Murahara, H., Onozuka, T. and Seki, S.
Osaka J. Math.
57 (2020), 647–653

BOWMAN-BRADLEY TYPE THEOREM FOR FINITE MULTIPLE ZETA VALUES IN $\mathcal{A}_2$

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(Received October 26, 2018, revised March 22, 2019)

Abstract

Bowman and Bradley obtained a remarkable formula among multiple zeta values. The formula states that the sum of multiple zeta values for indices which consist of the shuffle of two kinds of the strings $\{1, 3, \ldots, 1, 3\}$ and $\{2, \ldots, 2\}$ is a rational multiple of a power of π^2. Recently, Saito and Wakabayashi proved that analogous but more general sums of finite multiple zeta values in an adelic ring $\mathcal{A}_1$ vanish. In this paper, we partially lift Saito-Wakabayashi's theorem from $\mathcal{A}_1$ to $\mathcal{A}_2$. Our result states that a Bowman-Bradley type sum of finite multiple zeta values in $\mathcal{A}_2$ is a rational multiple of a special element and this is closer to the original Bowman-Bradley theorem.

1. Introduction

For positive integers $k_1, \ldots, k_r$ with $k_r \geq 2$, the multiple zeta values (MZVs) and the multiple zeta-star values (MZSVs) are defined by

$$\zeta(k_1, \ldots, k_r) := \sum_{1 \leq n_1 < \cdots < n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}},$$

$$\zeta^\star(k_1, \ldots, k_r) := \sum_{1 \leq n_1 \leq \cdots \leq n_r} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}.$$

By convention, we set $\zeta(\varnothing) = \zeta^\star(\varnothing) = 1$ for the empty index. Let $\{a_1, \ldots, a_l\}^m$ denote the m-times repetition of $a_1, \ldots, a_l$, e.g. $\{2\}^2 = 2, 2$ and $\{1, 3\}^2 = 1, 3, 1, 3$. For MZVs, Bowman and Bradley [1] established the following result:

Theorem 1.1 (Bowman-Bradley [1, Corollary 5.1])**.** *For non-negative integers l and m, we have*

$$\sum_{\substack{m_0 + \cdots + m_{2l} = m \\ m_i \geq 0\, (0 \leq i \leq 2l)}} \zeta(\{2\}^{m_0}, 1, \{2\}^{m_1}, 3, \{2\}^{m_2}, \ldots, \{2\}^{m_{2l-2}}, 1, \{2\}^{m_{2l-1}}, 3, \{2\}^{m_{2l}})$$
$$= \binom{2l+m}{2l} \frac{\pi^{4l+2m}}{(2l+1) \cdot (4l+2m+1)!}.$$

A similar result for MZSVs is known by Kondo-Saito-Tanaka [4] and Yamamoto [11], i.e. the similar sum for MZSVs is also a rational multiple of π^{4l+2m}.

2010 Mathematics Subject Classification. 11M32.

The third author is supported in part by the Grant-in-Aid for JSPS Fellows (JP18J00151), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

Let us consider counterparts of these results for finite multiple zeta values. For a positive integer n, we define the $\mathbb{Q}$-algebra $\mathcal{A}_n$ by

$$\mathcal{A}_n := \Bigl(\prod_p \mathbb{Z}/p^n\mathbb{Z}\Bigr) \Big/ \Bigl(\bigoplus_p \mathbb{Z}/p^n\mathbb{Z}\Bigr),$$

where p runs over prime numbers. For positive integers $k_1, \ldots, k_r$ and n, the finite multiple zeta values (FMZVs) and the finite multiple zeta-star values (FMZSVs) in $\mathcal{A}_n$ are defined by

$$\zeta_{\mathcal{A}_n}(k_1, \ldots, k_r) := \Biggl(\sum_{1\le n_1<\cdots<n_r\le p-1} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \bmod p^n\Biggr)_p \in \mathcal{A}_n,$$

$$\zeta^{\star}_{\mathcal{A}_n}(k_1, \ldots, k_r) := \Biggl(\sum_{1\le n_1\le\cdots\le n_r\le p-1} \frac{1}{n_1^{k_1}\cdots n_r^{k_r}} \bmod p^n\Biggr)_p \in \mathcal{A}_n.$$

We set $\zeta_{\mathcal{A}_n}(\varnothing) = \zeta^{\star}_{\mathcal{A}_n}(\varnothing) = 1$. For details, see Rosen [6] and Seki [10]. Recently, Saito and Wakabayashi [7] obtained Bowman-Bradley type results in a strong sense for finite multiple zeta values in $\mathcal{A}_1$. The following is a part of their results:

