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ON NORMS OF TRIGONOMETRIC POLYNOMIALS ON $SU(2)$

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A conjecture about the L^4-norms of trigonometric polynomials on $SU(2)$ is discussed and some partial results are proved.

1. Introduction.

If G is a compact abelian group, an elementary argument shows that $M_p(G) = M_q(G)$ where $M_p(G)$ denotes the space of L^p-multipliers on G and p and q are conjugate indices. Oberlin [7] found a nonabelian totally disconnected compact group G for which $M_p(G) \neq M_q(G)$. Herz [4] conjectured that inequality holds for all those infinite nonabelian compact groups G whose degrees of the irreducible representations are unbounded. However, for compact connected groups, the situation is still unresolved, even for $SU(2)$.

The present paper arose from an attempt to study the Herz conjecture for $SU(2)$. In his unpublished M.Sc. thesis [8], S. Roberts formulated a conjecture for $SU(2)$which, if proved, would settle the Herz conjecture for all compact connected groups. we believe that Robert's conjecture is interesting in its own right as it makes a rather delicate statement connecting the L^p-norms of noncentral trigonometric polynomials with the growth of the Clesh-Gordon coefficients.

We have pursued this interesting conjecture and make some partial progress towards settling it. Our results open the way to a detailed study of some new aspects of L^p analysis on compact Lie groups.

In Section 2, we establish our notation. We state the conjecture in Section 3 and proove some partial results (Theorem 3.2). In Section 4, we show the relevance of the conjecture to Herz's conjecture.

2. Notation and remarks.

2.1. Irreducible representations of $SU(2)$. We summarise some notation and definitions from [6] concerning the irreducible representation of $SU(2)$.

Let n be a rational number of the form $k/2$, where $k \in \mathbb{N}$ and H_n be the space of homogeneous polynomials on $\mathbb{C}^2$ of degree $2n$; i.e. of functions of the form

$$f(z_1, z_2) = \sum_{i=-n}^{+n} a_i z_1^{n-i} z_2^{n+i}. \tag{2.1}$$

Let (f, g) be the inner product on H_n given by the formula

$$\begin{aligned}\left(\sum_{i=-n}^{+n} a_i z_1^{n-i} z_2^{n+i}, \sum_{j=-n}^{+n} b_j z_1^{n-j} z_2^{n+j}\right) \\ = \sum_{i=-n}^{+n} (n-i)!(n+i)! a_i \bar{b}_i.\end{aligned} \tag{2.2}$$

Let $U(H_n)$ denote the set of unitary operators on H_n with respect to the inner product (2.2). The mapping $T_n : SU(2) \to U(H_n)$ given by

$$\left(T_n \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} f\right)(z_1, z_2) = f(\alpha z_1 - \bar{\beta} z_2, \beta z_1 + \bar{\alpha} z_2) \tag{2.3}$$

is an irreducible representation of $SU(2)$and in fact the set $\{T_n : n = 0, 1/2, \dots\}$ forms a complete set of inequivalent irreducible representations of $SU(2)$.

To each operator $T_n(x)$, $x \in SU(2)$, there corresponds a unitary matrix (relative to the natural orthonormal basis of H_n) whose elements will be denoted by t_{jk}^n $(-n \leq j, k \leq n)$. These matrix elements are continuous functions on $SU(2)$. We shall be estimating their norms as convolution operators on L^p.

There are many results on the L^p multiplier norms of central trigonometric polynomials - see for example [**2**], or the more recent optimal results of Sogge on Riesz kernels on arbitrary compact manifolds (c.f. also [**9**], [**10**]). However, the t_{jk}^n's considered here are non-central.

A word about the geometric significance of the matrix coefficients t_{jk}^n is in order. By the Peter-Weyl theorem, $L^2(G)$ decomposes as a direct sum of the irreducible representations of G, each one occuring with multiplicity equal to its dimension. These isotypic components represent the eigenspaces of the Laplace-Beltrami operator, and convolution by $(2n+1)\chi_n$, where χ_n is the characcter of the nth irreducible representation, is the projection onto this space.

For each j $(-n \leq j \leq n)$, the functions $\{t_{jk}^n : -n \leq k \leq n\}$ span one of the above copies of the representation space of degree $2n+1$. Convolution on the left by $(2n+1)t^n_{jj}$ is a projection of $L^2(G)$ onto this copy. Convolution by the function $(2n+1)t^n_{jk}$ are the natural isometries between the various copies of the nth irreducible representation inside the isotypic component.

2.2. Expressions of products of functions t^n_{jk}: The tensor product of any two nontrivial irreducible unitary representations of $SU(2)$is always reducible. If one decomposes such a tensor product into its irreducible components, then the coefficients which appear in the decomposition are known as the Clebsch-Gordan coefficients.

