EXISTENCE AND BEHAVIOR OF THE RADIAL LIMITS OF A BOUNDED CAPILLARY SURFACE AT A CORNER

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Consider a bounded capillary surface defined on a two-dimensional region Ω that has a corner point at O, with opening angle 2α. If the contact angle is bounded away from 0 and π, then the radial limits exist as O is approached from any direction in Ω. If the contact angle approaches limiting values as O is approached along each portion of the boundary, then there exist "fans" of directions adjacent to the two tangent directions at O in which the radial limits are constant. Other properties of the radial limit function are given and these results are used to show continuity of the solution up to O under certain conditions. For a convex corner, the solution is continuous up to O when the limiting angles γ_0^+, γ_0^- satisfy $|\pi - \gamma_0^+ - \gamma_0^-| < 2\alpha$ and $2\alpha + |\gamma_0^+ - \gamma_0^-| \le \pi$.

1. Introduction and Statement of Main Theorems.

Consider the capillary problem

$$(1) \qquad\qquad Nf = \kappa f + \lambda \quad \text{in } \Omega$$

$$(2) \qquad\qquad Tf \cdot \nu = \cos\gamma \quad \text{on } \partial\Omega$$

where Ω is a region in $I\!\!R^2$ with a corner at O, $O \in \partial\Omega$, $Nf = \nabla \cdot Tf$, $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$, κ and λ are constants, ν is the exterior normal on $\partial\Omega$, and $\gamma = \gamma(s)$ is a function of position on $\partial\Omega$, $0 \le \gamma(s) \le \pi$. The surface $z = f(x, y)$ describes the shape of the static liquid-gas interface in a vertical cylindrical tube of cross-section Ω; see Finn [8] for background. The cases $\kappa > 0$, $\kappa = 0$, $\kappa < 0$ correspond to positive, zero, and negative gravity, respectively. If the liquid, gas, and tube are each homogeneous then γ is constant. A general existence theory that covers the case $\kappa \ge 0$ is presented in [8]. This theory provides existence of a bounded variational solution (which may be shown to have further regularity). Theorem 4 in Siegel [27] provides existence of a classical solution in the case $\kappa > 0$ under less restrictive hypotheses (although the solution need not be bounded). An existence theorem for $\kappa < 0$ and κ small was given by Huisken [16].

We are interested in the behavior of solutions to (1), (2) in the neighborhood of a corner point of the boundary. We take the corner point to be $O = (0,0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(O)$, where $B_{\delta^*}(O)$ is the ball of radius δ^* about O. Polar coordinates relative to O will be denoted by r and θ. We assume that $\partial\Omega$ is piecewise smooth and that $\partial\Omega \cap B_{\delta^*}(O)$ consists of two arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point O is approached (see Figure 1). The points where $\partial B_{\delta^*}(O)$ intersect $\partial\Omega$ are labeled A and B; also, $\Gamma^* = \partial B_{\delta^*}(O) \cap \overline{\Omega}$. Let $\gamma^+(s)$ and $\gamma^-(s)$ denote γ along the arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, respectively, where $s = 0$ corresponds to the point O. For $0 \leq \alpha < \pi/2$, the corner will be said to be *convex* and for $\pi/2 \leq \alpha \leq \pi$, the corner will be said to be *nonconvex*. When $\alpha = 0$ or π, the region has a cusp. The case of a smooth boundary where the contact angle is discontinuous at the point O is included under $\alpha = \pi/2$.

The positive gravity case ($\kappa > 0$) with constant contact angle γ and straight boundary segments $\partial^\pm\Omega^*$ has received much attention. For a convex corner ($0 < \alpha < \pi/2$), Concus and Finn [2] showed that (a) if $|\pi/2 - \gamma| \leq \alpha$ then f is bounded in Ω^* and (b) if $|\pi/2 - \gamma| > \alpha$ then f is unbounded in Ω^*. In case (b) they also gave the first term in an asymptotic expansion of f. The bound on the error was later improved by Miersemann [22], [23]. Also for a convex corner, Simon [28] proved that f is C^1 up to O if $|\pi/2 - \gamma| < \alpha$ and Tam [31] proved that f and the normal vector are continuous up to O in the borderline case $|\pi/2 - \gamma| = \alpha$. On the other hand, for a nonconvex corner, Korevaar [17] gave examples of capillary surfaces that are discontinuous at O, for any $\alpha, \pi/2 < \alpha < \pi$, and any $\gamma, \gamma \neq \pi/2$. The results on boundedness at the beginning of the paragraph do not require that $\partial^\pm\Omega^*$ be line segments.

In the zero gravity case ($\kappa = 0$) with constant contact angle γ and straight boundary segments $\partial^\pm\Omega$ forming a convex corner ($0 < \alpha < \pi/2$), Concus and Finn [1] show that a solution can exist in a neighborhood of O only if $|\pi/2 - \gamma| \leq \alpha$. Conversely, for such an α and γ and $\lambda \neq 0$, there exists a solution in a domain Ω with $\partial^\pm\Omega^* \subset L^\pm$. For $\lambda = 0$, $|\pi/2 - \gamma| < \alpha$ suffices for existence. Regularity up to O holds as in the previous paragraph whenever a solution exists. The same necessary condition for a solution to exist also holds when $\partial^\pm\Omega^*$ are not line segments. In recent papers [3] and [11], Concus and Finn extend their analysis to the case when $\gamma^\pm$ are different constants. They show that a solution in a neighborhood of O can exist only if $|\pi - \gamma^+ - \gamma^-| \leq 2\alpha$. Conversely, for (γ^-, γ^+) satisfying this condition, there is some domain Ω with corner angle 2α and a λ for which there is a solution. If $|\pi - \gamma^+ - \gamma^-| < 2\alpha$ and $2\alpha + |\gamma^+ - \gamma^-| < \pi$ then λ can be prescribed.

In the present paper, we will prove the existence of radial limits of a bounded solution f to (1) that satisfies (2) on the smooth portions of $\partial\Omega$

provided that γ is bounded away from 0 and π, and for a convex corner an additional condition is satisfied coupling γ^+ and γ^-. The radial limits of f will be denoted by $Rf(\theta) = \lim_{r\to 0+} f(r\cos\theta, r\sin\theta), -\alpha < \theta < \alpha$ and $Rf(\pm\alpha) = \lim_{\mathbf{x}\to O, \mathbf{x}\in\partial^\pm\Omega^*} f(\mathbf{x}), \mathbf{x} = (x, y)$, which are the limits of the boundary values of f on the two sides of the corner.

Theorem 1. *Let f be a bounded solution to (1) satisfying (2) on $\partial^\pm\Omega^* \setminus O$, discontinuous at O, with $0 < \gamma_0 \le \gamma^\pm(s) \le \gamma_1 < \pi$. If $\alpha \ge \pi/2$ then $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. If $\alpha < \pi/2$ and there exist constants $\underline{\gamma}^\pm, \overline{\gamma}^\pm, 0 < \underline{\gamma}^\pm \le \pi/2, \pi/2 \le \overline{\gamma}^\pm < \pi$, satisfying*

$$\underline{\gamma}^+ + \underline{\gamma}^- > \pi - 2\alpha \quad and \quad \overline{\gamma}^+ + \overline{\gamma}^- < 2\alpha + \pi$$

so that $\underline{\gamma}^\pm \le \gamma^\pm(s) \le \overline{\gamma}^\pm$ for all $s, 0 < s < s_0$, for some s_0, then again $Rf(\theta)$ exists for all $\theta \in [-\alpha, \alpha]$. Furthermore, in either case, $Rf(\theta)$ is a continuous function on $[-\alpha, \alpha]$ which behaves in one of the following ways:

  (i) *There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ and strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$. Label these case(I) and case(D), respectively.*

  (ii) *There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha, \alpha_R = \alpha_L + \pi$, and Rf is constant on $[-\alpha, \alpha_1], [\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha]$ and either increasing on $[\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2]$ or decreasing on $[\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2]$. Label these case (ID) and case (DI), respectively.*

When $\kappa \ge 0$, f will automatically be bounded.

Proposition 1. *Let f be a solution to (1) satisfying (2) on $\partial^\pm\Omega^* \setminus O$, with γ satisfying the restrictions in Theorem 1 and $\kappa \ge 0$. Then f is bounded in a neighborhood of O.*

When the contact angle has limiting values on $\partial^\pm\Omega^*$ as O is approached then we can say more.

Theorem 2. *In addition to the hypotheses of Theorem 1 assume that $\lim_{s\to 0+} \gamma^\pm(s) = \gamma_0^\pm$ both exist. Then there exist fans of constant radial limits adjacent to each tangent direction at O: $\alpha_1 + \alpha \ge \gamma_0^-$ or $\pi - \gamma_0^-$ and $\alpha - \alpha_2 \ge \gamma_0^+$ or $\pi - \gamma_0^+$. Which lower bound on the size of the fan holds is described in terms of the cases labeled in Theorem 1:*

   $\alpha_1 + \alpha \ge \gamma_0^-$ *for (D) and (DI)*
   $\alpha_1 + \alpha \ge \pi - \gamma_0^-$ *for (I) and (ID)*
   $\alpha - \alpha_2 \ge \gamma_0^+$ *for (I) and (DI)*

$\alpha - \alpha_2 \geq \pi - \gamma_0^+$ *for* (D) *and* (ID).

Theorem 1 is obtained by use of parametric methods first introduced by Lancaster [18] and Elcrat and Lancaster [5] for the prescribed mean curvature equation and Dirichlet boundary condition. The contact angle boundary condition requires use of a different comparison principle, one due to Concus and Finn [2]. The presentation here we hope will clarify the method by bringing out the importance of a related problem in a region with a cusp and by more fully exploiting a boundary behavior result of Heinz [15].

Theorem 2 is obtained by a blow-up procedure employing the concept of generalized solution introduced by M. Miranda [24]. L. F. Tam [30], [31] has effectively exploited such arguments.

The proofs of Theorems 1 and Proposition 1 will be presented in Section 2. The proof of Theorem 2 will be presented in Section 3.

In Section 4 some consequences of Theorems 1 and 2 will be presented that give conditions under which f must be continuous at O. Corollary 4 states that for a convex corner where the limiting contact angles γ_0^+ and γ_0^- satisfy $|\pi - \gamma_0^+ - \gamma_0^-| < 2\alpha$ and $2\alpha + |\gamma_0^+ - \gamma_0^-| \leq \pi$, f must be continuous up to O. The rectangle of possible pairs (γ_0^-, γ_0^+) is the same rectangle that occurs in the work of Concus and Finn [3], [11] mentioned above. Corollary 2 seems to be the first result on continuity up to a nonconvex corner point.

Section 4 also contains some illustrative examples. In particular, Theorem 3 proves the existence of a bounded capillary surface without any radial limits with $\alpha = \pi/2$ (and γ not bounded away from 0 and π).

Extensions to other problems are discussed in Section 5.

2. Proofs of Theorem 1 and Proposition 1.

The proof of Theorem 1 is similar in outline to those employed in [5] for the prescribed mean curvature equation, [18], [19] for the minimal surface equation, and [21] for other elliptic equations, all subject to a Dirichlet condition.

In order to deal with the contact angle boundary condition, we will make use of the *Comparison Principle* of Concus and Finn (Theorem 5.1 of [8]). We state a somewhat specialized version which suffices for our purposes. Let Ω be a bounded domain in $I\!\!R^2$ with piecewise smooth boundary $\Sigma = \partial\Omega$.