Theorem 1.2 (Saito-Wakabayashi [7, Theorem 1.4])**.** *Let a and b be odd positive integers and c an even positive integer. For non-negative integers l and m with $(l, m) \neq (0, 0)$, we have*

$$\begin{aligned}
&\sum_{\substack{m_0+\cdots+m_{2l}=m\\ m_i\ge 0\,(0\le i\le 2l)}} \zeta_{\mathcal{A}_1}(\{c\}^{m_0}, a, \{c\}^{m_1}, b, \{c\}^{m_2}, \ldots, \{c\}^{m_{2l-2}}, a, \{c\}^{m_{2l-1}}, b, \{c\}^{m_{2l}})\\
&= \sum_{\substack{m_0+\cdots+m_{2l}=m\\ m_i\ge 0\,(0\le i\le 2l)}} \zeta^{\star}_{\mathcal{A}_1}(\{c\}^{m_0}, a, \{c\}^{m_1}, b, \{c\}^{m_2}, \ldots, \{c\}^{m_{2l-2}}, a, \{c\}^{m_{2l-1}}, b, \{c\}^{m_{2l}})\\
&= 0.
\end{aligned}$$

In this paper, we partially lift Saito-Wakabayashi's result from $\mathcal{A}_1$ to $\mathcal{A}_2$. In fact, we show that the Bowman-Bradley type sum of FMZ(S)Vs in $\mathcal{A}_2$ for the shuffle of $\{1, 3\}^l$ and $\{2\}^m$ is a rational multiple of the special element $\beta_{4l+2m+1}\boldsymbol{p}$. Here, $\boldsymbol{p}$ and β_k are defined to be $(p \bmod p^2)_p$ and $(B_{p-k}/k \bmod p^2)_p$ as elements of $\mathcal{A}_2$, respectively, where B_n is the nth Seki-Bernoulli number and k is an integer greater than 1. Then, our main theorem is the following:

Theorem 1.3 (Main theorem)**.** *For non-negative integers l and m with $(l, m) \neq (0, 0)$, we have*

$$\tag{1}\begin{aligned}
&\sum_{\substack{m_0+\cdots+m_{2l}=m\\ m_i\ge 0\,(0\le i\le 2l)}} \zeta_{\mathcal{A}_2}(\{2\}^{m_0}, 1, \{2\}^{m_1}, 3, \{2\}^{m_2}, \ldots, \{2\}^{m_{2l-2}}, 1, \{2\}^{m_{2l-1}}, 3, \{2\}^{m_{2l}})\\
&= (-1)^m\biggl\{(-1)^l 2^{1-2l}\binom{l+m}{l} - 4\binom{2l+m}{2l}\biggr\}\beta_{4l+2m+1}\boldsymbol{p},
\end{aligned}$$

$$\tag{2}\begin{aligned}
&\sum_{\substack{m_0+\cdots+m_{2l}=m\\ m_i\ge 0\,(0\le i\le 2l)}} \zeta^{\star}_{\mathcal{A}_2}(\{2\}^{m_0}, 1, \{2\}^{m_1}, 3, \{2\}^{m_2}, \ldots, \{2\}^{m_{2l-2}}, 1, \{2\}^{m_{2l-1}}, 3, \{2\}^{m_{2l}})\\
&= (-1)^l 2^{1-2l}\binom{l+m}{l}\beta_{4l+2m+1}\boldsymbol{p}.
\end{aligned}$$

Saito-Wakabayashi's theorem (Theorem 1.2) says that the sum of FMZ(S)Vs in $\mathcal{A}_1$ for the shuffle of $\{a,b\}^l$ and $\{c\}^m$ is zero for any odd positive integers a,b and any even positive integer c. On the other hand, by our computer calculations, it seems that the similar sum of FMZ(S)Vs in $\mathcal{A}_2$ *is not* a rational multiple of $\beta_{(a+b)l+cm+1}\boldsymbol{p}$, generally. For example, it is probable that $\zeta_{\mathcal{A}_2}(1,5,1,5)$ is not a rational multiple of $\beta_{13}\boldsymbol{p}$.

