Kodama, A. and Shimizu, S.
Osaka J. Math.
56 (2019), 739–757

TWO THEOREMS ON THE FOCK-BARGMANN-HARTOGS DOMAINS

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(Received April 16, 2018)

Abstract

In this paper, we prove two mutually independent theorems on the family of Fock-Bargmann-Hartogs domains. Let D_1 and D_2 be two Fock-Bargmann-Hartogs domains in $\mathbb{C}^{N_1}$ and $\mathbb{C}^{N_2}$, respectively. In Theorem 1, we obtain a complete description of an arbitrarily given proper holomorphic mapping between D_1 and D_2 in the case where $N_1 = N_2$. Also, we shall give a geometric interpretation of Theorem 1. And, in Theorem 2, we determine the structure of $\mathrm{Aut}(D_1 \times D_2)$ using the data of $\mathrm{Aut}(D_1)$ and $\mathrm{Aut}(D_2)$ for arbitrary N_1 and N_2.

1. Introduction

This is a continuation of our previous paper [9], and we retain the terminology and notation there.

In this paper, we prove two mutually independent theorems on the Fock-Bargmann-Hartogs domains

$$D_{n,m}(\mu) = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}^m \,;\, \|w\|^2 < e^{-\mu\|z\|^2}\right\}$$

in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$ introduced by Yamamori [16].

Let $D_1 = D_{n_1,m_1}(\mu_1)$, $D_2 = D_{n_2,m_2}(\mu_2)$ be two equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ and let $f : D_1 \to D_2$ be a proper holomorphic mapping. Then we know the following:

(†) *$f : D_1 \to D_2$ is necessarily a biholomorphic mapping from D_1 onto D_2, provided that $m_1 \geq 2$.*

In view of this, it would be natural to ask what happens when $m_1 = 1$. One of the main purposes of this paper is to clear up this matter. In fact, in our first Theorem 1, we clarify the structure of the space consisting of all proper holomorphic mappings between two equidimensional Fock-Bargmann-Hartogs domains. By the way, the fact (†) was first proved by Tu-Wang [15; Theorem 1.1]. After that, Kodama [9; Theorem 2] gave an alternative proof. In their proofs, it was a key point to verify that the complex Jacobian determinant J_f of f does not vanish everywhere on D_1. For the verification of this, Tu-Wang used some known facts in algebraic geometry and Kodama employed a technique in Lie group theory. Anyway, both the proofs are a little bit long and complicated. Taking this into account, we give

2010 Mathematics Subject Classification. Primary 32A07; Secondary 32M05.

*The second author is supported by Grant-in-aid for Science Research (C) 15K04913, Japan Society for the Promotion of Science.

a new proof of (†) in Theorem 1, which is a very short and simple one by making use of the well-known maximum principle for plurisubharmonic functions defined on a connected complex analytic subvariety of $\mathbb{C}^N$.

In order to state our precise result, let us here introduce the holomorphic self-mapping ρ_k of $\mathbb{C}^n \times \mathbb{C}$ given by

$$\rho_k(z, w) = \left(\sqrt{k}z, w^k\right) \quad \text{for } (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

where $k = 1, 2, \ldots$. It is obvious that the restriction of ρ_k to the Fock-Bargmann-Hartogs domain $D_{n,1}(\mu)$, say again ρ_k, gives rise to a proper holomorphic self-mapping of $D_{n,1}(\mu)$; and moreover, it is not an automorphism of $D_{n,1}(\mu)$ unless $k = 1$, i.e., $\rho_1 = \mathrm{id}_{D_{n,1}(\mu)}$. For given positive real numbers μ, ν, we also define the non-singular linear mapping

$$L_{\mu,\nu} : \mathbb{C}^n \times \mathbb{C}^m \to \mathbb{C}^n \times \mathbb{C}^m \quad \text{by} \quad L_{\mu,\nu}(z, w) = \left(\sqrt{\mu/\nu}z, w\right)$$

for $(z, w) \in \mathbb{C}^n \times \mathbb{C}^m$. Then $L_{\mu,\nu}$ induces a biholomorphic mapping, denoted by the same notation, $L_{\mu,\nu} : D_{n,m}(\mu) \to D_{n,m}(\nu)$.

