb{Z})$$

implies that each class $w \in H^2(X;\mathbb{Z}/2)$ is the reduction of some integral class, and hence each oriented bundle over $X$ reduces to some $\mathrm{Spin}^c$-bundle.

A manifold is said to have a $\mathrm{Spin}^c$ structure if its tangent bundle has this structure. The $\mathrm{Spin}^c$-structures on $X$ are parameterized by the group $2H^2(X;\mathbb{Z})$ corresponding to the various possible choices for the class $c$. If $\tau X$ is a complex vector bundle (for instance, if $X$ is a complex manifold) then

$$w_2(\tau X) \equiv c_1(\tau X) \qquad \mathrm{mod}\ 2$$

and hence $X$ has a $\mathrm{Spin}^c$-structure. If $X$ is a Spin manifold then it also inherits a $\mathrm{Spin}^c$-structure. As Lawson and Michaelson comment, it requires some searching to find an orientable manifold which is not $\mathrm{Spin}^c$. For example, $RP^{4n+1}$ is $\mathrm{Spin}^c$ although it is neither Spin nor complex. The simplest example, noticed by Stong and Landweber, is perhaps the 5-dimensional manifold $SU(3)/SO(3)$. Its only non-zero mod 2 cohomology classes are 1, $w_2$, $w_3$ and $w_2w_3$. Since $Sq^1(w_2) = w_3 \neq 0$, one concludes that $w_2$ is not the reduction of any integral class. See [**St**, p. 292-3] and [**LM**, Appendix D] for a general treatment of these matters.

**Definition 9.4.** Let $G$ be a compact Lie group. Suppose that $X$ is a $G$-manifold and $\xi$ is a real $n$-dimensional $G$-bundle over $X$. Let $J \to O(n)$ be some homomorphism of groups (the cases $J = O(n)$, $SO(n)$, $\mathrm{Spin}^c(n)$ are of central interest). We shall say that the bundle $\xi$ has a **G-invariant reduction to J** if there exists a space $Q$ with the following properties:

(1) $Q$ is a left $G$-space, a right $J$-space, and the actions commute. Give $Q/J$ its induced left $G$-action.

(2) $J \to Q \xrightarrow{\pi} Q/J$ is a principal $J$-bundle and $X \cong Q/J$ as $G$-spaces.

(3) There is an $\mathbb{R}$-vector space $V$ with a left $J$-action and an isomorphism of $O(n)$-bundles
$$E(\xi) \cong Q \times_J V.$$

(4) The isomorphism (3) is $G$-equivariant over $X$.

If $J = O(n)$ then this is simply a restatement of what it means for $\xi$ to be a $G$-vector bundle.

If a $G$-bundle $\xi$ has a $G$-invariant reduction to $\mathrm{Spin}^c$ then it has a $K^G$-orientation. The non-equivariant version of the following theorem is clearly set out in [**ABS**]. Petrie [**P 1**, §4] discusses the equivariant construction in somewhat different language but to the same effect. Certainly the result has been well-known for many years.

**Proposition 9.5.** *Let $G$ be a compact Lie group. Suppose that $X$ is a $G$-manifold and $\xi$ is a real $n$-dimensional $G$-bundle over $X$ with a $G$-invariant reduction to* $\mathrm{Spin}^c(n)$. *Then there is an associated $K^G$-orientation class $\mu \in \tilde{K}^n_G(T\xi)$.*

*Proof.* Suppose first that $n$ is even. By assumption there is a space $Q$ with a left $G$-action, a right $\mathrm{Spin}^c(n)$ action which commutes with the $G$-action, a principal $\mathrm{Spin}^c(n)$-bundle
$$\mathrm{Spin}^c(n) \longrightarrow Q \xrightarrow{\pi} X$$
with $\pi$ a $G$-map and an isomorphism of $O(n)$-bundles
$$E(\xi) \cong Q \times_{\mathrm{Spin}^c(n)} V.$$
Let $\Delta_+$ and $\Delta_-$ be the canonical $\mathrm{Spin}^c$-representations. Following [**ASi**, p. 489] and [**P 2**], define a $G \times \mathrm{Spin}^c(n)$-complex of vector bundles over $Q \times V$

of the form

$$\begin{array}{ccc} Q \times V \times \Delta_+ & \xrightarrow{\Phi} & Q \times V \times \Delta_- \\ \downarrow & & \downarrow \\ Q \times V & \xrightarrow{1} & Q \times V \end{array}$$