In the case of $SU(2)$the Clebsch-Gordan coefficients $C(n_1, n_2, n_3, j_1, j_2, j_3)$ make their appearance in this way in the formula

$$
\begin{aligned}
(2.4) \qquad t^{n_1}_{j_1 j_2} t^{n_2}_{k_1 k_2} = \sum_{m=|n_1-n_2|}^{n_1+n_2} & C(n_1, n_2, m, j_1, k_1, k_1 + j_1) \\
& \cdot C(n_1, n_2, m, j_2, k_2, k_2 + j_2) t^m_{j_1+k_1\, j_2+k_2}.
\end{aligned}
$$

While the Clebsch-Gordan coefficients are, in general, very complicated [8], there are simple formulas for them in certain situations. Two such cases are given below. they will be of interest in Section 3.

$$
(2.5)\ C(n, n, 2n, j, k, j+k) = \left(\frac{(2n+j+k)!(2n-j-k)!2n!2n!}{(n-j)!(n+j)!(n-k)!(n+k)!4n!}\right)^{1/2}
$$

(2.6)

$$
\begin{aligned}
C(n, n, 2m, j, j, 2j) = (-1)^{n-m} & \left(\frac{(4m+1)(2j+2m)!2n-2m!2m-2j!}{(2n+2m+1)!}\right)^{1/2} \\
& \times \frac{(m+n)!}{(m+j)!(m-j)!(n-m)!}
\end{aligned}
$$

if $n \geq m \geq |j|$ and 0 otherwise

$$
(2.7) \qquad C(n, n, 2m+1, j, j, 2j) = 0.
$$

We denote by C^{2m}_{ik} the Clebsch-Gordan coefficient $C(n, n, 2m, i, k, i+k)$ where n will considered be fixed throughout the argument.

In Section 4, we use the following convolution identity ([5], 27.20)

$$
t^n_{jk} * t^m_{pq} = \frac{1}{2n+1} \delta_{nm} \delta_{kp} t^n_{jq}.
$$

By $A_n \approx B_n$, $n > 1$ we mean that there exist positive constants α, β such that

$$
\beta B_n \leq A_n \leq \alpha B_n, \ \forall n \geq 1.
$$

The same symbol C may denote two different constants in two different lines.

3. The conjecture.

In this section we state the conjecture and prove some partial results.

Conjecture. *Denote by* $\underline{z}^{(n)} \in \mathbb{C}^{2n+1}$ *the vector with components* $\{z_i^{(n)}\}_{i=-n}^{+n}$. *Let*

$$A_n = \frac{1}{n^{1/8}} \sup_{\substack{\underline{z}^{(n)} \in \mathbb{C}^{2n+1} \\ \sum_{i=-n}^{+n} |z_i^{(n)}|=1}} \frac{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{0i}^n \right\|_4}{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{ni}^n \right\|_4}.$$

Then $A_n \to 0$ *as* $n \to \infty$.

Remark 3.1. For the motivation of the conjecture, see Section 4.

We will prove the following theorem which is a weaker version of the conjecture:

Theorem 3.2.

(A) *Let* $\underline{z}^{(n)} \in \mathbb{C}^{2n+1}$ *and* $\underline{z}_i^{(n)} \geq 0$, $\forall i = -n, \ldots, n$. *Define* $F_n(\underline{z}^{(n)}) = \{i |\, z_i^{(n)} \neq 0\}$. *Suppose that* $\sum_{i=-n}^{+n} \left| z_i^{(n)} \right| = 1$.

If

$$\left| F_n(\underline{z}^{(n)}) \right| \leq \frac{Cn^{1/3}}{(\log n)^{2/3}} \quad \forall n \geq 2, \qquad \textit{then}$$

$$\frac{1}{n^{1/8}} \frac{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{0i}^n \right\|_4}{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{ni}^n \right\|_4} \leq \frac{C}{(\log n)^{1/4}}.$$

(B) *Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of natural numbers such that* $p_n \geq 2$ $\forall n$. *Define* $j_n = \left[\frac{\log n}{\log p_n} \right]$. *Then*

$$\frac{1}{n^{1/8}} \sup_{\sum_{i=-n}^{+n} |z_i^n|=1} \frac{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{0p_n^i}^n \right\|_4}{\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{np_n^i}^n \right\|_4} \leq C \frac{\log n}{n^{1/4}}.$$

To prove Theorem 3.2, we first obtain an expression for $\left\| \sum_{i=-n}^{+n} z_i t_{ji}^n \right\|_4$ in terms of Clebsch-Gordan coefficients. Let $\underline{z} \in \mathbb{C}^{2n+1}$.