Comparison Principle. Suppose $\nabla \cdot Tu - \kappa u \geq \nabla \cdot Tv - \kappa v$ in Ω. Let $\Sigma = \Sigma_\alpha \cup \Sigma_\beta \cup \Sigma_0$, where Σ_α and Σ_β are unions of smooth arcs in Σ and Σ_0 is the set of points where the smooth arcs join. Suppose further that

$$v \geq u \text{ on } \Sigma_\alpha, \quad Tv \cdot \nu \geq Tu \cdot \nu \text{ on } \Sigma_\beta, \quad \text{and} \quad \kappa > 0 \text{ or } \Sigma_\alpha \neq \emptyset.$$

Then $v \geq u$ in Ω and equality holds at one point if and only if $v \equiv u$.

The Comparison Principle will be employed in one of two circumstances: either $Nu \geq \kappa u + \lambda, Nv \leq \kappa v + \lambda$ and $\kappa \geq 0$ or $Nu \geq Nv$ in Ω.

Although problem (1), (2) is non-parametric, it is useful to introduce a parametric description. Osserman [25] has other uses of parametric methods for non-parametric problems.

The proof of Theorem 1 will now be given in a series of five steps. The reader can get an overview of the proof by reading the statements to be proved listed at the start of each step. The relatively brief proof of Proposition 1 follows on completion of the proof of the theorem.

Let f satisfty (1), (2) and set

$$S_0^* = \{(x, y, f(x, y)) : (x, y) \in \Omega^*\},$$
$$\Gamma_0^* = \{(x, y, f(x, y)) : (x, y) \in \partial\Omega^* \setminus O\}.$$

Let T denote the z-axis. The parameter domain will first be the unit disk $E = \{(u, v) : u^2 + v^2 < 1\}$ and will then be the unit half-disk $B = \{(u, v) : u^2 + v^2 < 1, v > 0\}$, whose boundary is made up to two parts:

$$\partial''B = \{(u, 0) : -1 < u < 1\} \quad \text{and} \quad \partial'B = \{(u, v) : u^2 + v^2 = 1, v \geq 0\}.$$

Step 1. There is a parametric description of the surface S_0^*

$$X(u, v) = (x(u, v), y(u, v), z(u, v)) \in C^2(B : I\!R^3),$$

which has the following six properties.
  (i) X is a homeomorphism of B onto S_0^*.
 (ii) X maps $\partial'B$ strictly monotonically onto Γ_0^*.
(iii) X is conformal on B: $X_u \cdot X_v = 0, X_u^2 = X_v^2$ on B.
 (iv) $\triangle X := X_{uu} + X_{vv} = (\kappa z + \lambda)X_u \times X_v$.
  (v) $X \in C^0\left(\overline{B}\right)$ and $x = y = 0$ on $\partial''B$.
 (vi)   Writing $K(u, v) = (x(u, v), y(u, v))$, $K(\cos t, \sin t)$ moves clockwise about $\partial\Omega^*$ as t increases, $0 \leq t \leq \pi$, and K is orientation reversing on B.

Proof of Step 1. Following the proof in [5], from the existence of local conformal coordinates and the uniformization theorem, it follows that there is a global parametric description of $S_0^* : Y(u, v) \in C^2(E : I\!R^3), Y$ satisfied (iii) and (iv). Next, we show that the area of S_0^* is finite by an integration by parts argument (used in a different context by Finn, Lemma 5 in [6]).

Since f is assumed to be bounded then Nf is also bounded. Multiply by f and use the Divergence Theorem:

$$\int_{\Omega_0^*} fNf \, dx = -\int_{\Omega_0^*} \frac{|\nabla f|^2}{\sqrt{1 + |\nabla f|^2}} \, dx + \oint_{\partial\Omega_0^*} f \frac{\nabla f \cdot \nu}{\sqrt{1 + |\nabla f|^2}} \, ds.$$

Since the boundary integral is bounded, so is the first integral on the right hand side. Hence

$$\int_{\Omega_0^*} \sqrt{1 + |\nabla f|^2} \, dx = \int_{\Omega_0^*} \frac{1}{\sqrt{1 + |\nabla f|^2}} \, dx + \int_{\Omega_0^*} \frac{|\nabla f|^2}{\sqrt{1 + |\nabla f|^2}} \, dx$$

is bounded.

Set $Y(u,v) = (a(u,v), b(u,v), c(u,v)), G(u,v) = (a(u,v), b(u,v))$. Again following [5], since the Dirichlet integral $\int \int_E G_u^2 + G_v^2 \, du \, dv$ is twice the area of S_0^*, and hence is finite, it can be argued that $G \in C^0 \left(\overline{E} : I\!R^2\right)$. Define σ to be subset of ∂E which G maps onto $\partial\Omega^* \setminus O$. It can then be argued that σ is a connected arc of ∂E and Y maps σ strictly monotonically onto Γ_0^*.

Now, we prove $Y \in C^0 \left(\overline{E} : I\!R^3\right)$, which comes down to showing that $c(u,v) \in C^0 \left(\overline{E}\right)$. To do this we use the Courant–Lebesgue Lemma (Lemma 3.1 in [4]) and appropriate comparison functions to control the oscillation of $c(u,v)$ on $E \cap D_\delta$ where D_δ is any disk of radius δ. Since f is bounded then $|\kappa f + \lambda| \leq 2H_0, H_0$ a positive constant. Let $\mathcal{M}$ denote the set of continuous strictly increasing functions from the positive reals to the positive reals which are zero at zero. Moduli of continuity will be chosen in this class.

Let $\epsilon > 0; \delta > 0$ will be chosen below. For $\mathbf{u}_0 = (u_0, v_0)$, set $B_r = \left\{\mathbf{u} \in \overline{E} : |\mathbf{u} - \mathbf{u}_0| < r\right\}, C_r = \left\{\mathbf{u} \in \overline{E} : |\mathbf{u} - \mathbf{u}_0| = r\right\}, E_r = \partial B_r \setminus C_r$ and let l_r be the length of the image curve $Y(C_r)$. Also let $C_r' = G(C_r), B_r' = G(B_r)$ and $E_r' = G(E_r)$. The Courant–Lebesgue Lemma asserts that there is a function $p(\delta) \in \mathcal{M}$, and a number $\rho, \delta \leq \rho \leq \sqrt{\delta}$ so that $l_\rho \leq p(\delta)$. (The function is $p(\delta) = k/\sqrt{\ln(1/\delta)}$, for some constant k, taking $\delta < 1$.)

The cases $\alpha > \pi/2, \alpha = \pi/2$ and $\alpha < \pi/2$ require separate treatment.

Consider the case $\alpha > \pi/2$. Let $\hat{\gamma}_0 = \min\{\gamma_0, \pi - \gamma_1\}$ and choose $\gamma_0', 0 < \gamma_0' < \hat{\gamma}_0$. Let $L^\pm$ denote the tangent lines to $\partial^\pm\Omega^*$, respectively and let $d^\pm$ be the signed distance from $\mathbf{x} = (x,y)$ to $L^\pm$ where $d^\pm > 0$ for points on the positive x-axis. Let $h^\pm = h(d^\pm)$ be a the portion of a cylinder with mean curvature $-H_0$ with domain the strip $S^\pm = \{\mathbf{x} : -d_1 \leq d^\pm \leq d_2\}, d_1 = \frac{\cos\gamma_0'}{H_0}, d_2 = \frac{1-\cos\gamma_0'}{H_0}$ making a contact angle γ_0' on $L^\pm$ with respect to the direction of negative $d^\pm$ (see Figure 2). Since $\partial^\pm\Omega^*$ are C^1 we can choose δ^* so that $Th^\pm \cdot \nu \geq \cos\gamma_0$ on $\partial^\pm\Omega^*$, respectively. Let $h = h(\hat{r})$ denote an unduloid surface (see for example [9]) defined on the annulus

$\mathcal{A} = \{\mathbf{x} : r_1 \leq \hat{r} \leq r_2\}$ with constant mean curvature $-H_0$ that becomes vertical at $\hat{r} = r_1, r_2$, where $\hat{r}$ is the distance from $\mathbf{x}$ to $(-r_1, 0)$ and $r_1 = \frac{1-\sqrt{1+4c_0H_0}}{2H_0}, r_2 = \frac{1+\sqrt{1+4c_0H_0}}{2H_0}, c_0$ is arbitrary $-\frac{1}{4H_0} < c_0 < 0$, (see Figure 3). (The inclination angle ψ is given by $\sin\psi = -H_0\hat{r} + \frac{c_0}{r}$.) Let $\overline{h}$ denote an upper hemisphere of radius $\frac{1}{H_0}$ defined over a disk of the same radius. In view of the Comparison Principle the functions $h^{\pm} + k, h + k$ and $\overline{h} + k$, where k is a constant can be used to give upper bounds on f. Similarly, the functions $-h^{\pm} + k, -h + k$ and $-\overline{h} + k$ can be used to give lower bounds on f. Let moduli of continuity of the functions $h^{\pm}, h, \overline{h}$ be $q^+, q, \overline{q}$, respectively. This means that $|h(\mathbf{x}_1) - h(\mathbf{x}_2)| \leq q(|\mathbf{x}_1 - \mathbf{x}_2|)$, with the analogous statements for the other functions. The moduli of continuity are all taken to be in $\mathcal{M}$.

Let σ_N be the portion of σ corresponding to $\partial^+\Omega^* \cup \partial^-\Omega^*$ and let σ_D be the portion of σ corresponding $\Gamma^* = \partial B_{\delta^*} \cap \Omega$. Also let $\sigma_0 = \partial E \setminus \sigma$. There are five cases that can occur, depicted in Figures 4a, 4b, depending on the position of C_ρ relative to $\sigma_N, \sigma_D, \sigma_0$. In Figure 4a, σ_0 is the closed arc from o_1 to o_2, in the counterclockwise direction; in Figure 4b, σ_0 is the point o. In both figures, a and b are the points corresponding to A and B shown in Figure 1. In the case when σ_0 is an arc, take δ sufficiently small so that $p(\delta)$ is smaller than the distance between the two components of σ_N, so that the end points of C_ρ cannot be on different components of σ_N.