With these notations, our first result can be stated as follows:

Theorem 1. *Let $D_1 = D_{n_1,m_1}(\mu_1)$, $D_2 = D_{n_2,m_2}(\mu_2)$ be two equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ and let $f : D_1 \to D_2$ be a proper holomorphic mapping. Then we have the following*:

(I) *If $m_1 \geq 2$, then $f : D_1 \to D_2$ is necessarily a biholomorphic mapping from D_1 onto D_2. Moreover, we have $(n_1, m_1) = (n_2, m_2)$ in this case and, by putting $(n, m) = (n_j, m_j)$, $D_j = D_{n,m}(\mu_j)$ for $j = 1, 2$, $f : D_1 \to D_2$ can be written in the form*

$$f = g \circ L_{\mu_1,\mu_2} \quad \text{with some } g \in \mathrm{Aut}(D_2).$$

(II) *If $m_1 = 1$, then $m_2 = 1$ and hence $n_1 = n_2$. Moreover, by putting $n = n_j$, $D_j = D_{n,1}(\mu_j)$ for $j = 1, 2$, $f : D_1 \to D_2$ can be written in the form*

$$f = g \circ \rho_k \circ L_{\mu_1,\mu_2} \quad \text{with some } k \in \mathbb{N} \text{ and some } g \in \mathrm{Aut}(D_2).$$

In particular, $f : D_1 \to D_2$ is a biholomorphic mapping if and only if $k = 1$.

Therefore, together with the result in [6] (see Fact A in the next section), Theorem 1 gives us an explicit expression of any proper holomorphic mapping between two equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$.

Here we shall give a geometric interpretation of Theorem 1. (For the detailed arguments, see Section 3.) For the given two Fock-Bargmann-Hartogs domains $D_j = D_{n_j,m_j}(\mu_j)$ in $\mathbb{C}^N$ for $j = 1, 2$, let $C(D_1, \mathbb{C}^N)$ be the set of all continuous mappings from D_1 to $\mathbb{C}^N$ equipped with the compact-open topology. Note that in our case the compact-open topology coincides with the topology of uniform convergence on compact sets in D_1. Moreover, $C(D_1, \mathbb{C}^N)$ is a Hausdorff space satisfying the second axiom of countability. Now, let us denote by

$B(D_1, D_2)$ the set of all biholomorphic mappings from D_1 onto D_2; and

$P(D_1, D_2)$ the set of all proper holomorphic mappings from D_1 to D_2.

Then we have the natural inclusions: $B(D_1, D_2) \subset P(D_1, D_2) \subset C(D_1, \mathbb{C}^N)$. From now on, we assume that $P(D_1, D_2) \neq \emptyset$ and we always consider $P(D_1, D_2)$ as well as $B(D_1, D_2)$ as a topological space in the topology induced from that of $C(D_1, \mathbb{C}^N)$. Thus $B(D_1, D_2)$ and $P(D_1, D_2)$ are also Hausdorff spaces satisfying the second axiom of countability. Notice here that $\mathrm{Aut}(D_2)$ acts continuously on $P(D_1, D_2)$ by the natural action-mapping

$$\Phi : \mathrm{Aut}(D_2) \times P(D_1, D_2) \to P(D_1, D_2) \quad \text{given by} \quad \Phi(f, p) = f \cdot p$$

for $f \in \mathrm{Aut}(D_2)$ and $p \in P(D_1, D_2)$, where $f \cdot p$ is of course the composite mapping of f and p. It then follows immediately from Theorem 1 that $B(D_1, D_2)$ is just the $\mathrm{Aut}(D_2)$-orbit passing through the point $L_{\mu_1,\mu_2} \in P(D_1, D_2)$. Moreover, in the case where $m_1 = 1$, let us put, for $k = 1, 2, \ldots$,

$$p_k = \rho_k \circ L_{\mu_1,\mu_2} \quad \text{and} \quad P_k = \mathrm{Aut}(D_2) \cdot p_k,$$

the $\mathrm{Aut}(D_2)$-orbit passing through the point $p_k \in P(D_1, D_2)$. Then our Theorem 1 can be interpreted as follows: Each orbit P_k is open and closed in $P(D_1, D_2)$ and P_k with the relative topology from $P(D_1, D_2)$ is homeomorphic to the connected Lie group $\mathrm{Aut}(D_2)$. In particular, P_k is the connected component of $P(D_1, D_2)$ containing the point p_k and the space $P(D_1, D_2)$ can be decomposed into the connected components P_k:

$$P(D_1, D_2) = \begin{cases} B(D_1, D_2) & \text{if} \quad m_1 \geq 2, \\ \bigcup_{k=1}^{\infty} P_k & \text{if} \quad m_1 = 1. \end{cases}$$

Moreover, it can be seen that $P(D_1, D_2)$ is closed in $C(D_1, \mathbb{C}^N)$. Thus, considering the special case where $D_1 = D_2$, we have the following:

Let D be an arbitrary Fock-Bargmann-Hartogs domain in $\mathbb{C}^N$ and let $\{f_\nu\}$ be a sequence of proper holomorphic self-mappings of D. Assume that $\{f_\nu\}$ converges uniformly on every compact set in D to a mapping $f : D \to \mathbb{C}^N$. Then f is necessarily a proper holomorphic self-mapping of D. In particular, let $\{\varphi_\nu\}$ be a sequence in $\mathrm{Aut}(D)$ *which converges uniformly on compact subsets of D to a mapping $\varphi : D \to \mathbb{C}^N$. Then φ is a holomorphic automorphism of D.*

This would be interesting when we recall the following well-known theorem of H. Cartan [3]: Let W be a bounded domain in $\mathbb{C}^N$ and let $\{\varphi_\nu\}$ be a sequence in $\mathrm{Aut}(W)$ which converges uniformly on compact subsets of W to a mapping $\varphi : W \to \mathbb{C}^N$. Then the following three properties are equivalent:

(a) $\varphi \in \mathrm{Aut}(W)$ (b) $\varphi(W) \nsubseteq \partial W$ (c) $J_\varphi(p) \neq 0$ at some point $p \in W$.

Next let us consider two arbitrary Fock-Bargmann-Hartogs domains D_1 and D_2. Then, how can we describe the structure of $\mathrm{Aut}(D_1 \times D_2)$ using the data of $\mathrm{Aut}(D_1)$ and $\mathrm{Aut}(D_2)$? In connection with this, Peters [11; Satz 3.4] proved the following fact, which is a generalization of the theorem of H. Cartan [4] proved for bounded domains: Let X and Y be connected hyperbolic complex spaces in the sense of Kobayashi. Then the natural isomorphism from $\mathrm{Aut}(X) \times \mathrm{Aut}(Y)$ into $\mathrm{Aut}(X \times Y)$ induces an isomorphism

$$(\ddagger) \qquad \mathrm{Aut}^o(X) \times \mathrm{Aut}^o(Y) \cong \mathrm{Aut}^o(X \times Y),$$

where $\mathrm{Aut}^o(*)$ denotes the identity component of $\mathrm{Aut}(*)$.

Our second purpose is to establish the following theorem, which tells us that the analogue of ($\ddagger$) is still valid for the Fock-Bargmann-Hartogs domains:

Theorem 2. *Let $D_j = D_{n_j,m_j}(\mu_j)$ be an arbitrary Fock-Bargmann-Hartogs domain in $\mathbb{C}^{N_j}$, where $N_j = n_j + m_j$, for $j = 1, 2$. Then we have the following*:

(I) *If $(n_1, m_1) = (n_2, m_2)$, then*

$$\mathrm{Aut}(D_1 \times D_2) = (\mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)) \cup \{I \circ f\,;\, f \in \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)\},$$

where I is the involutive automorphism of $D_1 \times D_2$ defined by

$$I(z_1, w_1, z_2, w_2) = (\sqrt{\mu_2/\mu_1}\, z_2, w_2, \sqrt{\mu_1/\mu_2}\, z_1, w_1) \quad on \;\; D_1 \times D_2$$

under the natural identification $(z_1, w_1, z_2, w_2) = ((z_1, w_1), (z_2, w_2)) \in D_1 \times D_2$.

(II) *If $(n_1, m_1) \neq (n_2, m_2)$, then*

$$\mathrm{Aut}(D_1 \times D_2) = \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2).$$

Therefore, for any Fock-Bargmann-Hartogs domain D_j in $\mathbb{C}^{N_j}$ for $j = 1, 2$, every holomorphic automorphism of $D_1 \times D_2$ can be described explicitly in terms of the natural coordinate system in $\mathbb{C}^{N_1} \times \mathbb{C}^{N_2}$.

Finally it should be remarked that, since the Fock-Bargmann-Hartogs domains contain non-trivial complex Euclidean spaces, our Theorem 2 is not an immediate consequence of Peters [11]. However, using efficiently the fact (‡) by Peters and some result on algebraic automorphisms of Reinhardt domains in $\mathbb{C}^N$ due to Shimizu [13; Section 3], we will be able to prove Theorem 2.

After some preparations in the next Section 2, we prove our Theorem 1 and give a geometric interpretation of Theorem 1 in Section 3. And, Theorem 2 will be proved in the final Section 4.

2. Preliminaries

Throughout this paper, we usually consider the elements ζ of $\mathbb{C}^N$ as the row vectors. However we also think of ζ as the column vectors, as the need arises.

In this section, we collect some basic concepts and results on the Fock-Bargmann-Hartogs domains and Reinhardt domains. For later purpose, we also recall the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathcal{E})$ of an elementary Siegel domain $\mathcal{E}$.