Set

$$\varphi_j^n(\underline{z}) = \sum_{i=-n}^{+n} z_i t_{ji}^n$$

$$\begin{aligned}\left(\varphi_j^n(\underline{z})\right)^2 &= \left(\sum_{i=-n}^{+n} z_i t_{ji}^n\right)\left(\sum_{k=-n}^{+n} z_k t_{jk}^n\right)\\ &= \sum_{-n\le i,k\le n} z_i z_k t_{ji}^n t_{jk}^n\\ &= \sum_{-n\le i,k\le n} z_i z_k \left(\sum_{m=0}^{2n} C(n,n,m,j,j,2j)C(n,n,m,i,k,i+k)t_{2j\,i+k}^m\right).\end{aligned}$$

Using (2.6)-(2.7), we get

$$\begin{aligned}\left(\varphi_j^n(\underline{z})\right)^2 &= \sum_{-n\le i,k\le n} z_i z_k \sum_{m=|j|}^{n} C(n,n,2m,j,j,2j)C(n,n,2m,i,k,i+k)t_{2j\,i+k}^{2m}\\ &= \sum_{m=|j|}^{n} C_{j\,j}^{2m} \sum_{r=-2m}^{+2m}\left(\sum_{\substack{i+k=r\\ -n\le i,k\le n}} z_i z_k C_{i\,k}^{2m}\right) t_{2j\,r}^{2m}.\end{aligned}$$

Since $\left\{\left\{\sqrt{2n+1}\; t_{ij}^n\right\}_{-n\le i,j\le n}\right\}_{n=0,1/2,\dots}$ is an orthonormal set in $L^2(SU(2))$, we get

$$\left\|\left(\varphi_j^n(\underline{z})\right)^2\right\|_2^2 = \sum_{m=|j|}^{n} \frac{\left(C_{j\,j}^{2m}\right)^2}{(4m+1)} \sum_{r=-2m}^{+2m}\left|\sum_{\substack{i+k=r\\ -n\le i,k\le n}} z_i z_k C_{i\,k}^{2m}\right|^2.$$

In particular,

$$\|\varphi_0^n(\underline{z})\|_4 = \left[\sum_{m=-n}^{n} \frac{(C_{0\,0}^{2m})^2}{4m+1} \sum_{r=-2m}^{+2m}\left|\sum_{\substack{i+k=r\\ -n\le i,k\le n}} z_i z_k C_{i\,k}^{2m}\right|^2\right]^{1/4} \tag{3.3}$$

$$\begin{aligned}\|\varphi_n^n(\underline{z})\|_4 &= \left[\frac{(C_{n\,n}^{2n})^2}{(4n+1)} \sum_{r=-2n}^{+2n}\left|\sum_{\substack{i+k=r\\ -n\le i,k\le n}} z_i z_k C_{i\,k}^{2n}\right|^2\right]^{1/4} \qquad (3.4)\\ &= \left[\frac{1}{(4n+1)} \sum_{r=-2n}^{+2n}\left|\sum_{\substack{i+k=r\\ -n\le i,k\le n}} z_i z_k C_{i\,k}^{2n}\right|^2\right]^{1/4} \quad \text{as } C_{n\,n}^{2n}=1.\end{aligned}$$

Next we prove two Lemmas.

Lemma 3.5. *There exist constants $C_1, C_2 > 0$ satisfying: $(n \geq 2)$*

(i) $\frac{C_2}{n^{1/4}} \leq |C^{2n}_{0\,0}| \leq \frac{C_1}{n^{1/4}}$.

(ii) $\frac{C_2}{\sqrt{2n+1}} \leq |C^{0}_{0\,0}| \leq \frac{C_1}{\sqrt{2n+1}}$.

(iii) *Let $0 \leq |j| \leq n-1$. Then*

$$\frac{C_2 n^{1/4}}{(n+j)^{1/4}(n-j)^{1/4}} \leq \left|C^{2n}_{j\,j}\right| \leq \frac{C_1 n^{1/4}}{(n+j)^{1/4}(n-j)^{1/4}}.$$

(iv) *Let $|j|+1 \leq m \leq n-1$. Then*

$$\frac{C_2\sqrt{m}}{(m+n)^{1/4}(m+j)^{1/4}(m-j)^{1/4}(n-m)^{1/4}} \leq \left|C^{2m}_{j\,j}\right|$$

$$\left|C^{2m}_{j\,j}\right| \leq \frac{C_1\sqrt{m}}{(m+n)^{1/4}(m+j)^{1/4}(m-j)^{1/4}(n-m)^{1/4}}.$$

(v) *Let $1 \leq j \leq n-1$. Then*

$$\frac{C_2 j^{1/4}}{(n+j)^{1/4}(n-j)^{1/4}} \leq \left|C^{2j}_{j\,j}\right| \leq \frac{C_1 j^{1/4}}{(n+j)^{1/4}(n-j)^{1/4}}.$$