Case 1: $C_\rho \subset E$ or $C_\rho \cap \overline{E}$ consists of one point. Consider the image of D_ρ, D'_ρ. Since $l_\rho < p(\delta)$ then D'_ρ is contained in a disk of radius $p(\delta)$ and center $\mathbf{c}$. Let $h = h(r_c), r_c = |\mathbf{x} - \mathbf{c}|$ be an upper hemishpere, as described above. Take δ sufficiently small so that $p(\delta) < \frac{1}{H_0}$. The oscillation of f over D'_ρ can now be bounded in terms of the oscillation of f over C'_ρ by use of the functions $\pm\overline{h} + k$: from the Comparison Principle one has

$$\inf_{C'_\rho} f - \overline{h} < f < \sup_{C'_\rho} f + \overline{h} \quad \text{on} \quad D'_\rho,$$

where $\overline{h}$ is chosen to have a minimum 0 on $r_c = p(\delta)$. It follows that

$$(3) \qquad \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B'_\rho} f - \inf_{B'_\rho} f \leq 2\overline{q}(p(\delta)) + p(\delta).$$

Case 2: Both endpoints of C_ρ are on σ_D. Let q_D be a modulus of continuity for f relative to Γ_{δ^*}. Arguing as in Case 1, one has

$$\inf_{C'_\rho} f - \overline{h} - \inf_{E'_\rho} f < f < \sup_{C'_\rho} f + \overline{h} + \sup_{E'_\rho} f \quad \text{on} \quad D'_\rho,$$

from which follows

$$(4) \qquad \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B_\rho} f - \inf_{B_\rho} f \leq 2\overline{q}(p(\delta)) + p(\delta) + q_D(p(\delta)).$$

Case 3: One endpoint of C_ρ is on σ_N and the other is on either the same component of σ_N or is on σ_0. In this case for δ sufficiently small so that $p(\delta) < \min\{d_1, d_2\}$ then $G(B_\rho)$ is contained either in S^+ or S^-. The oscillation of f on $G(B_\rho)$ can now be bounded by use of the functions $h^\pm$. By the Comparison Principle one has

$$\inf_{C'_\rho} f - h^\pm < f < \sup_{C'_\rho} f + h^\pm \quad \text{on} \quad D'_\rho,$$

where $h^\pm$ has a minimum 0 for $r^\pm = p(\delta)$. It follows that

$$(5) \qquad \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B'_\rho} f - \inf_{B'_\rho} f \leq 2q^+(p(\delta)) + p(\delta).$$

Case 4: One endpoint of C_ρ is on σ_N and one is on σ_D. As in Case 3, take δ small enough that $p(\delta) < \min\{d_1, d_2\}$. Arguing as in Case 3, we obtain

$$\inf_{C_\rho} f - h^\pm - \inf_{E'_\rho \cap \Gamma^*} f < f < \sup_{C'_\rho} f + h^\pm + \sup_{E'_\rho \cap \Gamma^*} f \quad \text{on} \quad D'_\rho,$$

from which follows

$$(6) \qquad \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B'_\rho} f - \inf_{B'_\rho} f \leq 2q^+(p(\delta)) + p(\delta) + q_D(p(\delta)).$$

Case 5: σ_0 is a single point p and one end point of C_ρ is on each component of σ_N. Take δ so that $p(\delta) < \min\{d_1, d_2, r_2 - r_1\}$. The region B'_ρ consists of three parts: $I_A = B'_\rho \cap A, I^\pm = B'_\rho \cap \mathcal{H}^\pm$ where $\mathcal{H}^\pm = \{\mathbf{x} : \hat{r} < r_1, 0 < \pm y\}$; see Figure 5. First bound f on I_A by use of h which is chosen to have minimum value 0 for $\hat{r} = r_1 + p(\delta)$. By the Comparison Principle, using that $Th \cdot \nu = 1$ for $\hat{r} = r_1$, we have $f < \sup_{C'_\rho \cap A} f + h < \sup_{C'_\rho \cap A} f + h(O)$ on I_A. Now since $I^\pm \subset S^\pm$, f on $I^\pm$ can be bounded by use of $h^\pm$, which are chosen to to have minimum value 0 on $d^\pm = p(\delta)$. By the Comparison Principle and the previous bound, we have

$$f < \sup_{C'_\rho \cap S^\pm} f + h^\pm + h(O) < \sup_{C'_\rho} f + h(O) + h^+(O) \quad \text{on} \quad I^\pm,$$

and a similar bound from below,

$$f > \inf_{C'_\rho} f - h(O) - h^+(O) \quad \text{on} \quad I^\pm.$$

It follows that

$$(7) \qquad \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B'_\rho} f - \inf_{B'_\rho} f \leq 2[q(p(\delta)) + q^+(p(\delta))] + p(\delta).$$

Let $\hat{q}(\delta)$ be the maximum of the right hand sides of (3)–(7), and choose $\delta = \hat{q}^{-1}(\epsilon)$. Then $\sup_{B_\delta} c - \inf_{B_\delta} c < \epsilon$, giving the continuity of c.

The case $\alpha = \pi/2$ is simpler than the case $\alpha > \pi/2$. There is no need for an unduloid surface and only one boundary strip $S = S^\pm$ is necessary.

In the case $\alpha < \pi/2$, Cases 1–4 are exactly the same as above. In Case 5, the oscillation of f is bounded by hemispheres $\pm\overline{h}$ positioned in a manner to make an appropriate contact angle condition with $\partial^\pm\Omega^*$. This is done in the following way. Let $\overline{h}$ be defined on $\hat{D}(\mathbf{c})$, the disk of radius H_0^{-1} and center $\mathbf{c}$. Let $\overline{C} = \partial\hat{D}(\mathbf{c})$. Choose constants $\gamma_0^\pm, 0 < \underline{\gamma}_0^\pm < \gamma^\pm$, so that $\underline{\gamma}_0^+ + \underline{\gamma}_0^- > \pi - 2\alpha$. Choose $\mathbf{c} = (c_1, c_2)$ so that $\overline{C}$ makes an angle $\underline{\gamma}_0^-$ with L^-; see Figure 6. Moving $\mathbf{c}$ parallel to L^-, while decreasing c_1, we see that the angle $\overline{C}$ makes with L^+, β decreases and approaches $\pi - 2\alpha - \underline{\gamma}_0^-$. Thus there is a $\mathbf{c}$ for which $\beta < \underline{\gamma}_0^+$. Fix this position of $\mathbf{c}$. Let $\mathcal{C} = \overline{D}(\mathbf{c}) \cap \Omega^*$. For δ sufficiently small $B_\rho' \subset \mathcal{C}$. Further, as $\delta \to 0, C \to O$, so that $\mathrm{dist}(C_\rho', O) < g(\delta)$ with $g \in \mathcal{M}$. (In fact, g can be taken to be linear in δ.) Fix $\overline{h}$ by requiring $\overline{h} = 0$ for $r_c = g(p(\delta)) + |c|$. Now

$$T\overline{h} \cdot \nu > \cos\underline{\gamma}^\pm \quad \text{on} \quad \partial^\pm\Omega^*$$

sufficiently close to O. Thus, by the Comparison Principle

$$f < \sup_{C_\rho'} f + \overline{h} \quad \text{on} \quad B_\rho',$$

which implies $\sup_{B_\rho'} f \leq \sup_{C_\rho'} f + \overline{h}(O)$. Similarly, $\inf_{B_\rho'} f \geq \inf_{C_\rho'} f - \overline{h}(O)$. Therefore

$$(8) \qquad \sup_{B_\delta} c - \inf_{B_\delta} c \leq \sup_{B_\rho'} f - \inf_{B_\rho'} f \leq 2\overline{q}(p(\delta)) + p(\delta).$$

Thus c is continuous in all cases.

Finally (following [5]), when σ_0 is a single point, G is a homeomorphism from $\overline{E}$ to $\overline{\Omega}^*$, and G^{-1} is also a homeomorphism. Since $f = c \circ G^{-1}$, then f must be continuous at O. By hypothesis, this is not the case so σ_0 must be an arc. By a conformal transformation we can arrange to have B as the parameter domain and $\sigma_0 = \partial''B$, and satisfy the orientation convention (vi). The proof of Step 1 is complete. $\qquad\square$

Step 2. $X \in C^{1,\beta}(B \cup \partial''B)$ and there is at most one branch point (a point where $X_u = X_v = 0$) on $\partial''B$, and this occurs only if $z_u = 0$ at some point of $\partial''B$.

Proof of Step 2. Introduce complex notation: $w = u + iv, X_w = X_u - iX_v$. By a result of Heinz [15] $X \in C^{1,\beta}(B \cup \partial''B : I\!R^3)$ and about a branch point $w_0 \in$

$\partial''B$ there is an asymptotic representation $X_w = a(w - w_0)^l + o(|w - w_0|^l)$, where $a = (a_1, a_2, a_3)$ is a non-zero complex vector with $\sum a_i^2 = 0$. Since $x(u, 0) = y(u, 0) = 0$, then $x_u(u, 0) = y_u(u, 0) = 0$ and by $X_u \cdot X_v = 0$, $z_v(u, 0) = 0$. Therefore, $(u_0, 0)$ is a branch point only if $z_u(u_0, 0) = 0$.

Writing $a_i = b_i + ic_i, b_i$ and c_i real, we have

$$(9) \qquad\qquad b_1^2 - c_1^2 + b_2^2 - c_2^2 + b_3^2 - c_3^2 = 0$$

and

$$(10) \qquad\qquad b_1 c_1 + b_2 c_2 + b_3 c_3 = 0.$$

From $x_u(u, 0) = y_u(u, 0) = 0$, we have $b_1 = b_2 = 0$ and from $z_v(u, 0) = 0$ it follows that $c_3 = 0$. We note that (10) is automatically fulfilled. Also, (9) implies that $b_3^2 = c_1^2 + c_2^2$ and $b_3 \neq 0$. Using the asymptotic representation, a calculation gives

$$(11) \qquad\qquad \tilde{\mathbf{N}} = \frac{X_u \times X_v}{|X_u \times X_v|} = \left(\frac{c_2}{|b_3|}, -\frac{c_1}{|b_3|}, 0 \right) + o(1)$$

for the downward pointing normal $\tilde{\mathbf{N}}$ to the surface.

Let r_P and θ_P be polar cordinates in the parameter domain B, relative to the point w_0. The asymptotic relation for X_w can be integrated to yield

$$X = \operatorname{Re} \frac{a}{l+1} (w - w_0)^{l+1} + o(|w - w_0|^{l+1}),$$

from which we deduce that

$$(x, y) = -(c_1, c_2) \frac{r_P^{l+1}}{l+1} \sin((l+1)\theta_P) + o(r_P^{l+1}).$$

We now show that $l = 1$. Suppose that $l > 1$. Consider the behavior of $y(w)$ on rays through $w_0, \theta_P = \theta_j, \theta_j = \frac{\pi(2j-1)}{2(l+1)}, j = 1, 2, 3$. Take, for the sake of argument $c_2 < 0$, then y is positive on $\theta_P = \theta_1, \theta_3$, for r_P small, and y is negative on $\theta_P = \theta_2$, for r_P small. This is impossible. Therefore $l = 1$.

Let L be the line through O in the direction of (c_1, c_2); its inclination angle with the positive x-direction is θ_R. Since (x, y) as a function of w for r_P fixed is arbitrarily close to L, for small r_P, we can pick angles $\theta_L, \theta_R, \theta_R = \theta_L + \pi$ with $-\alpha \le \theta_L < \theta_R \le \alpha$. (By considering the image of a small half-disk about O under K, which is orientation reversing, we see that $c_2 < 0$.)

Now consider the images of vertical lines L_j given by $w = u_j, j = 1, 2$, for $u_1 < u_0 < u_2, u_1$ and u_2 close to u_0, and v small, starting at $v = 0$. From the asymptotic representaion $(x_v, y_v) = (c_1, c_2)r_P \cos \theta_P + o(r_P)$, we see that

$$(x_v(u, 0), y_v(u, 0)) = -(u - u_0)(c_1, c_2) + o(|u - u_0|)$$

for u close to u_0. Define $\Theta(u) = \arg(x_v(u,0) + iy_v(u,0))$, where the angle is chosen in the interval $[-\alpha, \alpha]$. We have that

$$(12) \quad \Theta(u) \to \theta_R \text{ as } u \to u_0+ \quad \text{and} \quad \Theta(u) \to \theta_L \text{ as } u \to u_0 - .$$

That $\Theta(u)$ is an increasing function of u follows from the fact that the map K is orietation reversing (this is the main reason for requiring (vi) of Step 1). Thus the image of S_{12}, the set of points of B between the lines L_1 and L_2, must contain the rays $\theta_L + \epsilon \leq \theta \leq \theta_R - \epsilon$ for $0 < r < r_0(\epsilon), \forall \epsilon > 0$. Any other branch point would have to have a different corresponding half-space of directions. This is impossible. Therefore, there can be at most one branch point on $\partial'' B$.   □

Step 3. $\Theta(u) = \arg[x_v(u,0) + iy_v(u,0)]$ is continuous except at a branch point $(u^*, 0) \in \partial'' B$, $\lim_{u \to u^*+} \Theta(u) = \lim_{u \to u^*-} \Theta(u) + \pi$, and $\Theta(u)$ is strictly increasing.