Let us start with recalling the structure of the Fock-Bargmann-Hartogs domain $D_{n,m}(\mu)$ in $\mathbb{C}^N = \mathbb{C}^n \times \mathbb{C}^m$. We set for a while

$$D = D_{n,m}(\mu), \quad \Delta_D = \{(z, w) \in D\,;\, w = 0\} \cong \mathbb{C}^n \;\text{ and }\; D^* = D \setminus \Delta_D.$$

Then we know that Δ_D coincides exactly with the degeneracy set

$$\{p \in D\,;\, d_D(p, q) = 0 \;\text{ for some point }\; q \neq p\}$$

for the Kobayashi pseudodistance d_D of D [9]. Hence, d_D induces a true distance on D^* and, in particular, D^* is hyperbolic in the sense of Kobayashi [7].

Concerning the automorphism group of D, we have the following result due to Kim-Ninh-Yamamori [6; Theorem 10]:

Fact A. *The automorphism group* Aut(D) *of the Fock-Bargmann-Hartogs domain D is generated by the following mappings*:

$$\varphi_A : (z, w) \mapsto (Az, w), \quad A \in U(n);$$
$$\varphi_B : (z, w) \mapsto (z, Bw), \quad B \in U(m);$$
$$\varphi_v : (z, w) \mapsto \left(z + v, e^{-\mu\langle z,v\rangle-(\mu/2)\|v\|^2} w\right), \quad v \in \mathbb{C}^n.$$

More precisely, every automorphism φ of D can be written as the composite mapping $\varphi = \varphi_v \circ \varphi_B \circ \varphi_A$ of automorphisms φ_A, φ_B and φ_v of the above type.

In particular, denoting by $\mathrm{Aut}(D) \cdot (z_o, w_o)$ the $\mathrm{Aut}(D)$-orbit passing through a given point $(z_o, w_o) \in D$, we have the following: $\mathrm{Aut}(D) \cdot (z_o, w_o)$ is a real analytic submanifold of D with

$$\dim_{\mathbb{R}} \mathrm{Aut}(D) \cdot (z_o, w_o) \;=\; \begin{cases} 2n & \text{if } \; w_o = 0, \\ 2(n+m) - 1 & \text{if } \; w_o \neq 0. \end{cases} \tag{2.1}$$

Let $T^N := (U(1))^N$ be the N-dimensional torus. Then T^N acts as a group of holomorphic automorphisms on $\mathbb{C}^N$ by the standard rule

$$\alpha \cdot \zeta \;=\; (\alpha_1\zeta_1, \dots, \alpha_N\zeta_N) \quad \text{for } \; \alpha = (\alpha_i) \in T^N, \; \zeta = (\zeta_i) \in \mathbb{C}^N.$$

Let D be an arbitrary Reinhardt domain in $\mathbb{C}^N$. Then, just by the definition, D is invariant under this action of T^N. Thus, each element $\alpha \in T^N$ induces an automorphism π_α of D given by $\pi_\alpha(\zeta) = \alpha \cdot \zeta$ for $\zeta \in D$, and the mapping ρ_D sending α to π_α is an injective continuous group homomorphism of T^N into $\mathrm{Aut}(D)$. The subgroup $\rho_D(T^N)$ of $\mathrm{Aut}(D)$ is denoted by $T(D)$. We have one more important topological subgroup $\mathrm{Aut}_{\mathrm{alg}}(D)$ of $\mathrm{Aut}(D)$ consisting of all elements φ of $\mathrm{Aut}(D)$ such that each component of φ is given by a Laurent monomial, that is, setting $\varphi = (\varphi_1, \dots, \varphi_N)$ by coordinates, φ_i are given by

$$\varphi_i(\zeta) = \alpha_i \zeta_1^{a_{i1}} \cdots \zeta_N^{a_{iN}}, \quad 1 \le i \le N,$$

where $(a_{ij}) \in GL(N, \mathbb{Z})$ and $(\alpha_i) \in (\mathbb{C}^*)^N$. We call $\mathrm{Aut}_{\mathrm{alg}}(D)$ the *algebraic automorphism group of D* and each element of $\mathrm{Aut}_{\mathrm{alg}}(D)$ is called an *algebraic automorphism of D*. It follows in particular from this definition that, if D contains the origin 0 of $\mathbb{C}^N$, then every algebraic automorphism φ of D reduces to a simple linear mapping of the form

$$\varphi(\zeta) \;=\; (\alpha_1\zeta_{\sigma(1)}, \dots, \alpha_N\zeta_{\sigma(N)}) \quad \text{for } \; \zeta = (\zeta_i) \in D, \tag{2.2}$$

where $(\alpha_i) \in (\mathbb{C}^*)^N$ and σ is a permutation of $\{1, \dots, N\}$. Moreover, concerning the algebraic automorphisms of Reinhardt domains in $\mathbb{C}^N$, we have the following result due to Shimizu [13; Section 3]:

Fact B. *Let φ be a holomorphic automorphism of a Reinhardt domain D in $\mathbb{C}^N$. Then φ is an algebraic automorphism of D if and only if φ has the property that $\varphi T(D)\varphi^{-1} = T(D)$.*

Next we recall the structure of the holomorphic automorphism group $\mathrm{Aut}(\mathcal{E})$ of the elementary Siegel domain

$$\mathcal{E} \;=\; \{(u, v) \in \mathbb{C} \times \mathbb{C}^n \,;\, \mathrm{Im}\, u - \|v\|^2 > 0\} \quad \text{in } \; \mathbb{C}^{n+1}.$$

This domain is holomorphically equivalent to the unit ball B^{n+1} in $\mathbb{C}^{n+1}$ via the correspondence $\phi : \mathcal{E} \to B^{n+1}$ given by

$$\phi(u,v) = \left(\frac{u-i}{u+i}, \frac{2v_1}{u+i}, \ldots, \frac{2v_n}{u+i}\right) \quad \text{for } (u,v) = (u, v_1, \ldots, v_n) \in \mathcal{E}. \tag{2.3}$$

Let $SU(n+1,1)$ be the special indefinite unitary group of signature $(n+1,1)$. Then it is well-known that every automorphism of B^{n+1} is a linear fractional transformation described by using some element of $SU(n+1,1)$, and we have $\mathrm{Aut}(\mathcal{E}) = \phi^{-1}\mathrm{Aut}(B^{n+1})\phi$. Hence, every automorphism F of $\mathcal{E}$ is also a linear fractional transformation of $\mathbb{C}^{n+1}$. In fact, expressing $F = (F_0, F_1, \ldots, F_n)$ with respect to the coordinate system $(u,v) = (u, v_1, \ldots, v_n)$ in $\mathbb{C}\times\mathbb{C}^n = \mathbb{C}^{n+1}$, we can see that each F_i has the form

$$F_i(u,v) = \frac{\alpha_{i0}u + \sum_{j=1}^n \alpha_{ij}v_j + \beta_i}{\gamma_0 u + \sum_{j=1}^n \gamma_j v_j + \delta}, \quad 0 \le i \le n; \tag{2.4}$$

where all the coefficients α_{ij}, β_i, γ_i $(0 \le i, j \le n)$ and δ are suitable complex constants. (For the precise description of $F \in \mathrm{Aut}(\mathcal{E})$, see [8; Section 3].) Let $\mathrm{Aff}(\mathbb{C}^{n+1})$ be the Lie group consisting of all non-singular complex affine transformations of $\mathbb{C}^{n+1}$ and set

$$\mathrm{Aff}(\mathcal{E}) = \{F \in \mathrm{Aff}(\mathbb{C}^{n+1})\,;\, F(\mathcal{E}) = \mathcal{E}\}.$$

Then $\mathrm{Aff}(\mathcal{E})$ is a closed subgroup of $\mathrm{Aff}(\mathbb{C}^{n+1})$. We call $\mathrm{Aff}(\mathcal{E})$ the *affine automorphism group of $\mathcal{E}$* and each element of $\mathrm{Aff}(\mathcal{E})$ is called an *affine automorphism of $\mathcal{E}$*. As for the group $\mathrm{Aff}(\mathcal{E})$, we know the following (cf. [10; Section 2]):

FACT C. *Every affine automorphism F of the elementary Siegel domain $\mathcal{E}$ in $\mathbb{C}\times\mathbb{C}^n$ can be written in the form*

$$F(u,v) = \left(ku + a + 2i\langle Bv, b\rangle + i\|b\|^2, Bv + b\right) \quad \text{for } (u,v) \in \mathcal{E},$$

where $a \in \mathbb{R}$, $b \in \mathbb{C}^n$ and $0 < k \in \mathbb{R}$, $B \in GL(n,\mathbb{C})$ with $k\|v\|^2 = \|Bv\|^2$ for all $v \in \mathbb{C}^n$ or $(1/\sqrt{k})B \in U(n)$.

3. Proof of Theorem 1 and a geometric interpretation of Theorem 1

Throughout this section, we denote by D_1, D_2 the equidimensional Fock-Bargmann-Hartogs domains in $\mathbb{C}^N$ and let $f : D_1 \to D_2$ be a proper holomorphic mapping as in Theorem 1.