(vi) *Let $1 \leq j \leq n-1$. Then*

$$\frac{C_2}{(n+j)^{1/4}(n-j)^{1/4}} \leq \left|C^{2j}_{0\,0}\right| \leq \frac{C_1}{(n+j)^{1/4}(n-j)^{1/4}}.$$

Proof. The easy proof using the following inequality

$$e^{7/8} \leq \frac{n!}{(n/e)^n n^{1/2}} \leq e \text{ for } n = 1, 2, 3, \ldots$$

is left to the reader. □

Lemma 3.6. *Let $n \geq 2$ be a natural number and $-n \leq i \leq n$. Then there exists a positive constant C such that*

$$\frac{1}{n^{1/8}} \frac{\|t^n_{0\,i}\|_4}{\|t^n_{n\,i}\|_4} \leq C\left(\frac{\log n}{n}\right)^{1/4}. \tag{3.7}$$

Also for every ϵ, $0 < \epsilon < 1$, there exists a $C_\epsilon > 0$ such that for $0 \leq |i| \leq n\epsilon$, we have

$$C_\epsilon\left(\frac{\log n}{n}\right)^{1/4} \leq \frac{1}{n^{1/8}} \frac{\|t^n_{0\,i}\|_4}{\|t^n_{n\,i}\|_4} \leq C\left(\frac{\log n}{n}\right)^{1/4}. \tag{3.8}$$

Proof. Using (3.3)-(3.4), we get

$$\text{(3.9)}\quad \|t^n_{0\,i}\|_4 = \left[\sum_{m=0}^{n} \frac{\left(C^{2m}_{0\,0}\right)^2}{(4m+1)} \left(C^{2m}_{i\,i}\right)^2\right]^{1/4} = \left[\sum_{m=|i|}^{n} \frac{\left(C^{2m}_{0\,0}\right)^2 \left(C^{2m}_{|i|\,|i|}\right)^2}{4m+1}\right]^{1/4}$$

as $C^{2m}_{i\,i} = C^{2m}_{|i|\,|i|}$ and $C^{2m}_{|i|\,|i|} = 0$ for $m < |i|$ and

$$\text{(3.10)}\qquad \|t^n_{n\,i}\|_4 = \frac{\sqrt{C^{2n}_{|i|\,|i|}}}{(4n+1)^{1/4}}.$$

From (3.9)-(3.10) we see that $\|t^n_{0\,i}\|_4 = \|t^n_{0\,-i}\|_4$ and $\|t^n_{n\,i}\|_4 = \|t^n_{n\,-i}\|_4$.

Therefore we assume that $0 \le i \le n$. We divide the rest of the proof in four steps:

Step 1. $i = 0$

$$\begin{aligned}
\frac{1}{n^{1/8}} \frac{\|t^n_{0\,0}\|_4}{\|t^n_{n\,0}\|_4} &= \left[\sum_{m=0}^{n} \frac{\left(C^{2m}_{0\,0}\right)^4}{4m+1} \frac{(4n+1)}{\sqrt{n}} \left(C^{2n}_{0\,0}\right)^2\right]^{1/4} \\
&\approx \left[\frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{(n+m)(n-m)(4m+1)} \frac{(4n+1)}{\sqrt{n}} \sqrt{n} + \frac{1}{n}\right]^{1/4} \\
&\approx \left[\frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{m(n-m)}\right]^{1/4} \\
&\approx \left[\frac{1}{n} + \sum_{m=1}^{n-1} \frac{1}{n}\left[\frac{1}{m} + \frac{1}{n-m}\right]\right]^{1/4} \\
&\approx \left[\frac{1}{n} + \frac{1}{n}\sum_{m=1}^{n-1} \frac{1}{n}\right]^{1/4} \\
&\approx \left(\frac{\log n}{n}\right)^{1/4}.
\end{aligned}$$

Step 2. $i = n$

$$\begin{aligned}
\frac{1}{n^{1/8}} \frac{\|t^n_{0\,n}\|_4}{\|t^n_{n\,n}\|_4} &= \left[\frac{4n+1}{\sqrt{n}} \frac{\left(C^{2n}_{0\,0}\right)^2}{(4n+1)}\right]^{1/4} \\
&\approx \frac{1}{n^{1/4}}.
\end{aligned}$$