with

$$\Phi(q, v, d) = (q, v, L(v)d)$$

where $L(v)$ denotes left Clifford multiplication on $\Delta_\pm$. The $G \times \mathrm{Spin}^c(n)$-action on $Q \times V$ is given by

$$(g, s)(q, v) = (gqs^{-1}, sv)$$

and the action on $Q \times V \times \Delta_\pm$ is given by

$$(g, s)(q, v, b) = (gqs^{-1}, sv, sb).$$

Since $(v, d) \mapsto (v, L(v)d)$ is an elliptic pairing, this complex defines an element

$$\mu \in K^0_{G \times \mathrm{Spin}^c(n)}(Q \times V) \cong K^0_G(Q \times_{\mathrm{Spin}^c(n)} V) \cong \tilde{K}^n_G(T\xi).$$

We claim that this class is a $K^G$-orientation. In fact this class induces the equivariant Thom isomorphism

$$K^*_G(X) \cong \tilde{K}^*_G(T\xi)$$

which is given explicitly by $x \mapsto x\mu$. Kasparov [**K 1**] shows that this map is an isomorphism by showing that the class $\mu$ is $KK^G$-invertible. Since $KK^G$-invertibility is preserved under restriction to closed subgroups and the class is natural with respect to restriction to subspaces of $X$, it follows that for any local Thom map $j : G/H \to X$ with fibre representation $W$ the induced map

$$\tilde{K}^*_G(T\xi) \longrightarrow \tilde{K}^*_G(T(j^*\xi)) \cong \tilde{K}^*_H(S^W)$$

sends $\mu$ to a $KK^H$-invertible element which must then be a generator of $K^*_H(S^W)$ as a free $\pi^H_*(K^G)$-module. This completes the argument when $n$ is even.

If $n$ is odd then apply the argument above to the (even-dimensional) bundle $\xi \times \mathbb{R}$. Then use the canonical $K^G$-orientation on the equivariant Thom space of the trivial line bundle to construct the corresponding $K^G$-orientation on $T\xi$. $\square$

Thus our attention turns to the circumstances under which a given $G$-bundle has a $G$-invariant reduction to $\mathrm{Spin}^c$. Suppose then that $G$ is a compact Lie group, $X$ is an oriented $G$-manifold, and $G$ acts on $X$. As remarked in the introduction, we may assume that the action of $G$ is implemented by orientation-preserving isometries on $X$. Suppose that $\xi$ is a real $n$-dimensional vector bundle over $X$ and that $\xi$ has given a $G$-invariant reduction to $SO(n)$. (In the applications $\xi = \tau M$.) Write

$$E(\xi) \cong Q \times_{SO(n)} V \tag{9.6}$$

as in Definition 9.4. We wish to determine when $\xi$ has a $G$-invariant reduction to $\mathrm{Spin}^c(n)$. If $\xi$ has a $G$-invariant reduction to $\mathrm{Spin}^c(n)$ then the class $w_2(\xi)$ must be the reduction of some integral class. Choose such a class $c \in H^2(X;\mathbb{Z})$. This corresponds to picking a (non-equivariant) $\mathrm{Spin}^c(n)$-reduction for $\xi$. More precisely, there is a right $\mathrm{Spin}^c(n)$-space P, a principal $\mathrm{Spin}^c(n)$-bundle

$$\mathrm{Spin}^c(n) \longrightarrow P \longrightarrow X$$

and an isomorphism of $SO(n)$-bundles

$$E(\xi) \cong P \times_{\mathrm{Spin}^c(n)} V. \tag{9.7}$$

Further, this structure covers (9.6) in the sense that there is a principal $S^1$-bundle

$$S^1 \longrightarrow P \longrightarrow Q$$

where $S^1$ is acting on $P$ on the right by regarding $S^1$ as the center of $\mathrm{Spin}^c(n)$ via the canonical extension

$$S^1 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n).$$

The bundle (9.7) is classified (as a principal $S^1$-bundle) by a class $\kappa \in H^2(Q;\mathbb{Z})$ which depends upon the bundle $\xi$ and upon the choice of the class $c \in H^2(X;\mathbb{Z})$ which specifies the non-equivariant $\mathrm{Spin}^c(n)$ -structure.

Fix a basepoint $q_o \in Q$. Define

$$\omega : G \longrightarrow Q$$

by $\omega(g) = gq_o$ and define

$$\hat{\omega} : G \times Q \to Q$$

by $\hat{\omega}(g,q) = gq$.