3.1. Proof of Theorem 1. First of all, by a result of Tu-Wang [15; Theorem 2.5], f extends holomorphically to a connected open neighborhood W of $\overline{D_1}$, the closure of D_1 in $\mathbb{C}^N$. We set

$$V_f = D_1 \cap \{\zeta \in W\,;\, J_f(\zeta) = 0\}.$$

To prove the assertion (I) of Theorem 1, we assume that $m_1 \ge 2$. Once it is shown that $f : D_1 \to D_2$ is a biholomorphic mapping, it follows from Tu-Wang [15; Theorem 1.2] or Kodama [9; Fact 5] that $(n_1, m_1) = (n_2, m_2)$ and f has the form $f = g \circ L_{\mu_1,\mu_2}$ with some $g \in \mathrm{Aut}(D_2)$. Thus, in order to complete the proof of the assertion (I), it suffices to show that $f : D_1 \to D_2$ is biholomorphic. To this end, note that D_2 is a simply connected domain in $\mathbb{C}^N$. Then we have only to verify the following:

Lemma 1. *The set V_f is contained in Δ_{D_1}. In particular, if $m_1 \geq 2$, then $V_f = \emptyset$. Moreover, if $m_1 = 1$ and $V_f \neq \emptyset$, then $V_f = \Delta_{D_1}$.*

Proof. For the verification of the first assertion, we may assume that $V_f \neq \emptyset$; and so V_f is a complex analytic subvariety of D_1 of $\dim_{\mathbb{C}} V_f = N - 1 > 0$. Moreover, $\overline{V_f} \cap \partial D_1 = \emptyset$ by the same method as in the proof of [2; Theorem 2] or [12; Lemma 1.3]; accordingly, V_f may be regarded as a closed complex analytic subvariety of $\mathbb{C}^N$ contained in D_1.

Choosing an irreducible component V of V_f arbitrarily, we wish to show that $V \subset \Delta_{D_1}$. To this end, we introduce the function h on V given by

$$h(z, w) = \|w\|^2 \quad \text{for} \ \ (z, w) \in V.$$

Then h is a continuous plurisubharmonic function on V and

$$h(z, w) = \|w\|^2 \ < \ e^{-\mu_1\|z\|^2} \leq 1 \quad \text{for all} \ \ (z, w) \in V \subset D_1.$$

Once it is shown that $h(z, w) \equiv 0$ on V, we conclude that $V \subset \Delta_{D_1}$. Assume that there exists a point $\zeta_o = (z_o, w_o) \in V$ such that $h(\zeta_o) = \|w_o\|^2 \neq 0$. Then

$$0 \ < \ \|w_o\|^2 = h(\zeta_o) \ \leq \ \sup\{h(\zeta)\,;\, \zeta \in V\} =: M \ \leq \ 1;$$

and hence, there is a sequence $\zeta_\nu = (z_\nu, w_\nu) \in V$, $\nu = 1, 2, \ldots$, such that

$$\|w_o\|^2/2 \leq \|w_\nu\|^2 \leq M, \quad \nu = 1, 2, \ldots, \quad \text{and} \quad \lim_{\nu\to\infty} h(\zeta_\nu) = M.$$

Passing to a subsequence, if necessary, we may assume that $\{w_\nu\}_{\nu=1}^{\infty}$ converges to some point $w^* \in \mathbb{C}^{m_1}$ with $\|w^*\|^2 = M$. Moreover, we have

$$\|z_\nu\|^2 \ < \ (-1/\mu_1)\log\left(\|w_o\|^2/2\right) \ < \ +\infty, \quad \nu = 1, 2, \ldots,$$

because $\zeta_\nu = (z_\nu, w_\nu) \in V \subset D_1$. Thus, passing again to a subsequence, we may further assume that $\{\zeta_\nu\}_{\nu=1}^{\infty}$ converges to a point $\zeta^* = (z^*, w^*) \in \overline{V}$. Since V is now a closed subset of $\mathbb{C}^N$ contained in D_1, it then follows that

$$\zeta^* \in V \subset D_1 \quad \text{and} \quad h(\zeta^*) = M.$$

Consequently, $h(\zeta) \equiv h(\zeta_o) > 0$ on V by the maximum principle for plurisubharmonic functions on a closed connected complex analytic subvariety of $\mathbb{C}^N$ (cf. [5; Chapter IX]). Thus V is contained in the bounded subset $\{(z, w) \in D_1\,;\, \|w\| = \|w_o\|\}$ of $\mathbb{C}^N$. Accordingly, V is a compact, irreducible complex analytic subvariety of $\mathbb{C}^N$ contained in D_1; and hence, $V = \{\zeta_o\} \subset D_1$. But this contradicts the fact $\dim_{\mathbb{C}} V > 0$. As a result, we have shown that $h(\zeta) \equiv 0$ on V and so $V \subset \Delta_{D_1}$, as desired.