Step 3. $1 \le i \le n-1$

$$\begin{aligned}
\frac{1}{n^{1/8}}\frac{\|t^n_{0\,i}\|_4}{\|t^n_{n\,i}\|_4} &= \left[\frac{4n+1}{\sqrt{n}}\sum_{m=i}^{n}\frac{\left(C^{2m}_{0\,0}\right)^2\left(C^{2m}_{i\,i}\right)^2}{(4m+1)\left(C^{2n}_{i\,i}\right)^2}\right]^{1/4}\\
&= \left[\frac{4n+1}{\sqrt{n}}\left\{\frac{\left(C^{2i}_{0\,0}\right)^2\left(C^{2i}_{i\,i}\right)^2}{(4i+1)\left(C^{2n}_{i\,i}\right)^2}+\sum_{m=i+1}^{n-1}\frac{\left(C^{2m}_{0\,0}\right)^2\left(C^{2m}_{i\,i}\right)^2}{(4m+1)\left(C^{2n}_{i\,i}\right)^2}+\frac{\left(C^{2n}_{0\,0}\right)^2}{(4n+1)}\right\}\right]^2\\
&= \left[\frac{4n+1}{\sqrt{n}}\left\{A_n+Z_n+B_n\right\}\right]^{1/4} \quad \text{(say)}\\
A_n &= \frac{\left(C^{2i}_{0\,0}\right)^2\left(C^{2i}_{i\,i}\right)^2}{(4i+1)\left(C^{2n}_{i\,i}\right)^2}\\
&\approx \frac{1}{(n+i)^{1/2}(n-i)^{1/2}}\frac{i^{1/2}}{(n+i)^{1/2}(n-i)^{1/2}}\frac{1}{(4i+1)}\frac{(n+i)^{1/2}(n-i)^{1/2}}{n^{1/2}}\\
&\approx \frac{1}{n\sqrt{n-i}\sqrt{i}}.
\end{aligned}$$

Hence $A_n \leq \frac{C}{n^{2/3}}$.

$$B_n = \frac{\left(C^{2n}_{0\,0}\right)^2}{(4n+1)} \approx \frac{1}{n^{3/2}}$$

$$\begin{aligned}
Z_n &= \sum_{m=i+1}^{n-1}\frac{\left(C^{2m}_{0\,0}\right)^2\left(C^{2m}_{i\,i}\right)^2}{(4m+1)\left(C^{2n}_{i\,i}\right)^2}\\
&\approx \sum_{m=i+1}^{n-1}\frac{m(n+i)^{1/2}(n-i)^{1/2}}{(n+m)^{1/2}(n-m)^{1/2}(n+m)^{1/2}(n-m)^{1/2}(m+i)^{1/2}(m-i)^{1/2}(4m+1)n^{1/2}}\\
&\approx \frac{1}{n}\sum_{m=i+1}^{n-1}\frac{(n-i)^{1/2}}{(n-m)(m+i)^{1/2}(m-i)^{1/2}}.
\end{aligned}$$

Now we further divide Step 3 into three cases:
a) $i = n-1$. Then

$$A_n \leq \frac{C}{n^{3/2}},\ B_n \approx \frac{1}{n^{3/2}},\ Z_n = 0.$$

Therefore $\frac{1}{n^{1/8}}\frac{\|t^n_{0\,i}\|_4}{\|t^n_{n\,i}\|_4} \approx \frac{1}{n^{1/4}}$.

b) $1 \leq i \leq \frac{n}{2}$. Then

$$\begin{aligned}
Z_n &\leq \frac{C}{\sqrt{n}} \sum_{m=i+1}^{n-1} \frac{1}{(n-m)(m-i)} \\
&= \frac{C}{\sqrt{n}} \sum_{m=i+1}^{n-1} \left[\frac{1}{n-m} + \frac{1}{m-i}\right] \frac{1}{(n-i)} \\
&= \frac{2C}{\sqrt{n}} \left[\sum_{j=1}^{n-i-1} \frac{1}{j}\right] \frac{1}{(n-i)} \\
&\leq \frac{4C}{n^{3/2}} \left[1 + \log(n-i-1)\right] \leq \frac{8C\,(\log n)}{n^{3/2}}.
\end{aligned}$$

Therefore $\frac{1}{n^{1/8}} \frac{\|t^n_{0,i}\|_4}{\|t^n_{n,i}\|_4} \leq C\left(\frac{\log n}{n}\right)^{1/4}$.

c) $\frac{n}{2} \leq i \leq n-2$. Then

$$\begin{aligned}
Z_n &\leq \frac{C}{n^{3/2}} \sum_{m=i+1}^{n-1} \frac{\sqrt{n-i}}{(n-m)\sqrt{m-i}} \\
&= \frac{C}{n^{3/2}\sqrt{n-i}} \sum_{m=i+1}^{n-1} \left[\frac{\sqrt{m-i}}{(n-m)} + \frac{1}{\sqrt{m-i}}\right] \\
&\leq \frac{C}{n^{3/2}} \left[\sum_{m=i+1}^{n-1} \frac{1}{(n-m)} + \frac{1}{\sqrt{n-i}} \sum_{m=i+1}^{n-1} \frac{1}{\sqrt{m-i}}\right] \\
&\leq \frac{C \log n}{n^{3/2}}.
\end{aligned}$$

Therefore $\frac{1}{n^{1/8}} \frac{\|t^n_{0,i}\|_4}{\|t^n_{n,i}\|_4} \leq C\left(\frac{\log n}{n}\right)^{1/4}$.