Proof of Step 3. From Step 2, $\Theta(u)$ is defined except at a branch point and is an increasing and continuous function. When there is a branch point the limit relation must hold. We now show that $\Theta(u)$ is strictly increasing. Argue by contradiction. If Θ were not strictly increasing there would be an interval $[u_1, u_2]$ on which Θ is constant. We may take $u_1 > u^*$ or $u_2 < u^*$.

By the Implicit Function Theorem, there are level curves C_j of $z(u,v)$ through $(u_j, 0), j = 1, 2$, near these points. Let $\mathcal{V}$ be the set of points between C_1 and C_2 for $0 < v < v_1$, v_1 sufficiently small, and let $\mathcal{V}' = K(\mathcal{V})$ be the image of $\mathcal{V}$. Let $\partial \mathcal{V}' = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with $\Gamma_j \subset C_j, j = 1, 2$. Since $z_u(u,0) \neq 0$ for $u \in [u_1, u_2], z(u_1, 0) \neq z(u_2, 0)$. Hence f satisfies

$$f = c_j \quad \text{on} \quad \Gamma_j \qquad j = 1, 2$$

where c_1 and c_2 are different constants and Γ_1 and Γ_2 have the same direction at O. We now show that f cannot exist. The reasoning is different for the cases $\kappa > 0, \kappa = 0$, and $\kappa < 0$. The surface $z = f(x,y), (x,y) \in \mathcal{V}'$, when described parametrically is smooth up to T (the z-axis), contains the segment $Z_1 Z_2, Z_j = (0, 0, c_j), j = 1, 2$, and the tangent plane to any point on this segment is a vetical plane $\mathcal{P}$.

Suppose $\kappa > 0$. Take $c_1 < c_2$ to fix the notation. Since the curvature of T is zero, the curve Γ_j has curvature $\hat{\kappa}_j = -(\kappa c_j + \lambda), j = 1, 2$, at O relative to the xy-projection of the downward pointing normal $\tilde{N}_O = (N_1(O), N_2(O))$. Since $\hat{\kappa}_1 > \hat{\kappa}_2$ and Γ_1 and Γ_2 are both orthogonal to $\tilde{N}_O, \Gamma_1$ lies above Γ_2, relative to the the direction $\tilde{N}_O$, near O. It is seen that the xy-projection of the downward pointing normal $\nabla f / \sqrt{1 + |\nabla f|^2}$ tends to $-\tilde{N}_O$, as $\mathbf{x} \to O$, giving a contradiction.

Suppose $\kappa = 0$. One can argue as follows (see [21]). Look at the surface near T as the graph of a function $g(\tau, z)$ defined on a rectangle $R = [0, b] \times [c_1, c_2]$ in the tangent plane $\mathcal{P}$. We have that g satisfies a Cauchy problem:

$$Ng = \lambda \quad \text{in} \quad R$$

$$g = 0 \quad \text{and} \quad g_\tau = 0 \quad \text{for} \quad \tau = 0.$$

This problem has a unique solution $g = g(\tau)$ which is a cylinder, which implies that f must represent a vertical cylinder, a contradiction.

Suppose $\kappa < 0$. Again rotate the problem as for $\kappa = 0$. By the Cauchy-Kovalevskaya Theorem, there is a unique solution to

$$Ng = \kappa z + \lambda \quad \text{in} \quad R = [0, b] \times [c_1, c_2]$$

$$g = 0, g_\tau = 0 \quad \text{for} \quad \tau = 0,$$

for b sufficiently small. Now let $\hat{g}$ be the solution to the same problem with c_2 replaced by $d_2, d_2 > c_2$. Choose d_2 larger than the assumed bound on f. The surface given by $\hat{g}$ projects simply onto the xy-plane for b sufficiently small, since the curves $z = z_0, z_0 \in [c_1, d_2]$ have the same tangent direction but different curvatures $-(\kappa z_0 + \lambda)$ for $\tau = 0$. Let $z = F(x, y)$ be a description of the surface given by g. The domain of definition is a "larger" cusp region $\hat{\mathcal{V}}: \hat{\mathcal{V}}_\rho \supset \mathcal{V}_\rho$, where $\hat{\mathcal{V}}_\rho = \hat{\mathcal{V}} \cap B_\rho(O)$ and $\mathcal{V}_\rho = \mathcal{V} \cap B_\rho(O)$. Choose ρ so that $\hat{\mathcal{V}}_\rho \subset \Omega^*$. Then $f = F$ but F achieves the value d_2 at some point, giving a contradiction.

Step 4. Let $\omega_\theta(t) = K^{-1}(t \cos \theta, t \sin \theta)$. Then $\lim_{t \to 0+} \omega_\theta(t) = (U(\theta), 0)$ exists for $\theta \in (-\alpha, \alpha)$, and $U(\theta)$ is continuous and increasing in θ, and $\lim_{\theta \to \pm \alpha} U(\theta) = \pm 1$. This implies that $Rf(\theta)$ exists and is continuous.

Proof of Step 4. We first show that $\lim_{t \to 0+} \omega(t)$ exists for $\theta \in (-\alpha, \alpha)$. The second coordinate of $\omega_\theta(t)$ tends to zero as $t \to 0+$, so that if $\omega_\theta(t)$ does not tend to a limit, then it must have an interval of limit points on $\partial'' B, \mathcal{I} = \{(u, 0) : u_1 \le u \le u_2\}, u_1 < u_2$. Take $\mathcal{I}$ to contain no branch point. Let L_i be the vertical line through $(u_i', 0), i = 1, 2, u_1 < u_1' < u_2' < u_2$. We must have that $\omega_\theta(t)$ intersects L_1 and L_2 infinitely often for $t \in (0, t_0)$, for any t_0. The image curves $L_i' = K(L_i)$ have inclination angles $\Theta(u_i')$ at $O, \Theta(u_1') < \Theta(u_2')$. Thus the straight line $(t \cos \theta, t \sin \theta), 0 < t < t_0$ cannot intersect both L_1 and L_2 for t_0 small. This is a contradiction. Therefore $U(\theta)$ exists.

Next, we show that $U(\theta)$ is an increasing function. Consider the region $R' = \{\mathbf{x} : \theta_1 < \theta < \theta_2, 0 < r < r_0\}$ for small r_0. The image of R' under K^{-1} must have $U(\theta_2) > U(\theta_1)$, since K^{-1} is orientation reversing.

We now prove that U and Θ are inverse functions:

$$(13) \qquad\qquad U(\Theta(u)) = u \quad \text{for} \quad u \in (-1,1), u \neq u_0.$$

If there is no branch point then (13) holds for all $u \in (-1,1)$.

Suppose that $U(\Theta(u_1)) > u_1, u_1 \neq u_0$. Take $u_2, u_1 < u_2 < U(\Theta(u_1)), u_2 \neq u_0$. Then $\Theta(u_2) > \Theta(u_1)$. Consider L_1, L_2 as above and their respective images L_1', L_2'. But L_2' should be between the line $\theta = \Theta(u_1)$ and the curve L_1', for $0 < r < r_0, r_0$ small, giving a contradiction. The argument is similar if $U(\Theta(u_1)) < u_1, u_1 \neq u_0$.

Let $\alpha_1 = \lim_{u \to -1+} \Theta(u)$ and $\alpha_2 = \lim_{u \to 1-} \Theta(u)$. Clearly, $-\alpha \leq \alpha_1 < \alpha_2 \leq \alpha$ and $\alpha_1 < \alpha_L, \alpha_R < \alpha_2$ if there is a branch point. Statement (13) implies that $U(\theta)$ is continuous and strictly increasing on (α_1, α_2) in the case of no branch point on $\partial''B$ and on $(\alpha_1, \alpha_L) \cup (\alpha_R, \alpha_2)$ if there is a branch point on $\partial''B$. Since $U(\theta)$ is increasing, by (13) $\lim_{\theta \to \alpha_1+} U(\theta) = -1$ and $\lim_{\theta \to \alpha_2-} U(\theta) = 1$. Therefore, $U(\theta) = -1$ for $\theta \in (-\alpha, \alpha_1]$ and $U(\theta) = 1$ for $\theta \in (\alpha_2, \alpha]$.

In the case of no branch point, $U(\theta)$ is continuous on $(-\alpha, \alpha)$, constant on $(-\alpha, \alpha_1]$ and (α_2, α) and strictly increasing on $[\alpha_1, \alpha_2]$.

In the case with a branch point, from (13) and (12), $U(\theta_L) = u_0$ and $U(\theta_R) = u_0$ and therefore $U(\theta) = u_0$ for $\theta \in [\theta_L, \theta_R]$. We thus have that $U(\theta)$ is continuous on $(-\alpha, \alpha)$, constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$, and strictly increasing on $[\alpha_1, \alpha_2]$.

Now $Rf(\theta) = \lim_{t \to 0+} f(\omega_\theta(t)) = z(U(\theta))$ for $\theta \in (-\alpha, \alpha)$ and

$$\lim_{\mathbf{x} \to O, \mathbf{x} \in \partial^+ \Omega^*} f(\mathbf{x}) = \lim_{\phi \to 0+} z(\cos\phi, \sin\phi) = z(1,0)$$

$$\lim_{\mathbf{x} \to O, \mathbf{x} \in \partial^- \Omega^*} f(\mathbf{x}) = \lim_{\phi \to \pi-} z(\cos\phi, \sin\phi) = z(-1,0).$$

We see that $Rf(\theta)$ is continuous for $\theta \in [-\alpha, \alpha]$.

As a convenience we extend the definition of $U(\theta)$ so that $U(\pm\alpha) = \pm 1$. Then $Rf(\theta) = z(U(\theta))$ for all $\theta \in [-\alpha, \alpha]$. $\qquad\square$

Step 5. When there is no branch point on $\partial''B$ case (I) or case (D) holds. When there is one branch point on $\partial''B$ case (ID) or case (DI) holds.

Proof of Step 5. The proof follows easily from results established in Step 4. Recall that $Rf(\theta) = z(U(\theta))$ for all $\theta \in [-\alpha, \alpha]$.

Suppose that there is no branch point on $\partial''B$. Then either $z_u > 0$ or $z_u < 0$ on $\partial''B$, and these give case (D) and case (I), respectively.

Suppose that there is a single branch point $(u_0, 0), -1 < u_0 < 1$. Then either $z_u(u,0) > 0$, for $u < u_0$ and $z_u(u,0) < 0$, for $u > u_0$ or $z_u(u,0) < 0$,

for $u < u_0$ and $z_u(u,0) > 0$, for $u > u_0$. From the behavior of $U(\theta)$, we obtain either case (ID) or case (DI), respectively.

The proof is complete.   □

Proof of Proposition 1. The cases $\kappa = 0$ and $\kappa > 0$ are handled differently.

Suppose $\kappa = 0$. Since $Nf = \lambda$, the mean curvature has a positive upper bound H_0 as was needed in Step 2 of the proof of Theorem 1. The same comparison functions give a bound on f in Ω^* in terms of a bound on f on $\Gamma^* = \partial B_{\delta^*}(O) \cap \Omega$. Since f is assumed to be smooth up to the boundary, except for corner points, this latter quantity is bounded.