The following theorem is the key to the rest of this section of the paper; I am most grateful to Dan Gottlieb for calling it to my attention. His result applies to principal torus bundles but we state only the case which we shall use.

**Theorem 9.8** (D. Gottlieb [**G**]). *Let $G$ be a compact connected Lie group. Let*

$$S^1 \to P \xrightarrow{\pi} Q$$

*be a principal $S^1$-bundle classified by $\kappa \in H^2(Q;\mathbb{Z})$. Suppose that $Q$ is a left $G$-space. Define*

$$\Gamma(\kappa) = \hat{\omega}^*(\kappa) - (1 \times \kappa) \in H^2(G \times Q;\mathbb{Z}).$$

*Then the action of $G$ on $Q$ lifts to an action of $G$ on $P$ with $\pi$ a $G$-map if and only if $\Gamma(\kappa) = 0$.*

Gottlieb comments that in general

$$\Gamma(\kappa) = \omega_1 + (\omega^*(\kappa) \times 1)$$

where $\omega_1 \in H^1(G;H^1(Q;\mathbb{Z}))$. He notes the following:

(1) $\omega_1 = 0$ whenever either $H_1(G)$ or $H_1(Q)$ is a torsion group.

(2) $\omega^*(\kappa) = 0$ if the $G$-action on $Q$ has a fixed point.

(3) $\chi(Q)\omega^* = 0$ if $Q$ is compact.

(4) $\omega^* = 0$ if $H_1(G;\mathbb{Z})$ is a free abelian group and $H_1(Q;\mathbb{Z})$ is a torsion group.

The conditions of (4) also imply that $\omega_1 = 0$.

**Corollary 9.9.** *Let $G$ be a compact connected Lie group. Let $X$ be an oriented $G$-manifold. Suppose that $\xi$ is a real $n$-dimensional vector bundle over $X$ and that $\xi$ has a given $G$-invariant reduction to $SO(n)$ and a (non-equivariant) further reduction to* $\mathrm{Spin}^c(n)$ *with associated principal bundle*

$$S^1 \to P \xrightarrow{\pi} Q$$

*classified by $\kappa$ as above. Then the $G$-action on $Q$ lifts to a $G$-action on $P$ with $\pi$ a $G$-map if and only if $\Gamma(\kappa) = 0$. Further, if $H_1(G;\mathbb{Z})$ is free abelian and $H_1(Q;\mathbb{Z})$ is a torsion group then $\Gamma(\kappa) = 0$ and hence there is always a $\pi$-compatible lifting of the $G$-action.*

**Corollary 9.10.** *Let $G$ be a Hodgkin group. Let $X$ be a connected, simply connected oriented $G$-manifold. Suppose that $\xi$ is a real $n$-dimensional vector bundle over $X$ and that $\xi$ has a given $G$-invariant reduction to $SO(n)$ and a (non-equivariant) further reduction to* $\mathrm{Spin}^c(n)$ *with associated principal bundle*

$$S^1 \to P \xrightarrow{\pi} Q$$

*classified by $\kappa$ as above. Then the $G$-action on $Q$ lifts to a $G$-action on $P$ with $\pi$ a $G$-map.*

*Proof.* The homotopy sequence of the principal bundle

$$SO(n) \longrightarrow Q \longrightarrow X$$

implies that $\pi_1(Q) = \mathbb{Z}/2$ or 0 and in either case $H_1(Q)$ is a torsion group. The group $H_1(G;\mathbb{Z}) \cong \pi_1(G)$ is torsionfree and finitely generated, hence free, and so the conditions of (9.9) are satisfied. □

This is not quite the end of the story, since the left $G$-action on $P$ which is constructed above does not necessarily commute with the right principal $SO(n)$-action on $P$. We shall deform the left $G$ action constructed above in order to repair this defect. The following proposition was proved in the case $G = S^1$ by Petrie [**P 1**, p. 117] and his argument generalizes immediately to the present situation.