Step 4. Let $0 < \epsilon < 1$ and $0 \leq i \leq n\epsilon$. In this case,

$$Z_n \approx \frac{1}{\sqrt{n}} \sum_{l=i+1}^{n-1} \frac{1}{(n-l)(l+i)^{1/2}(l-i)^{1/2}}.$$

Therefore

$$
\begin{aligned}
Z_n \geq & \frac{1}{\sqrt{n}} \sum_{l=i+1}^{n-1} \frac{1}{(n-l)(l+i)} \\
= & \frac{1}{\sqrt{n}(n+i)} \sum_{l=i+1}^{n-1} \left[\frac{1}{(n-l)} + \frac{1}{(l+i)}\right] \\
\geq & \frac{1}{n^{3/2}} \sum_{j=1}^{n-i-1} \frac{1}{j} \geq \frac{\log(n-i)}{n^{3/2}} \\
\geq & C_\epsilon \frac{\log n}{n^{3/2}}.
\end{aligned}
$$

Hence $\frac{1}{n^{1/8}} \frac{\|t_{0\,i}^n\|_4}{\|t_{n\,i}^n\|_4} \geq C_\epsilon \left(\frac{\log n}{n}\right)^{1/4}$.

Now by Step 1 and Step 3 we have

$$
\frac{1}{n^{1/8}} \frac{\|t_{0\,i}^n\|_4}{\|t_{n\,i}^n\|_4} \leq C \left(\frac{\log n}{n}\right)^{1/4} \quad \text{for all } 0 \leq i \leq n-1.
$$

Therefore $C_\epsilon \left(\frac{\log n}{n}\right)^{1/4} \leq \frac{1}{n^{1/8}} \frac{\|t_{0\,i}^n\|_4}{\|t_{n\,i}^n\|_4} \leq C \left(\frac{\log n}{n}\right)^{1/4}$. □

Proof of Theorem 3.2.

(A) Let $\underline{z}^{(n)}$ be as in the hypothesis of Theorem 3.2(A). Consider

$$
\begin{aligned}
\left\| \sum_{i=-n}^{+n} z_i^{(n)} t_{n\,i}^n \right\|_4 = & \left[\frac{1}{(4n+1)} \sum_{r=-2n}^{+2n} \left| \sum_{\substack{i+k=r \\ -n \leq i,k \leq n}} z_i^{(n)} z_k^{(n)} C_{i\,k}^{2n} \right|^2 \right]^{1/4} \\
\geq & \left[\frac{1}{(4n+1)} \sum_{i=-n}^{+n} \left(C_{i\,i}^{2n}\right)^2 \left| z_i^{(n)} \right|^4 \right]^{1/4} \quad \text{as } C_{i\,k}^{2n} \geq 0, \ \forall i,k \\
= & \left[\sum_{i=-n}^{+n} \left(z_i^{(n)}\right)^4 \|t_{n\,i}^n\|_4^4 \right]^{1/4}.
\end{aligned}
$$

Now

$$\begin{aligned}
\frac{1}{n^{1/8}}\frac{\left\|\sum_{i=-n}^{+n} z_i^{(n)} t_{0\,i}^{n}\right\|_4}{\left\|\sum_{i=-n}^{+n} z_i^{(n)} t_{n\,i}^{n}\right\|_4} &\leq \frac{1}{n^{1/8}}\frac{\sum_{i=-n}^{+n} z_i^{(n)}\left\|t_{0\,i}^{n}\right\|_4}{\left[\sum_{i=-n}^{+n}\left(z_i^{(n)}\right)^4\left\|t_{n\,i}^{n}\right\|_4^4\right]^{1/4}} \\
&\leq \frac{1}{n^{1/8}}\frac{\left(\sum_{i=-n}^{+n}\left(z_i^{(n)}\right)^4\left\|t_{0\,i}^{n}\right\|_4^4\right)^{1/4}\left|F_n\left(\underline{z}^{(n)}\right)\right|^{3/4}}{\left(\sum_{i=-n}^{+n}\left(z_i^{(n)}\right)^4\left\|t_{n\,i}^{n}\right\|_4^4\right)^{1/4}} \\
&\leq C\left(\frac{\log n}{n}\right)^{1/4}\left(\frac{n^{1/3}}{(\log n)^{2/3}}\right)^{3/4} \leq \frac{C}{(\log n)^{1/4}},
\end{aligned}$$

(by Lemma 3.6).