Zhao conjectures that the dimension of the $\mathbb{Q}$-vector space spanned by MZVs of weight k coincides with the dimension of the $\mathbb{Q}$-vector space spanned by FMZVs in $\mathcal{A}_2$ of weight k ([12, Conjecture 9.6]). However, this conjecture doesn't mean that a correspondence $\zeta(k_1,\dots,k_r) \mapsto \zeta_{\mathcal{A}_2}(k_1,\dots,k_r)$ gives an isomorphism between these two spaces. In this situation, it is worth emphasizing that there exists a similarity between Bowman-Bradley type theorems for MZ(S)Vs and FMZ(S)Vs in $\mathcal{A}_2$, i.e. the sum of MZ(S)Vs for the shuffle of $\{1,3\}^l$ and $\{2\}^m$ is a rational multiple of π^{4l+2m} and the similar sum of FMZ(S)Vs in $\mathcal{A}_2$ is a rational multiple of $\beta_{4l+2m+1}\boldsymbol{p}$. Note that these two rational coefficients are different.

We prove our main theorem in §2 and §3.

2. Preliminaries

We prepare some notation and lemmas in this section. Let $\mathfrak{H}^1$ be the Hoffman algebra $\mathbb{Q} + \mathbb{Q}\langle x,y\rangle y$. We define two kinds of shuffle products $ш$ and $\widetilde{ш}$ on $\mathfrak{H}^1$ as in [5, §2]. We call a tuple of positive integers an index. Let $\mathfrak{R} = \bigoplus_{r=0}^{\infty} \mathbb{Q}[\mathbb{Z}_{>0}^r]$ be the $\mathbb{Q}$-vector space spanned by all indices. Then, we use the same notation $ш$ and $\widetilde{ш}$ on $\mathfrak{R}$ by the correspondence $(k_1,\dots,k_r) \mapsto x^{k_r-1}y\cdots x^{k_1-1}y$ between $\mathfrak{R}$ and $\mathfrak{H}^1$. Note that, for the definition of MZVs, the order of indices in [5] is reverse to ours. For example, $(1,2)\,ш\,(1) = 3(1,1,2)+(1,2,1)$ and $(2,3)\,\widetilde{ш}\,(1) = (1,2,3)+(2,1,3)+(2,3,1)$. Then, the summations in Theorem 1.3 are written as $\zeta_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{ш}\,(\{2\}^m))$ and $\zeta^\star_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{ш}\,(\{2\}^m))$, respectively. Here, we extend $\zeta_{\mathcal{A}_2}$ and $\zeta^\star_{\mathcal{A}_2}$ to functions on $\mathfrak{R}$, linearly.

Lemma 2.1. *For non-negative integers l and m, we have*

$$4^l\left\{(\{1,3\}^l)\,\widetilde{ш}\,(\{2\}^m)\right\} = (\{2\}^{l+m})\,ш\,(\{2\}^l) - \sum_{k=0}^{l-1} 4^k\binom{2l+m-2k}{l-k}\left\{(\{1,3\}^k)\,\widetilde{ш}\,(\{2\}^{2l+m-2k})\right\}.$$

Proof. This follows from [5, Proposition 2 (1)]. □

The following lemma is the shuffle relation for FMZVs in $\mathcal{A}_2$.

Lemma 2.2. *For indices $\boldsymbol{k}$ and $\boldsymbol{l} = (l_1,\dots,l_s)$, we have*

$$\begin{aligned}&\zeta_{\mathcal{A}_2}(\boldsymbol{k}\,ш\,\boldsymbol{l})\\ &= (-1)^{l_1+\cdots+l_s}\sum_{\substack{e_1+\cdots+e_s=0,1\\ e_1,\dots,e_s\geq 0}}\prod_{j=1}^{s}\binom{l_j+e_j-1}{e_j}\zeta_{\mathcal{A}_2}(\boldsymbol{k},l_s+e_s,\dots,l_1+e_1)\boldsymbol{p}^{e_1+\cdots+e_s}.\end{aligned}$$

Proof. This follows from [9, Theorem 6.4] which is also proved independently by Jarossay in [3, Lemma 4.17] by taking $\varprojlim_n \mathcal{A}_n \twoheadrightarrow \mathcal{A}_2$. □

Lemma 2.3. *For a positive integer r, we have*

$$\zeta_{\mathcal{A}_2}(\{2\}^r) = (-1)^{r-1} 2\beta_{2r+1}\boldsymbol{p}, \tag{3}$$

$$\zeta^{\star}_{\mathcal{A}_2}(\{2\}^r) = 2\beta_{2r+1}\boldsymbol{p}. \tag{4}$$