Next, consider the case where $m_1 \geq 2$. Then $\dim_{\mathbb{C}} \Delta_{D_1} = n_1 \leq N - 2$ and $\dim_{\mathbb{C}} V_f = N - 1$, provided that $V_f \neq \emptyset$. Hence, $V_f \subset \Delta_{D_1}$ can only happen when $V_f = \emptyset$. Moreover, if $m_1 = 1$ and $V_f \neq \emptyset$, then V_f is a complex analytic subvariety of $\Delta_{D_1} \cong \mathbb{C}^{n_1}$ with $\dim_{\mathbb{C}} V_f = n_1$; consequently, we conclude that $V_f = \Delta_{D_1}$, as asserted. □

Eventually we have completed the proof of the assertion (I) of Theorem 1.

Remark 1. With exactly the same argument as in the proof of Lemma 1, one can prove the following:

Proposition. *Let V be an irreducible complex analytic subvariety of the Fock-Bargmann-Hartogs domain $D = D_{n,m}(\mu)$ in $\mathbb{C}^N$ with $\dim_{\mathbb{C}} V > 0$ and $\overline{V} \subset D$. Then V is contained in Δ_D.*

Before undertaking the proof of (II), we show the following:

Lemma 2. *Assume that $m_j = 1$, $\mu_j = 1$ for $j = 1, 2$. Then, putting $n = n_j$, $D = D_j$ for $j = 1, 2$, we have $f(\Delta_D) = \Delta_D$ and $f(D^*) = D^*$, where $D^* = D \setminus \Delta_D$.*

Proof. If f is a holomorphic automorphism of D, then this lemma is an immediate consequence of the fact that the Kobayashi pseudodistance d_D is invariant under f and that Δ_D is just the degeneracy set for d_D.

We now consider the case where f is not a holomorphic automorphism of D. Then V_f is a complex analytic subvariety of D of $\dim_{\mathbb{C}} V_f = n$. In order to prove that $f(D^*) \subset D^*$, consider here the proper holomorphic mapping $F := f \circ f : D \to D$. Then, since Lemma 1 remains true for any proper holomorphic mapping from D_1 onto D_2, we have $V_f \subset V_F = \Delta_D$; and hence, $V_f = \Delta_D$. Assume that there exists a point $\zeta_o \in D^*$ such that $f(\zeta_o) \in \Delta_D$. Then $J_F(\zeta_o) = J_f(f(\zeta_o))J_f(\zeta_o) = 0$ and $\zeta_o \in V_F = \Delta_D$, a contradiction. Therefore we have $f(D^*) \subset D^*$.

Next we assert that $f(\Delta_D) = \Delta_D$. Indeed, assume that there exists a point $p \in \Delta_D$ with $f(p) \in D^*$. Then there exists a small open Euclidean ball $B(p)$ with center p such that $f(B(p)) \subset D^*$. Recall that the Kobayashi pseudodistance d_D of D is identically zero on Δ_D and d_D is a true distance on D^*. Then, by the distance-decreasing property of d_D under holomorphic mappings, we have

$$d_D(f(p), f(q)) \leq d_D(p, q) = 0 \quad \text{for all} \;\; q \in B(p) \cap \Delta_D;$$

which implies that $f(q) = f(p)$ for all $q \in B(p) \cap \Delta_D$. Thus $f(\Delta_D) = \{f(p)\}$ by analytic continuation. Anyway, in such a case, $f^{-1}(f(p))$ is not a finite subset of D. However this contradicts the fact that $f : D \to D$ is a proper holomorphic mapping. Therefore $f(\Delta_D) \subset \Delta_D$. Since $f(\Delta_D)$ is also a complex analytic subvariety of D of $\dim_{\mathbb{C}} f(\Delta_D) = n$ by Remmert's proper mapping theorem, we conclude that $f(\Delta_D) = \Delta_D$. Accordingly, we obtain that $f(D^*) = D^*$ because $f(D) = D$; proving the Lemma 2. □

We can now prove the assertion (II) of Theorem 1. First consider the case where $f : D_1 \to D_2$ is a biholomorphic mapping. It then follows that $f(\Delta_{D_1}) = \Delta_{D_2}$ and f induces a biholomorphic mapping from $\Delta_{D_1} \cong \mathbb{C}^{n_1}$ onto $\Delta_{D_2} \cong \mathbb{C}^{n_2}$ because the degeneracy sets for Kobayashi pseudodistances are invariant under biholomorphic mappings, in general. Hence $n_1 = n_2$ and so $m_2 = m_1 = 1$. Moreover, putting $n = n_j$ and $D_j = D_{n,1}(\mu_j)$ for $j = 1, 2$, we know by [15; Theorem 1.2] or [9; Fact 5] that f has the form $f = g \circ L_{\mu_1,\mu_2}$ with some $g \in \mathrm{Aut}(D_2)$. Therefore we obtain the assertion (II) in the case where $f : D_1 \to D_2$ is a biholomorphic mapping, since $\rho_1 = \mathrm{id}_{D_2}$.