This completes the proof of part (A).

(B) Consider

$$\begin{aligned}
\left\|\sum_{i=0}^{j_n} z_i^{(n)} t_{n\,p_n^i}^{n}\right\|_4 &= \left[\frac{1}{(4n+1)}\sum_{r=0}^{+2n}\left|\sum_{\substack{p_n^i+p_n^k=r\\ 0\leq i,k\leq j_n}} z_i^{(n)} z_k^{(n)} C_{p_n^i\,p_n^k}^{2n}\right|^2\right]^{1/4} \\
&\geq \left[\frac{1}{(4n+1)}\sum_{i=0}^{j_n}\left|z_i^{(n)}\right|^4\left(C_{p_n^i\,p_n^i}^{2n}\right)^2\right]^{1/4} \quad \text{as}
\end{aligned}$$

$$\left|\sum_{p_n^i+p_n^k=r} z_i^{(n)} z_k^{(n)} C_{p_n^i\,p_n^k}^{2n}\right|^2 = \left|z_l^{(n)}\right|^4\left(C_{p_n^l\,p_n^l}^{2n}\right)^2 \quad \text{if } r = 2p_n^l.$$

Therefore $\left\|\sum_{i=0}^{j_n} z_i^{(n)} t_{n\,p_n^i}^{n}\right\|_4 \geq \left[\sum_{i=0}^{j_n}\left|z_l^{(n)}\right|^4\left\|t_{n\,p_n^i}^{n}\right\|_4^4\right]^{1/4}$.

Hence

$$\begin{aligned}\frac{1}{n^{1/8}}\frac{\left\|\sum_{i=0}^{j_n} z_i^{(n)} t^n_{0\,p_n'}\right\|_4}{\left\|\sum_{i=o}^{j_n} z_i^{(n)} t^n_{n\,p_n'}\right\|_4} &\leq \frac{1}{n^{1/8}}\frac{\sum_{i=0}^{j_n}\left|z_i^{(n)}\right|\left\|t^n_{0\,p_n'}\right\|_4}{\left[\sum_{i=0}^{j_n}\left|z_i^{(n)}\right|^4\left\|t^n_{n\,p_n'}\right\|_4^4\right]^{1/4}}\\
&\leq \frac{1}{n^{1/8}}\frac{\left[\sum_{i=0}^{j_n}\left|z_i^{(n)}\right|^4\left\|t^n_{0\,p_n'}\right\|_4^4\right]^{1/4} j_n^{3/4}}{\left[\sum_{i=0}^{j_n}\left|z_i^{(n)}\right|^4\left\|t^n_{n\,p_n'}\right\|_4^4\right]^{1/4}}\\
&\qquad \text{(by Hölder's inequality)},\\
&\leq C\left(\frac{\log n}{n}\right)^{1/4}(\log n)^{3/4}\\
&= C\frac{\log n}{n^{1/4}}.\end{aligned}$$

This completes the proof of the Theorem.

□

Remark 3.11. The following inequality can be proved by using the ideas of the proof of Theorem 3.2(B):

$$\frac{1}{n^{1/8}}\frac{\left\|z_1 t^n_{0\,p} + z_2 t^n_{0\,q}\right\|_4}{\left\|z_1 t^n_{n\,p} + z_2 t^n_{n\,q}\right\|_4} \leq C\left(\frac{\log n}{n}\right)^{1/4}$$
$$\text{for} -n \leq p,\ q \leq n.$$

4.

Let G be a compact group and let Γ be the dual object of G, the set of equivalence classes of irreducible unitary representations of G. For each $\sigma \in \Gamma$, select a representation $U_\sigma \in \sigma$, let H_σ be the Hilbert space on which U_σ acts, and let d_σ be the dimension of H_σ. Let $B(H_\sigma)$ denote the space of linear operators on H_σ and $\mathcal{C}(\Gamma)$ denote the space $\prod_{\sigma\in\Gamma} B(H_\sigma)$.

Definition. Fix $p \in [1, \infty]$. Let$m$ be an element of $\mathcal{C}(\Gamma)$, so that for each σ, $m(\sigma) \in B(H_\sigma)$. The function m is a (left) multiplier of $L^p(= L^p(G))$ if for each $f \in L^p$, the series

$$\sum_{\sigma\in\Gamma} d_\sigma \operatorname{tr}\left[m(\sigma)\hat{f}(\sigma)U_\sigma(x)\right]$$

is the Fourier series of some function $L_m f \in L^p$. The collection of all such m is denoted by $M_p(G)$ or simply M_p.

For each $m \in M_p$, the map $f \to L_m f$ defines a bounded linear operator on L^p, an operator which commutes with left translations by the elements of G. we regard M_p as a Banach space under the operator norm.