Suppose $\kappa > 0$ and $\alpha \geq \pi/2$. We no longer know that Nf is bounded. However, a lower hemishpere $\underline{h}$ of radius δ, defined on $B_\delta(P)$, whose lowest value is $\underline{h}(P) = \frac{2}{\kappa}\delta - \frac{\lambda}{\kappa}$ will satisfy a supersolution condition $N\underline{h} \leq \kappa\underline{h} + \lambda$. Choose δ so that each point in a fixed neighborhood of O, Ω', is in some disk $B_\delta(P)$ which is either contained in Ω^* or meets one of $\partial^\pm\Omega^*$ and satisfies $T\underline{h}\cdot\nu > \cos\gamma_0$ on $\partial^\pm\Omega^*$. The reasoning is similar to that given in the proof of Step 1 of Theorem 1, where $h^\pm + k$ are used as comparison functions. By the Comparison Principle, $f \leq \frac{2}{\kappa\delta} - \frac{\lambda}{\kappa} + \delta$ for $\mathbf{x} \in \Omega'$. Similarly, $f \geq -\frac{2}{\kappa\delta} - \frac{\lambda}{\kappa} - \delta$ in Ω'. This gives us that f is bounded in a neighborhood of O. (If $\partial^\pm\Omega^*$ are assumed to be C^2 up to O then only disks $B_\delta(P)$ which are contained in Ω^* are needed. The present argument requires that $\partial^\pm\Omega^*$ be C^1 up to O.)

For $\alpha < \pi/2$ the argument is similar to one used in the proof of Step 1, Case 5 of Theorem 1. Constants $\gamma_0^\pm, 0 < \underline{\gamma}_0^\pm < \gamma^\pm$, are chosen so that $\underline{\gamma}_0^+ + \underline{\gamma}_0^- > \pi - 2\alpha$. Choose a ball $B_\delta(\mathbf{c})$ so that its boundary circle goes through O and makes an angle $\underline{\gamma}_0^-, 0 < \underline{\gamma}_0^- < \gamma^-$, with L^-, the tangent line to $\partial^-\Omega^*$, see Figure 7. Then, the angle β that $\partial B_\delta(\mathbf{c})$ makes with L^+ is $\beta = \pi - 2\alpha - \underline{\gamma}_0^-$. This implies that $\beta < \underline{\gamma}_0^+$. Now the lower hemisphere $\underline{h}$ defined above is a supersolution that satisfies $T\underline{h}\cdot\nu > \cos\gamma^\pm$ on $\partial^\pm \Omega^*\cap B_\delta(\mathbf{c})$, for δ sufficiently small. Thus, by the Comparison Principle

$$-\frac{2}{\kappa\delta} - \frac{\lambda}{\kappa} - \delta < f < \frac{2}{\kappa\delta} - \frac{\lambda}{\kappa} + \delta$$

on $B_\delta(\mathbf{c}) \cap \Omega^*$.

The proof of Proposition 1 is complete.   □

3. Proof of Theorem 2.

The proof of Theorem 2 is similar in outline to that employed in §2.2-2.3 of [29] for capillary surfaces with nonnegative gravity and constant contact angle in piecewise smooth domains which have convex corners. Since we are interested in nonconvex as well as convex corners, variable contact angle

(which need not be continuous), and negative as well as nonnegative gravity, some details in [**29**] require modification.

We first require a more general version of Lemma 6.1 of Finn [**8**]. As in [**8**], p. 141, consider Ω as a metric space with distance function $d(p, q) =$ infinum of Euclidean lengths of curves joining p to q in Ω and suppose its closure $\overline{\Omega}$ is covered by a (finite) partition of unity with certain properties. In particular, we suppose $\overline{\Omega}$ is covered by N open (in $\overline{\Omega}$) sets Ω_i and each set Ω_i is associated with a function $\varphi_i \in C_0^\infty(\Omega_i)$ such that $0 \leq \varphi_i \leq 1$ and $\Sigma_{i=1}^N \varphi_i = 1$ on Ω. Further, we assume:

i)   There is a finite (possibly empty) set of points $P \subset \Sigma(= \partial\Omega)$ and an associated integer $N_P < N$ such that $P = \{p_k : k = N_P + 1, \cdots, N_P + K\}$ with $K \leq N - N_P$, $\Sigma \cap \Omega_j \neq \emptyset$ for each $j \leq N_P$, the set $\Sigma_j = \Sigma \cap \Omega_j$ is open and connected in the relative topology of Σ, and $\Sigma \backslash P = \cup_{j=1}^{N_P} \Sigma_j$.

ii)   For each $k = N_P + 1, \cdots, N_P + K$, we have $p_k \in \Omega_k$, $p_k \notin \Omega_j$ for $j \neq k$, $1 \leq j \leq N$, and $\Sigma_k = \Sigma \cap \Omega_k \neq \emptyset$ meets exactly two (adjacent) sets Σ_j, $1 \leq j \leq N_P$ (see Fig 6.8 of [**8**]).

iii) There exists $\tau > 0$ such that Σ_j ($j = 1, \cdots, N_P$) can be represented (after a rigid motion $F_j : I\!R^2 \to I\!R^2$) over some interval $a_j < x < b_j$ with $a_j < 0 < b_j$ by a Lipschitz function $y = \psi_j(x)$ such that ψ_j has Lipschitz constants L_j^- on $[a_j, 0]$ and L_j^+ on $[0, b_j]$, the set $T_j = \{(x, y + \psi_j(x)) : a_j < x < b_j, -\tau < y < 0\}$ lies in Ω, and, if $j_1, j_2 \in \{1, \cdots, N_P\}$ with $j_1 \neq j_2$ and $p_k \in \overline{\Omega}_{j_1} \cap \overline{\Omega}_{j_2}$ for some $k \in \{N_P + 1, \cdots, N_P + K\}$, then $T_{j_1} \cap T_{j_2} = \emptyset$.

The points $p_j, 1 \leq j \leq N_P$, will be our "convex cornvers" and $p_j, N_P + 1 \leq j \leq N_P + K$, will be our "reentrant or nonconvex corners."

Lemma 1.   *Under the above conditions, let β be a bounded measurable function on Σ and, in the notation of* iii)*, let* $\beta_j^+ = \text{ess.} \sup\limits_{0 \leq x \leq b_j} | \beta(x, \psi_j(x)) |$ *and* $\beta_j^- = \text{ess.} \sup\limits_{a_j \leq x \leq 0} | \beta(x, \psi_j(x)) |$ *for* $j = 1, \cdots, N_P$*. Let* $\mathcal{A}_\delta \subseteq \Omega$ *be the strip of width δ adjacent to Σ, i.e.*

$$\mathcal{A}_\delta = \cup_{j=1}^{N_P} F_j^{-1}\left(\{(x, y + \psi_j(x)) : a_j \leq x \leq b_j, -\tau \leq y \leq 0\}\right).$$

Then for any $f \in BV(\Omega)$, there holds

$$\left| \int_\Sigma \beta f \, ds \right| \leq \mu \int_{\mathcal{A}_\delta} | Df | + \Upsilon(\Omega, \delta) \int_{\mathcal{A}_\delta} |f| \, ;$$

here $\mu = \max\left\{ \beta_j^- \sqrt{1 + (L_j^-)^2}, \beta_j^+ \sqrt{1 + (L_j^+)^2} \right\}$*, taken over all*

$j \in \{1, \cdots, N_P\}$ *for which* $\Omega_j \cap \mathcal{A}_\delta \cap$ supp $f \neq \emptyset$.

The proof of Lemma 6.1 of [8] begins by considering $f_j = f\varphi_j$. When $j > N_P$, the proof of Lemma 1 is the same as that of Lemma 6.1. When $j \leq N_P$, we see that applying Finn's argument separately to $\Sigma_j^+ = F_j^{-1}(\{(x, \psi_j(x)) : 0 \leq x \leq b_j\})$ and $\Sigma_j^- = F_j^-(\{(x, \psi_j(x)) : a_j \leq x \leq 0\})$, we obtain in place of (6.18), [8],

$$\left| \int_{\Sigma_j^{\pm}} \beta_j f \varphi_j \, ds \right| \leq \beta_j^{\pm} \sqrt{1 + (L_j^+)^2} \int_{\mathcal{A}_\delta \cap \overline{\Omega_j^{\pm}}} |Df| \, \varphi_j$$
$$+ \beta_j^{\pm} \sqrt{1 + (L_j^{\pm})^2} \, \Upsilon_j \int_{\mathcal{A}_\delta \cap \overline{\Omega_j^{\pm}}} |f| \, ,$$

where $\Omega_j^+ = F_j^{-1}(\{(x, y + \psi_j(x)) : 0 < x < b_j \, , \, \tau < y < 0\})$, etc. Lemma 1 then follows in the same manner as does Lemma 6.1 of [8].

Let us consider the proof of Theorem 2.

Step 1. We will assume $\alpha < \frac{\pi}{2}$, $\gamma_0^- \leq \frac{\pi}{2}$, $\gamma_0^+ \leq \frac{\pi}{2}$, and case (D) of Theorem 1 holds and prove Theorem 2 in this case.

Notice that $\gamma_0^+ \geq \underline{\gamma}^+$ and $\gamma_0^- \geq \underline{\gamma}^-$ and so $\gamma_0^+ + \gamma_0^- \geq \underline{\gamma}^+ + \underline{\gamma}^- > \pi - 2\alpha$. Also, $\gamma_0^+ \leq \overline{\gamma}^+$ and $\gamma_0^- \leq \overline{\gamma}^-$ and so $\gamma_0^+ + \gamma_0^- \leq \overline{\gamma}^+ + \overline{\gamma}^- < \pi + 2\alpha$. Let $0 < \sigma_j < \frac{\pi}{2} (j = 1, 2)$ such that $\sigma_1 + \sigma_2 = \pi - 2\alpha$, $\sigma_1 < \gamma_0^+$, and $\sigma_2 < \gamma_0^-$. Rotate Ω about O through an angle of $\pi - \sigma_1 - \alpha$; let us denote by $\tilde{\Omega}, \tilde{f}, \tilde{\gamma}$, etc. the sets, functions, etc. obtained from Ω, f, γ, etc. by this rotation. Then $\partial\tilde{\Omega}$ is tangent to the rays $\theta = \sigma_2$ and $\theta = \pi - \sigma_1$ at O. Let $\tilde{\Sigma}^j = \{(x, y) \in \partial\tilde{\Omega} \cap B_{\delta^*}(O) : (-1)^j x \leq 0\}$ be the graph of a (Lipschitz) function $\psi_j(x)$, $j = 1, 2$. If we choose $\delta^* > 0$ small enough, we may assume the Lipschitz constants L_j of ψ_j, $j = 1, 2$, satisfy

$$L_1 < |\tan(\gamma_0^+)| \quad \text{and} \quad L_2 < |\tan(\gamma_0^-)| \, .$$

Let $(0, 0, z)$ belong to the closure of the graph of f and let $\{\epsilon_j\}$ be any sequence in (0,1) converging to 0. Set $\tilde{\Omega}_j = \left\{(x, y) \in I\!R^2 : (\epsilon_j x, \epsilon_j y) \in \tilde{\Omega}\right\}$ and define $\tilde{f}_j \in C^\infty\left(\tilde{\Omega}\right)$ by

$$\tilde{f}_j(x, y) = \frac{1}{\epsilon_j}\left(\tilde{f}(\epsilon_j x, \epsilon_j y) - z\right)$$

for $(x, y) \in \tilde{\Omega}_j$. Let $\tilde{\gamma}_j$ be defined on $\partial\tilde{\Omega}_j \backslash O$ by $\tilde{\gamma}_j(x, y) = \tilde{\gamma}(\epsilon_j x, \epsilon_j y)$ and let $\nu_j = \nu_j(x, y)$ denote the outward unit normal to $\partial\tilde{\Omega}_j$. Then $\tilde{f}_j$ satisfies

the capillary problem

$$N\tilde{f}_j(x,y) = \epsilon_j \kappa \tilde{f}(\epsilon_j x, \epsilon_j y) + \epsilon_j \lambda, \qquad\qquad (x,y) \in \tilde{\Omega}_j$$

$$T\tilde{f}_j \cdot \nu_j = \cos(\tilde{\gamma}_j) \qquad\qquad \text{on} \quad \partial\tilde{\Omega}_j \backslash O.$$

Set $\Omega_\infty = \lim\limits_{j\to\infty} \tilde{\Omega}_j = \{(r\cos(\theta), r\sin(\theta)) : r > 0, \sigma_2 < \theta < \pi - \sigma_1\}$. If we argue as in §2.2 of [29] and in [30], using our Lemma 1 in place of Lemma 1.4 of [29] and, for example, using an appropriately modified version of (2.13) of [29], we can find a subsequence of $\{\tilde{f}_j\}$, which we continue to denote $\{\tilde{f}_j\}$, which converges locally to a generalized solution f_∞ (in the sense of Miranda [24] and Guisti [14], [13]) of the functional

$$\mathcal{F}_\infty(g) = \iint\limits_{\Omega_\infty} \sqrt{1 + |Dg|^2}\, dx - \int_{\Sigma_\infty^1} \cos(\gamma_0^+) g\, ds - \int_{\Sigma_\infty^2} \cos(\gamma_0^-) g\, ds,$$

where $\Sigma_\infty^1 = \{(r\cos(\pi - \sigma_1), r\sin(\pi - \sigma_1)) : r > 0\}$ and $\Sigma_\infty^2 = \{(r\cos(\sigma_2), r\sin(\sigma_2)) : r > 0\}$.