Proof. The equality (3) is a special case of the second congruence in the last remark of [13]. The equality (4) is obtained by (3) and [8, Corollary 3.16 (42)]. □

Lemma 2.4 (Hessami Pilehrood-Hessami Pilehrood-Tauraso [2, Theorem 4.1])**.** *For non-negative integers a and b, we have*

$$\zeta_{\mathcal{A}_1}(\{2\}^a, 3, \{2\}^b) = \frac{(-1)^{a+b} 2(a-b)}{a+1} \binom{2a+2b+3}{2b+2} \beta_{2a+2b+3}.$$

Here, we regard $\beta_{2a+2b+3}$ as an element of $\mathcal{A}_1$ by the projection $\mathcal{A}_2 \twoheadrightarrow \mathcal{A}_1$.

Lemma 2.5. *For non-negative integers l and m with $(l, m) \neq (0, 0)$, we have*

$$\zeta_{\mathcal{A}_2}((\{2\}^{l+m}) ⧢ (\{2\}^l)) = (-1)^m 2 \left\{ 1 - 2\binom{4l+2m}{2l} \right\} \beta_{4l+2m+1}\boldsymbol{p}.$$

Proof. By Lemma 2.2, 2.3 (3), and 2.4, we have

$$\begin{aligned}
&\zeta_{\mathcal{A}_2}((\{2\}^{l+m}) ⧢ (\{2\}^l)) \\
&= \zeta_{\mathcal{A}_2}(\{2\}^{2l+m}) + 2\sum_{j=0}^{l-1} \zeta_{\mathcal{A}_2}(\{2\}^{l+m+j}, 3, \{2\}^{l-j-1})\boldsymbol{p} \\
&= (-1)^{m-1} \left\{ 2\beta_{4l+2m+1}\boldsymbol{p} + 4\sum_{j=0}^{l-1} \frac{m+2j+1}{l+m+j+1} \binom{4l+2m+1}{2l-2j} \beta_{4l+2m+1}\boldsymbol{p} \right\}.
\end{aligned}$$

Since $\frac{a-2b}{a}\binom{a}{b} = \binom{a-1}{b} - \binom{a-1}{b-1}$, by putting $a = 4l+2m+2$ and $b = 2j$, we have

$$\begin{aligned}
&\sum_{j=0}^{l} \frac{m+2j+1}{l+m+j+1} \binom{4l+2m+1}{2l-2j} = \sum_{j=0}^{l} \frac{2l+m-2j+1}{2l+m+1} \binom{4l+2m+2}{2j} \\
&= \sum_{j=0}^{l} \left\{ \binom{4l+2m+1}{2j} - \binom{4l+2m+1}{2j-1} \right\} = \sum_{j=0}^{2l} (-1)^j \binom{4l+2m+1}{j} \\
&= \sum_{j=0}^{2l} (-1)^j \left\{ \binom{4l+2m}{j} + \binom{4l+2m}{j-1} \right\} = \binom{4l+2m}{2l}.
\end{aligned}$$

Hence, we obtain the desired formula. □

Lemma 2.6. *For non-negative integers l and m, we have*

$$\sum_{k=0}^{l} (-1)^k \binom{2l+m-2k}{l-k} \binom{2l+m-k}{k} = 1,$$

$$\sum_{k=0}^{l} 4^k \binom{2l+m-2k}{l-k} \binom{2l+m}{2k} = \binom{4l+2m}{2l}.$$

Proof. Since $\binom{a-b}{c-b}\binom{a}{b} = (-1)^{a-c}\binom{c}{b}\binom{-c-1}{a-c}$, by putting $a = 2l + m - k$, $b = k$, and $c = l$, we have

$$\sum_{k=0}^{l}(-1)^k\binom{2l+m-2k}{l-k}\binom{2l+m-k}{k}$$
$$= (-1)^{l+m}\sum_{k=0}^{l+m}\binom{l}{k}\binom{-l-1}{l+m-k} = (-1)^{l+m}\binom{-1}{l+m} = \binom{l+m}{l+m} = 1$$

by the Chu-Vandermonde identity. Next, we prove the second equality. Let $\binom{n}{a,b,c} := n!/(a!b!c!)$. Since

$$(1+Y)^{4l+2m} = (1+2Y+Y^2)^{2l+m} = \sum_{\substack{a+b+c=2l+m\\ a,b,c\geq 0}}\binom{2l+m}{a,b,c}(2Y)^bY^{2c}$$

holds, by comparing the coefficient of Y^{2l}, we have

$$\binom{4l+2m}{2l} = \sum_{j=0}^{l}\binom{2l+m}{j+m, 2l-2j, j}2^{2l-2j} = \sum_{k=0}^{l}4^k\binom{2l+m-2k}{l-k}\binom{2l+m}{2k}.$$