When G is abelian, an easy argument shows that if $\frac{1}{p} + \frac{1}{q} = 1$, then $M_p = M_q$. It is known that for $1 < p < 2$, $M_p \neq M_q$ ($\frac{1}{p} + \frac{1}{q} = 1$) for many nonabelian groups G (see [**1**, **2**, **3**, **4**, **6**]).

For connected compact non-abelian group G and for $1 < p < 2$, it is an open problem whether $M_p = M_q$, $\frac{1}{p} + \frac{1}{q} = 1$.

S.G. Roberts has shown in [**8**] that if the conjecture is true then $M_p(G) \neq M_q(G)$ for every connected compact non-abelian group and $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$.

We give an easy proof that if the conjecture is true then $M_p(SU(2)) \neq M_q(SU(2))$ for $1 < p < 2$, $\frac{1}{p} + \frac{1}{q} = 1$. This proof is essentially due to Roberts [**8**], but has never to the best of our knoledge been published. We present it here for completeness. The result will follow if we show that

$$\frac{|||t_{0n}^n|||_p}{|||t_{0n}^n|||_q} \to \infty \text{ as } n \to \infty \tag{4.1}$$

where $|||t_{0n}^n|||_p$ denotes the norm of t_{0n}^n as an element in M_p.

To prove (4.1), we use the following norm estimates for t_{0n}^n and t_{nn}^n which are easy to establish (see [**8**]).

$$\|t_{0n}^n\|_p \approx \frac{1}{n^{1/4+1/2p}}, \ \|t_{nn}^n\|_p = \frac{1}{(np+1)^{1/p}}$$
$$|||t_{0n}^n|||_1 = \|t_{0n}^n\|_1 \approx \frac{1}{n^{3/4}}, \ |||t_{0n}^n|||_2 = \frac{1}{2n+1}.$$

Now by Riesz convexity theorem, we get

$$|||t_{0n}^n|||_p \leq |||t_{0n}^n|||_1^{\alpha} \, |||t_{0n}^n|||_2^{1-\alpha}$$

where

$$\alpha = \frac{2-p}{p}.$$

Hence

$$|||t_{0n}^n|||_p \leq \frac{C}{n^{(5/4)-(1/2p)}}.$$

Also

$$\left\|t^n_{0\,n}\right\|_p \geq \frac{\left\|t^n_{0\,n} * t^n_{n\,n}\right\|_p}{\left\|t^n_{0\,n}\right\|_p} = \frac{1}{(2n+1)} \frac{\left\|t^n_{0\,n}\right\|_p}{\left\|t^n_{n\,n}\right\|_p} \geq \frac{C}{n^{(5/4)-(1/2p)}}.$$

Therefore (4.1) is true if

$$n^{(5/4)-(1/2p)} \left\|t^n_{0\,n}\right\|_q \to 0 \text{ as } n \to \infty. \tag{4.2}$$

A routine argument using Riesz convexity theorem shows that (4.2) is true if

$$n^{7/8} \left\|t^n_{0\,n}\right\|_4 \to 0 \text{ as } n \to \infty. \tag{4.3}$$

Now

$$\begin{aligned}\left\|t^n_{0\,n}\right\|_4 &= \sup_{\substack{f\in L^4\\ f\neq 0}} \frac{\left\|t^n_{0\,n} * f\right\|_4}{\|f\|_4}\\ &= \sup_{\substack{f\in L^4\\ f\neq 0}} \frac{\left\|t^n_{0\,n} * t^n_{n\,n} * f\right\|_4}{\|f\|_4}(2n+1)\end{aligned}$$

and

$$\begin{aligned}(2n+1)\left\|t^n_{n\,n} * f\right\|_4 &\leq (2n+1)\left\|t^n_{n\,n}\right\|_1 \|f\|_4\\ &= \frac{(2n+1)}{(n+1)}\|f\|_4 \leq 2\|f\|_4.\end{aligned}$$

So

$$\begin{aligned}\left\|t^n_{0\,n}\right\|_4 &\leq 2 \sup_{\substack{f\in L^4\\ f\neq 0}} \frac{\left\|t^n_{0\,n} * t^n_{n\,n} * f\right\|_4}{\left\|t^n_{n\,n} * f\right\|_4}\\ &= \frac{2}{(2n+1)} \sup_{\sum_{i=-n}^{+n}|z_i^{(n)}|\neq 0} \frac{\left\|\sum_{i=-n}^{+n} z_i^{(n)} t^n_{0\,i}\right\|_4}{\left\|\sum_{i=-n}^{+n} z_i^{(n)} t^n_{n\,i}\right\|_4}.\end{aligned}$$

Therefore (4.3) is true if the conjecture is true. Hence (4.1) is true if the conjecture is true.

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