Let us now define the sets

$$\mathcal{P} = \{(x,y) \in \Omega_\infty : f_\infty(x,y) = \infty\}$$

and

$$\mathcal{N} = \{(x,y) \in \Omega_\infty : f_\infty(x,y) = -\infty\}.$$

These sets have a special structure which follows from the fact that $\mathcal{P}$ minimizes the functional

$$\Phi(A) = \iint\limits_{\Omega_\infty} |D\chi_A| - \cos(\gamma_0^+)\int_{\Sigma_\infty^1} \chi_A dH^1 - \cos(\gamma_0^-)\int_{\Sigma_\infty^2} \chi_A dH^1$$

$$= H^1(\Omega_\infty \cap \partial A) - \cos(\gamma_0^+)H^1(\Sigma_\infty^1 \cap \partial A) - \cos(\gamma_0^-)H^1(\Sigma_\infty^2 \cap \partial A)$$

and $\mathcal{N}$ minimizes the functional

$$\Psi(A) = \iint\limits_{\Omega_\infty} |D\chi_A| + \cos(\gamma_0^+)\int_{\Sigma_\infty^1} \chi_A dH^1 + \cos(\gamma_0^-)\int_{\Sigma_\infty^2} \chi_A dH^1$$

$$= H^1(\Omega_\infty \cap \partial A) + \cos(\gamma_0^+)H^1(\Sigma_\infty^1 \cap \partial A) + \cos(\gamma_0^-)H^1(\Sigma_\infty^2 \cap \partial A)$$

in the appropriate sense (e.g. [8], [13], [30]). After modification on a set of measure zero, we may assume that $\partial\mathcal{P} \cap \Omega_\infty$ and $\partial\mathcal{N} \cap \Omega_\infty$ each consists of straight lines which do not intersect inside Ω_∞ (e.g. [30]).

We claim that $Rf(\cdot)$ has a fan of constant radial limits of width γ_0^- adjacent to $\theta = -\alpha$; this equivalent to $R\tilde{f}(\theta) = Rf(-\alpha)$ for each $\theta \in [\sigma_2, \lambda]$ with $\lambda - \sigma_2 \geq \gamma_0^-$. Recall that we are in case (D) and choose our blow-up point $(0, 0, z)$ such that $z = R\tilde{f}(\lambda)$ for some $\lambda \in (\sigma_2, \pi - \sigma_1)$ with $R\tilde{f}(\pi - \sigma_1) < R\tilde{f}(\lambda) < R\tilde{f}(\sigma_2)$. Notice then that Theorem 1 implies

$$\mathcal{P} = \{(r\cos\theta, r\sin\theta) : r > 0, \sigma_2 \leq \theta < \lambda\}$$

and

$$\mathcal{N} = \{(r\cos\theta, r\sin\theta) : r > 0, \lambda < \theta \leq \pi - \sigma_1\}.$$

We shall show that the fact $\mathcal{P}$ minimizes $\Phi(\cdot)$ implies $\mathcal{P}$ is a sector of angle at least γ_0^- and so $\lambda \geq \sigma_2 + \gamma_0^-$. Since this will hold for all $\lambda \in [\sigma_2, \pi - \sigma_1]$ for which $R\tilde{f}(\lambda) < R\tilde{f}(\sigma_2)$, we see that $R\tilde{f}(\cdot)$ is constant on $[\sigma_2, \sigma_2 + \gamma_0^-]$.

As in [30] and [31], we see that

$$\mathcal{P} = \{(r\cos\theta, r\sin\theta) : r > 0, \sigma_2 \leq \theta < \sigma_2 + \beta\}$$

for some $\beta > 0$ (see Figure 8). Now let $\delta \in \left(0, \frac{\pi}{2}\right)$ and let

$$\mathcal{P}_\delta = \left\{(r\cos\theta, r\sin\theta) : \sigma_2 < \theta < \sigma_2 + \beta, r > \frac{\sin(\delta)}{\sin(\delta + \beta - \theta)}\right\}$$

be the portion of $\mathcal{P}$ remaining after removing the triangle with interior angles β, δ, and $\pi - (\beta + \delta)$ and side length 1 on $\theta = \sigma_2 + \beta$ as indicated in Figure 8. If we label the length of the side of the triangle on $\theta = \sigma_2$ by ϵ and the length of the remaining side by b, then $\epsilon = \dfrac{\sin(\delta)}{\sin(\beta + \delta)}$ and $b = \dfrac{\sin(\beta)}{\sin(\beta + \delta)}$. Since $\mathcal{P}$ minimizes $\Phi(\cdot)$, we see that $\Phi(\mathcal{P}) \leq \Phi(\mathcal{P}_\delta)$ in the sense that

$$H^1(\partial\mathcal{P} \cap \Omega_\infty \cap B_R(O)) - \cos(\gamma_0^+)H^1(\Sigma_\infty^1 \cap \partial\mathcal{P} \cap B_R(O))$$
$$- \cos(\gamma_0^-)H^1(\Sigma_\infty^2 \cap \partial\mathcal{P} \cap B_R(O))$$
$$\leq H^1(\partial\mathcal{P}_\delta \cap \Omega_\infty \cap B_R(O)) - \cos(\gamma_0^+)H^1(\Sigma_\infty^1 \cap \partial\mathcal{P}_\delta \cap B_R(O))$$
$$- \cos(\gamma_0^-)H^1(\Sigma_\infty^2 \cap \partial\mathcal{P}_\delta \cap B_R(O))$$

for each $R > \sec(\beta)$. In this case, we obtain

$$1 - \epsilon\cos(\gamma_0^-) \leq b \quad \text{or} \quad \sin(\beta + \delta) - \sin(\delta)\cos(\gamma_0^-) \leq \sin(\beta).$$

Then $\cos(\gamma_0^-) \geq \dfrac{\sin(\beta + \delta) - \sin(\beta)}{\sin(\delta)} = \cos(\beta) + \sin(\beta)\dfrac{\cos(\delta) - 1}{\sin(\delta)}$. If we take the limit as $\delta \to 0+$, we see that $\cos(\gamma_0^-) \geq \cos(\beta)$ and, since $0 < \gamma_0^-, \beta < \frac{\pi}{2}$,

$\beta \geq \gamma_0^-$. A similar argument for the set $\mathcal{N}$, which minimizes $\Psi(\cdot)$, shows that $\lambda \leq (\pi - \sigma_1) - (\pi - \gamma_0^+) = \gamma_0^+ - \sigma_1$ if $R\tilde{f}(\lambda) > R\tilde{f}(\pi - \sigma_1)$ and so $R\tilde{f}(\cdot)$ is constant on $[\pi - \sigma_1 - (\pi - \gamma_0^+), \pi - \sigma_1]$. This completes Step 1.

Step 2.   Suppose $\alpha < \frac{\pi}{2}$. Notice the condition

$$\pi - 2\alpha < \gamma_0^+ + \gamma_0^- < \pi + 2\alpha$$

implies that there exist angles $\sigma_1, \sigma_2 \in (0, \frac{\pi}{2})$ such that $\sigma_1 + \sigma_2 = \pi - 2\alpha$, $\sigma_1 < \min(\gamma_0^+, \pi - \gamma_0^+)$, and $\sigma_2 < \min(\gamma_0^-, \pi - \gamma_0^-)$ and so, setting $\gamma_1 = \gamma_0^+$ and $\gamma_2 = \gamma_0^-$,

$$| \cos(\gamma_j) | \sqrt{1 + \tan^2(\sigma_j)} < 1.$$

The proof in Step 1, with slight modification, then yields the existence of fans adjacent to $\theta = \sigma_2$ and $\theta = \pi - \sigma_1$ of the appropriate size.

Step 3.   Suppose $\alpha \geq \frac{\pi}{2}$. Then the existence of fans as indicated in Theorem 2 follows as in Steps 1 and 2 after observing that we may consider the two sides of a reentrant corner separately and so obtain in, for example, (1.18) of [**30**] and Lemma 1 Lipschitz constants L arbitrary close to zero. The proof then follows (without any need for angle conditions).

The proof of Theorem 2 is complete.   □

4. Further Results and Examples.

This section contains applications of Theorems 1 and 2 and illustrative examples. We start with two corollaries of Theorems 1 and 2 where γ has the same limit γ_0 on both sides of the corner (which includes the situation in which γ is constant).

Corollary 1.   *Let f be a bounded solution to* (1) *satisfying* (2) *on $\partial^\pm \Omega^* \setminus O$, with $\gamma_0 = \lim_{s \to 0+} \gamma^+(s) = \lim_{s \to 0+} \gamma^-(s), 0 < \gamma_0 < \pi$. Then for $\alpha \geq \pi/2$, case (ID) cannot occur when $\alpha + \gamma_0 \leq \frac{3\pi}{2}$ and case (DI) cannot occur when $\alpha \leq \gamma_0 + \pi/2$. For $\alpha < \pi/2$ and $|\pi/2 - \gamma_0| < \alpha$ or if $\alpha = \pi/2$ then f must be continuous up to O.*

Proof. Suppose that $\alpha + \gamma_0 \leq \frac{3\pi}{2}$ and case (ID) holds. Then by Theorem 2, $\alpha - \alpha_2 \geq \pi - \gamma_0$ and $\alpha_1 + \alpha \geq \pi - \gamma_0$. This leads to the contradiction $2\alpha > 2(\pi - \gamma_0) + \pi$. A similar argument shows that for $\alpha \leq \gamma_0 + \pi/2$ case (DI) cannot hold.

For $\alpha \leq \pi/2$ suppose that f is not continuous at O then either case (I) or case (D) holds. For case (I), by Theorem 2, $\alpha - \alpha_2 \geq \gamma_0$ and $\alpha_1 + \alpha \geq \pi - \gamma_0$. This leads to the contradiction $2\alpha > \gamma_0 + (\pi - \gamma_0) = \pi$.   □

This can be applied to Korevaar's example [17].

Example 1. Let $\kappa > 0, \lambda = 0, \gamma$ is constant $0 < \gamma < \pi/2, \alpha > \pi/2$. Let Ω be the region depicted in Figure 9. For ϵ sufficiently small, the solution f to (1),(2) is discontinuous at O since a lower bound on f in Ω_1 is greater than an upper bound for f in Ω_0. Thus $Rf(-\alpha) > Rf(\alpha)$ and case (D) cannot hold. By Corollary 1, case(ID) cannot hold and under the assumption $\alpha \leq \gamma + \pi/2$, case (DI) cannot occur. We may conclude that case (D) holds provided that $\alpha \leq \gamma + \pi/2$.