This concludes the proof. □

Lemma 2.7. *For non-negative integers l and m, we have*

$$\zeta^{\star}_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{ш}\,(\{2\}^m))$$
$$= \sum_{\substack{2i+k+u=2l\\ j+n+v=m}}(-1)^{j+k}\binom{k+n}{k}\binom{u+v}{u}\zeta_{\mathcal{A}_2}((\{1,3\}^i)\,\widetilde{ш}\,(\{2\}^j))\zeta^{\star}_{\mathcal{A}_2}(\{2\}^{k+n})\zeta^{\star}_{\mathcal{A}_2}(\{2\}^{u+v}),$$

where parameters i, j, k, n, u, v are non-negative integers.

Proof. This follows from [11, Theorem 2.1]. □

3. Proof of the main theorem

Proof of Theorem 1.3. First, we prove (1) by induction on l. We see that the case $l = 0$ holds by Lemma 2.3 (3). For the general case, let l be a positive integer and m a non-negative integer. By Lemma 2.1, we have

$$\zeta_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{ш}\,(\{2\}^m))$$
$$= 4^{-l}\zeta_{\mathcal{A}_2}((\{2\}^{l+m})\,ш\,(\{2\}^l)) - \sum_{k=0}^{l-1}4^{k-l}\binom{2l+m-2k}{l-k}\zeta_{\mathcal{A}_2}((\{1,3\}^k)\,\widetilde{ш}\,(\{2\}^{2l+m-2k})).$$

Hence, by Lemma 2.5 and the induction hypothesis, we have

$$\begin{aligned}
&\zeta_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{\text{ш}}\,(\{2\}^m))\\
&= (-1)^m 2^{1-2l}\left\{1-2\binom{4l+2m}{2l}\right\}\beta_{4l+2m+1}\boldsymbol{p}\\
&\quad -\sum_{k=0}^{l-1}4^{k-l}\binom{2l+m-2k}{l-k}\cdot(-1)^m\left\{(-1)^k 2^{1-2k}\binom{2l+m-k}{k}-4\binom{2l+m}{2k}\right\}\beta_{4l+2m+1}\boldsymbol{p}.
\end{aligned}$$

By Lemma 2.6, we can simplify as

$$\begin{aligned}
\zeta_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{\text{ш}}\,(\{2\}^m)) &= (-1)^m 2^{1-2l}\left\{1-2\binom{4l+2m}{2l}\right\}\beta_{4l+2m+1}\boldsymbol{p}\\
&\quad -(-1)^m 2^{1-2l}\left\{1-(-1)^l\binom{l+m}{l}\right\}\beta_{4l+2m+1}\boldsymbol{p}\\
&\quad +(-1)^m 4^{1-l}\left\{\binom{4l+2m}{2l}-4^l\binom{2l+m}{2l}\right\}\beta_{4l+2m+1}\boldsymbol{p}\\
&= (-1)^m\left\{(-1)^l 2^{1-2l}\binom{l+m}{l}-4\binom{2l+m}{2l}\right\}\beta_{4l+2m+1}\boldsymbol{p}.
\end{aligned}$$

Next, we prove (2). By the equality (1) and Lemma 2.3 (4), many terms in the right-hand side of the equality in Lemma 2.7 vanish and we have

$$\begin{aligned}
&\zeta^{\star}_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{\text{ш}}\,(\{2\}^m))\\
&= (-1)^m\zeta_{\mathcal{A}_2}((\{1,3\}^l)\,\widetilde{\text{ш}}\,(\{2\}^m))+2\binom{2l+m}{2l}\zeta^{\star}_{\mathcal{A}_2}(\{2\}^{2l+m})\\
&= (-1)^l 2^{1-2l}\binom{l+m}{l}\beta_{4l+2m+1}\boldsymbol{p}.
\end{aligned}$$

This finishes the proof. □

Acknowledgements. The authors would like to thank Doctor Hisatoshi Kodani for valuable comments. They also would like to express their gratitude to the anonymous referee for useful suggestions.

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