**Proposition 9.11** (Petrie). *Let $G$ be a compact connected Lie group. Suppose given an oriented connected manifold $Q$ with a right $SO(n)$-action and with $G$ acting on the left. Let*

$$S^1 \to P \xrightarrow{\pi} Q$$

*be the associated principal bundle, where* $\mathrm{Spin}^c(n)$ *acts on $P$ on the right, and $S^1$ acts on $P$ by regarding it as the center of* $\mathrm{Spin}^c(n)$*. Suppose further that $P$ is a left $G$-space and that the map $\pi$ is a $G$-map. Then the action of $G$ on $P$ may be deformed to a new left action of $G$ on $P$ with respect to which $\pi$ is still a $G$-map and the new action of $G$ on $P$ commutes with the right* $\mathrm{Spin}^c(n)$*-action.*

*Proof.* Define $\hat{\psi} : G \times P \times \mathrm{Spin}^c(n) \longrightarrow S^1$ by

$$(gp)s \;=\; g(ps)\hat{\psi}(g,p,s).$$

This makes sense since the left $G$-action on $Q$ commutes with the right $SO(n)$-action. Then

$$\hat{\psi}(g,pt,s) = \hat{\psi}(g,p,s)$$

for $t \in S^1$ since $S^1$ is being regarded as the center of $\mathrm{Spin}^c(n)$. Hence $\hat{\psi}$ passes to quotients: define

$$\psi : G \times Q \times \mathrm{Spin}^c(n) \longrightarrow S^1$$

by

$$\psi(g, \pi(p), s) = \hat{\psi}(g, p, s).$$

The map $\psi$ satisfies the following conditions:

(1) $\qquad \psi(1_G, q, s) = \psi(g, q, 1_{\mathrm{Spin}^c(n)}) = 1_{S^1}$

(2) $\qquad \psi(g_1 g_2, p, s) = \psi(g_1, g_2 p, s)\psi(g_2, p, s)$

(3) $\qquad \psi(g, p, s_1 s_2) = \psi(g, p s_1, s_2)\psi(g, p, s_1)$.

Choose a basepoint $q_o$ for $Q$. Condition (1) implies that $\psi \equiv 1$ on the wedge

$$G \vee Q \vee \mathrm{Spin}^c(n)$$

and thus $\psi$ factors through a based map

$$\bar{\psi} : G \wedge Q \wedge \mathrm{Spin}^c(n) \longrightarrow S^1 \cong K(\mathbb{Z}, 1).$$

Since $G$, $Q$, and $\mathrm{Spin}^c(n)$ are connected, the Künneth Theorem in ordinary cohomology implies that $H^1(G \wedge Q \wedge \mathrm{Spin}^c(n); \mathbb{Z}) = 0$ and hence $\bar{\psi}$ is null-homotopic. Thus there is a unique lift

$$\Psi : G \times Q \times \mathrm{Spin}^c(n) \longrightarrow \mathbb{R}$$

of $\psi$ such that

$$\Psi\left(1_G, q_o, 1_{\mathrm{Spin}^c(n)}\right) = 1_{\mathbb{R}}.$$

Define $\gamma : G \times Q \to \mathbb{R}$ by

$$\gamma(g, q) = \int_{\mathrm{Spin}^c(n)} \Psi(g, qs, s^{-1}) ds$$

where the integration is with respect to normalized Haar measure on $\mathrm{Spin}^c(n)$. Let $\bar{\gamma}$ be the composite

$$G \times Q \xrightarrow{\gamma} \mathbb{R} \longrightarrow S^1.$$

Define a new action of $G$ on $P$ by

$$g \bullet p = g p \bar{\gamma}(g, \pi(p)).$$

Then the invariance properties (2) and (3) of $\psi$ imply that

(1) $\qquad (g_1 g_2) \bullet p = g_1 \bullet (g_2 \bullet p)$

(2) $\qquad (g \bullet p)s = g \bullet (ps)$

(3) $\qquad \pi(g \bullet p) = \pi(gp)$

for all $g$, $g_1$, $g_2 \in G$, $p \in P$, $s \in SO(n)$. This completes the proof of the theorem. $\square$

The results above immediately imply the following theorem.