Corollary 2. *Let Ω be symmetric about the x-axis, $\alpha \geq \pi/2, \gamma(x, -y) = \gamma(x, y)$ for all $\mathbf{x} \in \partial\Omega$, and $\gamma_0 = \lim_{s \to 0+} \gamma^+(s) = \lim_{s \to 0+} \gamma^-(s)$, satisfying*

$$\alpha - \pi/2 \leq \gamma_0 \leq \pi/2 \quad or \quad \pi/2 \leq \gamma_0 \leq 3\pi/2 - \alpha.$$

Let $f(x, y)$ be a bounded solution to (1), (2) that is even in y. Then f must be continuous up to O. The condition on the symmetry of f is automatic when $\kappa \geq 0$.

Proof. When $\kappa > 0$ ($\kappa = 0$), the Comparison Principle guarantees a unique (unique up to an additive constant) solution to (1), (2). Thus, the symmetry of the boundary value problem implies that $f(x, y)$ is even in y.

Suppose that f is not continuous at O. Then in view of Theorem 1, cases (I) and (D) cannot hold and $\alpha_L = -\pi/2$ and $\alpha_R = \pi/2$. By Corollary 1, case (DI) holds when $\gamma_0 \leq \pi/2$ and case (ID) holds when $\gamma_0 \geq \pi/2$. By Theorem 2, $\alpha - \alpha_2 \geq \gamma_0$ for $\gamma_0 \leq \pi/2$ and $\alpha - \alpha_2 \geq \pi - \gamma_0$ for $\gamma_0 \geq \pi/2$. These inequalities both imply that $\alpha_2 \leq \pi/2 = \alpha_R$, which contradicts Theorem 1. $\square$

The following example shows that the condition for continuity in Corollary 2 is sharp. The construction is similar to that used by Korevaar [17].

Example 2. Let $\kappa > 0, \lambda = 0$, and γ constant, $0 < \gamma < \alpha - \pi/2$ ($\alpha > \pi/2$). Consider the symmetric region Ω depicted in Figure 10. We let R be fixed and we make ϵ sufficiently small so that the solution to (1),(2) must be discontinuous at O.

Consider the rectangle Ω_0 (see Figure 10a). Each point in Ω_0 is in a ball of radius δ which is contained in Ω. Thus ([2])

$$f \leq \frac{2}{\kappa\delta} + \delta \quad \text{in} \quad \Omega_0.$$

Consider the region Ω^+ bounded by two straight sides and two circles which is symmetric about the dashed line (see Figure 10b). Each circle has radius $\frac{\epsilon}{a}, a = \cos\gamma - \sin(\frac{3}{2}\pi - 2\alpha)$, and $\hat{\gamma} = 2(\pi - \alpha) + \gamma$. Let g be the

portion of a torus obtained by rotating one of the circular arcs about the axis of symmetry by π radians, above the xy-plane (thinking of the xy-plane as part of $I\!\!R^3$) and adjusting the height as follows. Twice the mean curvature of g has the lower bound: $Ng \geq \frac{a}{\epsilon} - \frac{1}{R-\epsilon}$. Adjust the height of g so that $\max g = \frac{1}{\kappa}(\frac{a}{\epsilon} - \frac{1}{R-\epsilon})$. Then $Ng \geq \kappa g$, so that g is a subsolution. Note that $Tg \cdot \nu = \cos \gamma$ on the straight side $\partial^+ \Omega^*$ and $Tg \cdot \nu = \cos \hat{\gamma}$ on the other straight side, while $Tf \cdot \nu = -1$ on the two circular sides. It follows from the Comparison Principle that

$$f \geq g \geq \frac{1}{\kappa} \left(\frac{a}{\epsilon} - \frac{1}{R-\epsilon} \right) - R \quad \text{in} \quad \Omega^+.$$

By making ϵ sufficiently small we can make the lower bound in Ω^+ greater than the upper bound in Ω_0, forcing f to be discontinuous at O. Note that $-f$ has contact angle $\gamma' = \pi - \gamma$, so that $3\pi/2 - \alpha < \gamma' < \pi$.

We next consider the situation in which the contact angle has different limits $\gamma_0^\pm$ along the two sides of the corner.

Corollary 3. *Let f be a bounded solution to (1) satisfying (2) on $\partial^\pm \Omega^* \backslash O$, with $\gamma_0^\pm = \lim_{s \to 0+} \gamma^\pm(s), 0 < \gamma_0^\pm < \pi$. For $\alpha < \pi/2$ assume in addition*

$$\pi - 2\alpha < \gamma_0^+ + \gamma_0^- < \pi + 2\alpha.$$

Then
   Case (I) cannot hold if $2\alpha + \gamma_0^- - \gamma_0^+ \leq \pi$.
   Case (D) cannot hold if $2\alpha - \gamma_0^- + \gamma_0^+ \leq \pi$.
   For $\alpha > \pi/2$, case (ID) cannot hold if $2\alpha + \gamma_0^- + \gamma_0^+ \leq 3\pi$.
   For $\alpha > \pi/2$, case (DI) cannot hold if $2\alpha - \gamma_0^- - \gamma_0^+ \leq \pi$.

The proof follows from Theorem 2 as in Corollaries 1 and 2, so will be omitted.

By specializing to the case of a convex corner, we obtain a condition for continuity up to the corner point.

Corollary 4. *Let f be a bounded solution to (1) satisfying (2) on $\partial^\pm \Omega^* \backslash O$, with $\gamma_0^\pm = \lim_{s \to 0+} \gamma^\pm(s), 0 < \gamma_0^\pm < \pi, \alpha < \pi/2$,*

$$(14) \qquad\qquad \pi - 2\alpha < \gamma_0^+ + \gamma_0^- < \pi + 2\alpha,$$

$$(15) \qquad\qquad 2\alpha + |\gamma_0^+ - \gamma_0^-| \leq \pi.$$

Then f must be continuous at O.

Proof. By Corollary 3 we see that both cases (I) and (D) cannot occur. Hence f must be continuous at O.   □

It is easy to see that the set of (γ_0^-, γ_0^+) satisfying conditions (14), (15) forms a rectangle in the first quadrant of the $\gamma_0^- \gamma_0^+$-plane (see Figure 11). It is not known whether conditions (14), (15) for continuity are sharp.

As discussed in Section 1, Concus and Finn [3], [11] have encountered the same rectangle in their investigation of the existence or nonexistence of a solution in the neighborhood of O in the case that $\kappa = 0$ and $\gamma^\pm$ are constants. In addition to the results described in Section 1, they show that a necessary condition for a solution to have a unit normal that is continuous up to O is that (γ^-, γ^+) belongs to the closed rectangle pictured in Figure 11.

We next present an example where $\gamma^\pm$ are different constants, $\alpha = \pi/2$ ($\partial\Omega$ is smooth) and case (D) (or case (I)) holds. This is a modification of an example of Finn (see Section 4 in [10]).

Example 3. Let $\gamma^\pm$ be two constants $0 < \gamma^- < \gamma^+ < \pi/2$ and $\kappa > 0$, $\lambda = 0$. Consider the closed off strip region depicted in Figure 12; Γ is the straight segment not contianing O. Let $\gamma(s) = \gamma^\pm$ on $\partial^\pm\Omega^*, \gamma = \pi/2$ on Γ and any values on the remainder of the boundary so that γ is a smooth except at O. (In Finn's example there are two points of discontinuity of γ.) Let $\Omega^\pm$ be the indicated regions, each having two straight sides and two circular sides. The circular sides make angles $\gamma^\pm$ and $\pi/2$ with the straight sides as indicated.

As in Example 2, one obtains a bound from below in the region Ω^-:

$$f > \frac{1}{\kappa} \left(\frac{\cos\gamma^-}{\epsilon} - \frac{1}{R - \epsilon} \right) - R \quad \text{in} \quad \Omega^-.$$

Revolving a circular arc bounding Ω^+ about its axis of symmetry by π radians below the xy-plane and adjusting the height of the resulting torus, one obtains an analogous upper bound:

$$f < \frac{1}{\kappa} \left(\frac{\cos\gamma^+}{\epsilon} + \frac{1}{R + \epsilon} \right) + R \quad \text{in} \quad \Omega^+.$$

For ϵ sufficiently small, the lower bound exceeds the upper bound, implying that f is discontinuous at O. By Corollary 3, case (I) cannot hold, so therefore case (D) must hold. (Of course for $-f$ case (I) holds.)

Finally, we prove the existence of a bounded capillary surface defined in a disk, smooth up to the boundary except one point O, for which no radial

limits exist as O is approached. The construction is a modification of the "gliding hump" argument used by Lancaster [20].

Theorem 3. *Let Ω be the disk of radius 1 centered at $(1,0)$. Then there exists a solution to $Nf = \frac{1}{2}f$ in $\Omega, |f| \leq 2, f \in C^2(\Omega) \cap C^1\left(\overline{\Omega} \setminus O\right), O = (0,0)$ so that no radial limits $Rf(\theta)$ exist $(\theta \in [-\pi/2, \pi/2])$.*

By an existence theorem of Serrin [26], the Dirichlet problem

$$(16) \qquad\qquad Ng = \frac{1}{2}g \quad \text{in} \quad \Omega, \qquad g = g_0 \quad \text{on} \quad \partial\Omega$$

can be solved for all continuous functions g_0, with $\max |g_0| \leq 2$ (since the curvature of $\partial\Omega$ is 1). Furthermore if $g_0 \in C^{1,\beta}$ then $g \in C^2(\Omega) \cap C^{1,\beta'}\left(\overline{\Omega}\right)$. Before giving the proof of Theorem 3, we need to state a "localization" lemma.

Lemma 2. *For $\epsilon > 0, 0 < \delta < 1, \Omega$ the disk of radius 1 about $(1,0)$ and $Nh = \frac{1}{2}h$ in $\Omega, h = h_0$ on $\partial\Omega, h_0 \in C^{1,\beta}, |h_0| \leq 2$, there exists a g satisfying (16), $g_0 \in C^{1,\beta}$, with $g(O) = 2, |g| \leq 2, g_0 = h_0$ on $\partial\Omega \setminus B_\delta$, and $|g - h| < \epsilon$ on $\overline{\Omega}_\delta = \overline{\Omega} \setminus B_\delta, B_\delta$ is the disk of radius δ about O. (The same result holds with $g(O) = -2$.)*

Proof. For $0 < \delta' < \delta$ let $g_0 = h_0$ on $\partial\Omega \setminus B_{\delta'}, g_0(O) = 2, g_0 \in C^{1,\beta}$, and $|g_0| \leq 2$ on $\partial\Omega$. By the maximum principle, $|g| \leq 2$ on $\overline{\Omega}$. By known interior and boundary estimates (these are used, for example, in the proof of Theorem 4 of [27]) g tends to h, pointwise on Ω and uniformly in the C^1 norm on $\overline{\Omega}_\delta$ as δ' tends to zero. The conclusion follows. □

Proof of Theorem 3. Using Lemma 2, we construct a sequence $\{f_n\}$ satisfying $Nf_n = \frac{1}{2}f_n$ in $\Omega, |f_n| \leq 2$ in $\overline{\Omega}$ and f_n even in y. Each f_n is determined by its boundary values, so these will be chosen to be symmetric with respect to the x-axis. Let $f_1 = -2$ on $\partial\Omega$. Then there exists $r_1 \in (0,1)$ with $f_1(r_1, 0) < -1$. Let $\epsilon_1 = -(1 + f_1(r_1, 0))$. By Lemma 2 there exists f_2 with $f_2(O) = 2, |f_2| \leq 2$, and $|f_2 - f_1| < \epsilon_1$ in $\overline{\Omega}_{r_1}$. This implies that $f_2(r_1, 0) < -1$ and hence, by continuity, there exists and r_2 so that $0 < r_2 < r_1, r_2 < \frac{1}{2}$ and $f_2(r_2, 0) > 1$. Define f_n and r_n inductively: set $\epsilon_n = \min_{1 \leq k \leq n} |f_n(r_k, 0) - (-1)^k|$, pick f_{n+1}, r_{n+1} so that $0 < r_{n+1} < \min\{r_n, \frac{1}{n+1}\}$ and $|f_{n+1} - f_n| < \epsilon_n$ on $\overline{\Omega}_{r_n}$, where $f_{n+1}(O) = 2(-1)^{n+1}$ and f_{n+1} agrees with f_n on $\partial\Omega \setminus B_{r_n}$. It follows that $f_n(r_k, 0) > 1$ if k is even and $f_n(r_k, 0) < -1$ if k is odd, for $1 \leq k \leq n$. In view of the interior and boundary extimates invoked in the proof of Lemma 2, a subsequence of $\{f_n\}$ converges to a solution f satisfying $Nf = \frac{1}{2}f$ in $\Omega, |f| \leq 2, f \in C^2(\Omega) \cap C^1\left(\overline{\Omega} \setminus O\right)$, the convergence being uniform in the

C^1 norm on each $\overline{\Omega}_{r_n}$. Since we take each f_n to be even in y, then f is even in y. From previous inequalities, we have $f(r_n, 0) \geq 1$ if n is even and $f(r_n, 0) \leq -1$ if n is odd, for all n. Therefore, $Rf(0) = \lim_{r \to 0+} f(r, 0)$ does not exist. Now, if any other radial limit $Rf(\theta_0)$ did exist ($\theta_0 \neq 0$), then by symmetry so would $Rf(-\theta_0)$ and $Rf(-\theta_0) = Rf(\theta_0)$. Considering the region $\Omega' = \{x \in \Omega \ : \ |\theta| < |\theta_0|\}$, f has continuous boundary values on $\partial\Omega'$. If $\theta_0 = \pm\pi/2$, then $\Omega' = \Omega$, and f must by continuous on $\overline{\Omega}$, giving a contraditcion. If $|\theta_0| < \pi/2$ then portions of cylinders can be used as upper and lower barriers to establish the continuity of f on $\overline{\Omega}'$, again giving a contradicton. $\qquad\square$

5. Extensions.

We now make a few remarks about extensions of our results. Theorems 1 and 2 apply to variational solutions which are continuous up to smooth portions of the boundary since the Comparison Principle holds for weak solutions (Theorem 7.7 in [8]).

If we change the equation to $Nf = 2H(\mathbf{x}, f)$ then Theorems 1 and 2 are again true provided either H is strictly increasing in f or if H depends only on f, is analytic, strictly decreasing, and unbounded from one side. An example of the latter is $H = e^{-f}$. These conditions on f come into play only in ruling out "cusp" solutions (see Step 3 in the proof of Theorem 1). It is not known if these "cusp" solutions can be ruled out more generally.
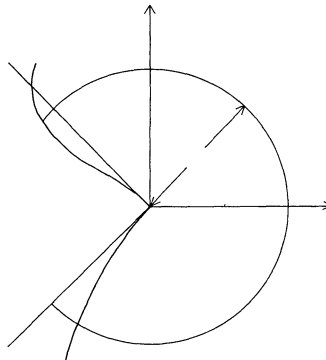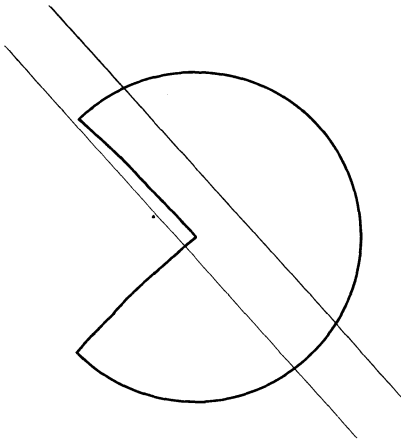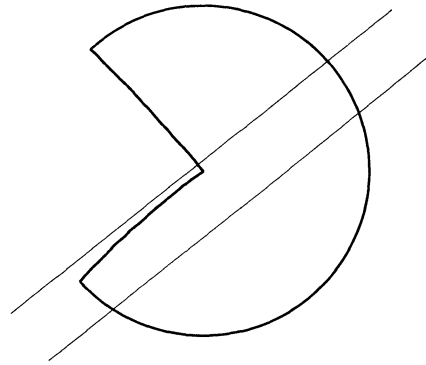
Figure 1.

Figure 2a.   **Figure 2b.**
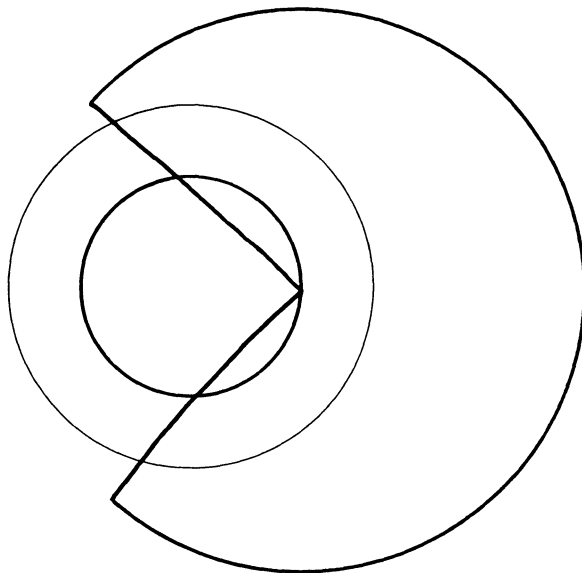
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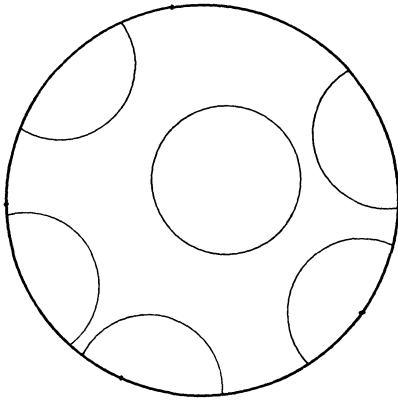
Figure 4a.

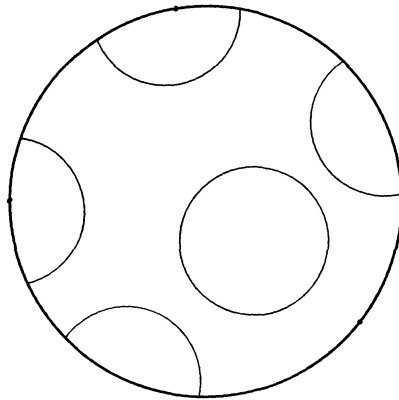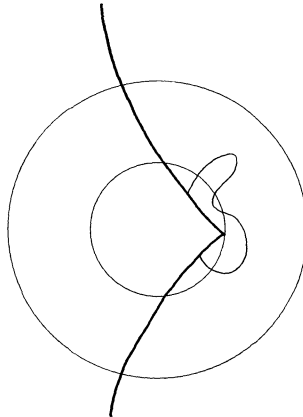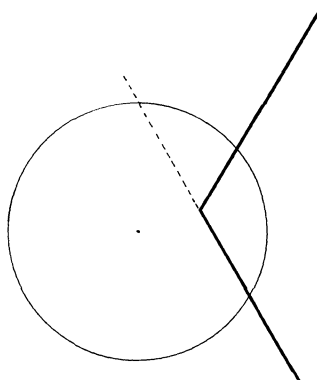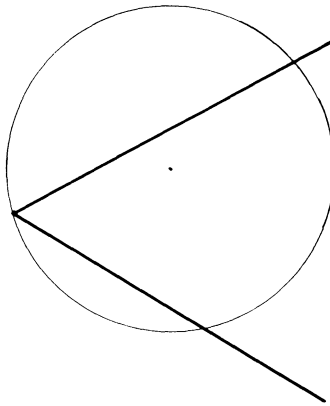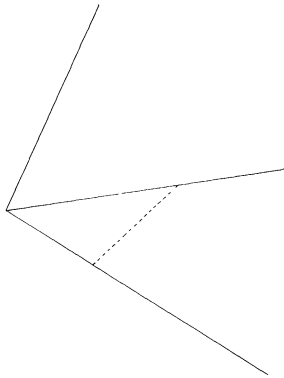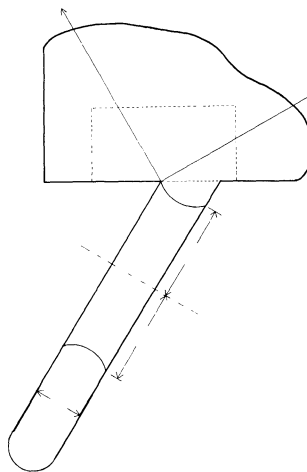
Figure 4b.

Figure 5.

Figure 6.

Figure 7.

Figure 8.

Figure 9.
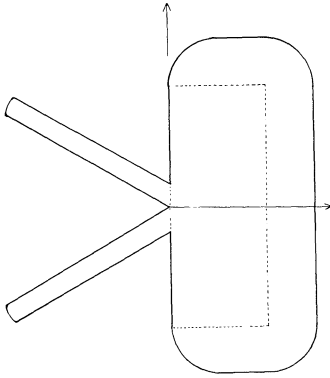
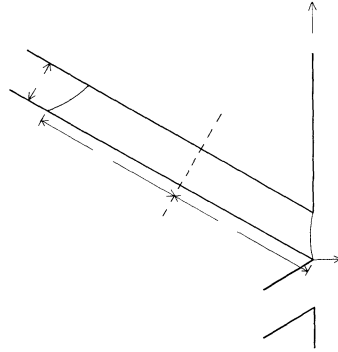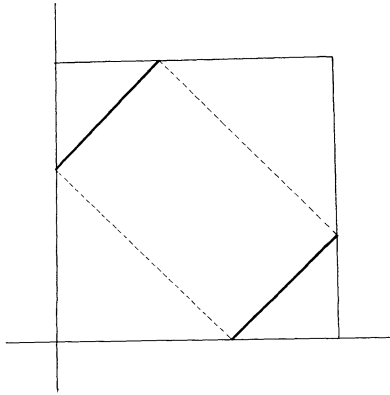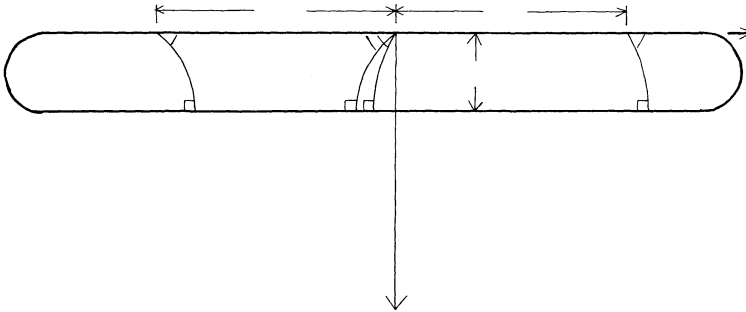**Figure 10a.**   **Figure 10b.**

Figure 11.

Figure 12.

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