**Theorem 9.12.** *Suppose that $G$ is a Hodgkin group. Let $X$ be a connected, simply connected $G$-manifold. Suppose that $w_2(X)$ is the reduction of an integral class, or, equivalently, that $\tau X$ has a (non-equivariant)* $\mathrm{Spin}^c$ *structure. Then $\tau X$ has a $G$-invariant reduction to* $\mathrm{Spin}^c$ *which is (non-equivariantly) equivalent to the given* $\mathrm{Spin}^c$*-structure, and thus $X$ has a $K^G$-orientation.*

*Proof.* By assumption the tangent bundle $\tau X$ has a (non-equivariant) $\mathrm{Spin}^c$ structure. Corollary 9.10 of Gottlieb's theorem applies, and so there is a $G$-action on the total space $P$ of the associated $\mathrm{Spin}^c$-bundle. This action may not commute with the given $\mathrm{Spin}^c$-structure, but Proposition 9.11 allows us to deform the initial $G$ action on $P$ to a new $G$ action which does commute with the $\mathrm{Spin}^c$ action. Then $\tau X$ has a $G$-invariant reduction to $\mathrm{Spin}^c$ and Proposition 9.5 implies that $X$ has a $K^G$-orientation. $\square$

**Remark 9.13.** The next question which might be asked is to determine the exact relationship between $K^G$-orientations and invariant $\mathrm{Spin}^c$-structures. Stong [**St**, p. 301] shows that every $K$-oriented vector bundle has a stable $\mathrm{Spin}^c$-structure, but the $K$-orientation which arises from the $\mathrm{Spin}^c$-structure may not coincide with the original $K$-orientation. Of course the two may differ at most by a unit. Similarly, it is natural to ask how many of the $K^G$-orientations of a bundle arise in this fashion. The situation for Spin-manifolds and $KO$-orientations has been determined by Atiyah and Hirzebruch [**AH2**]. It turns out that half of the $KO$-orientations come from Spin-structures. We presume that the analogous statements are true equivariantly.

In principle the results in Section 9 should be derived from equivariant bundle theory directly, rather then via the arguments we have made which handle the non-equivariant and equivariant obstructions separately. Corresponding to the sequence

$$\mathbb{Z}/2 \longrightarrow \mathrm{Spin}^c(n) \longrightarrow SO(n) \times S^1$$

there is a sequence of equivariant classifying spaces [**M**]

$$B(\mathbb{Z}/2, G \times \mathrm{Spin}^c(n)) \longrightarrow B_G(\mathrm{Spin}^c(n)) \xrightarrow{\eta^G} B_G(SO(n) \times S^1). \tag{9.14}$$

May's techniques imply that as a $G$-space,

$$B(\mathbb{Z}/2, G \times \mathrm{Spin}^c(n)) \cong K(\mathbb{Z}/2, 1)$$

where $G$ acts trivially on $K(\mathbb{Z}/2,1)$. Hence (9.14) becomes the sequence

$$K(\mathbb{Z}/2,1) \longrightarrow B_G(\mathrm{Spin}^c(n)) \xrightarrow{\eta^G} B_G(SO(n) \times S^1) \xrightarrow{\kappa} C_{(\eta^G)} \tag{9.15}$$

which is analogous to (9.3). Determination of the cofibre $C_{(\eta^G)}$ and the map $\kappa$ might lead to an equivariant analysis of the results of this section. However this is apparently out of reach of current technology. The general area is now under study by Costanoble, Kriz, May, and Waner.

## 10. Conclusion.

In this short section we pull together the results of the previous sections in order to establish the main theorem stated in the Introduction.

**Theorem 10.1.** *Suppose that $X$ and $Y$ are connected, simply connected* $\mathrm{Spin}^c$*-manifolds of the same dimension. Let $G$ be a Hodgkin group which acts upon $X$ and $Y$ such that $X^G$ and $Y^G$ are non-empty and consist entirely of isolated fixed points. Suppose that $f : X \to Y$ is a smooth $G$-map such that the induced map $f^* : K_G^*(Y) \to K_G^*(X)$ is an isomorphism. Then:*

(1) *If $X$ and $Y$ are of the same even dimension then for each fixed point $x \in X^G$, the local representations of $G$ at $x$ and at $f(x)$ are equivalent.*

(2) *If $uf : uX \to uY$ is an equivalence then $uf^* : H^*(uY;\mathbb{Q}) \to H^*(uX;\mathbb{Q})$ preserves Pontryagin classes.*

*Proof.* The space $X$ is a connected, simply connected manifold with the Hodgkin group $G$ acting by orientation-preserving isometries and with a (non-equivariant) $\mathrm{Spin}^c$-structure. Theorem 9.12 implies that $\tau X$ has a $G$-equivariant reduction to $\mathrm{Spin}^c$ and hence $X$ has a $K^G$-orientation. The same argument shows that $Y$ also has a $K^G$-orientation. Then we may apply Theorem 7.5 (which is the version of Theorem 10.1 for $K^G$-oriented manifolds) to complete the proof of the theorem. □

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