Consider next the case where $f : D_1 \to D_2$ is not a biholomorphic mapping. To prove that $m_2 = 1$, assume to the contrary that $m_2 \geq 2$. Then $\dim_{\mathbb{C}} \Delta_{D_2} = n_2 \leq N - 2$. On the other hand, $f(\Delta_{D_1})$ is a complex analytic subvariety of D_2 of $\dim_{\mathbb{C}} f(\Delta_{D_1}) = n_1 = N - 1$ by Remmert's proper mapping theorem. Thus Δ_{D_2} is too small to contain $f(\Delta_{D_1})$; and hence, there exists a point $p \in \Delta_{D_1}$ with $f(p) \in D_2^* := D_2 \setminus \Delta_{D_2}$. Choose a small open Euclidean ball $B(p) \subset D_1$ with center p such that $f(B(p)) \subset D_2^*$. Then, with exactly the same argument as in the proof of Lemma 2, it can be seen that $f^{-1}(f(p))$ is not a finite subset of D_1, a contradiction. Therefore we conclude that $m_2 = 1$ and $n_1 = n_2$, as required.

From now on, we set $n = n_j$ for $j = 1, 2$. The proof will be divided into two cases as follows:

CASE 1. $(\mu_1, \mu_2) = (1, 1)$: In this case, we put $D = D_j$ for $j = 1, 2$ and use the following notation: For the domain $D^* = D \setminus \Delta_D$, we set

$$\mathcal{S} = \left\{(z, w) \in \mathbb{C}^n \times \mathbb{C}\,;\, |w|^2 e^{\|z\|^2} = 1\right\} = \partial D,$$

the subset of ∂D^* consisting of all C^ω-smooth strictly pseudoconvex boundary points of D^*. We also set

$$\mathcal{E} = \left\{(u, v) \in \mathbb{C} \times \mathbb{C}^n\,;\, \operatorname{Im} u - \|v\|^2 > 0\right\},$$

the elementary Siegel domain in $\mathbb{C}^{n+1}$. Note that $f(\mathcal{S}) \subset \mathcal{S}$ and f can be regarded as a proper holomorphic self-mapping of D^* by Lemma 2; and, via the mapping ϕ given in (2.3), $\mathcal{E}$ is now biholomorphically equivalent to the unit ball B^{n+1} in $\mathbb{C}^{n+1}$.

Consider here a holomorphic mapping ϖ from $\mathcal{E}$ into $\mathbb{C}^n \times \mathbb{C}^*$ defined by

$$\varpi(u, v) = \left(v, e^{iu/2}\right) \quad \text{for } (u, v) \in \mathcal{E}.$$

Then it is easily seen that $\varpi(\mathcal{E}) = D^*$ and $\mathcal{E}$ is the universal covering of D^* with the covering projection ϖ. Clearly, ϖ is, in fact, defined on $\mathbb{C} \times \mathbb{C}^n$ and $\varpi(\partial\mathcal{E}) = \mathcal{S}$.

Now, pick a point $p_1 \in \mathcal{S}$ arbitrarily and put $p_2 = f(p_1) \in \mathcal{S}$. Notice that $f|_{\mathcal{S}} \neq \mathrm{id}_{\mathcal{S}}$ because $f \neq \mathrm{id}_D$. Thus we may assume that $p_1 \neq p_2$. Let $q_1, q_2 \in \partial\mathcal{E}$ such that $\varpi(q_j) = p_j$ for $j = 1, 2$. Since $\overline{V_f} \cap \mathcal{S} = \emptyset$, there exist connected open neighborhoods V_1, V_2 of p_1, p_2, respectively, such that f gives rise to a biholomorphic mapping, say again f, from V_1 onto V_2. Moreover, since ϖ is a covering projection from $\mathbb{C} \times \mathbb{C}^n$ onto $\mathbb{C}^n \times \mathbb{C}^*$ with $\varpi(\partial\mathcal{E}) = \mathcal{S}$, we can find connected open neighborhoods W_1, W_2 of q_1, q_2, respectively, such that both the restrictions

$$\Pi_j := \varpi|_{W_j} : W_j \to V_j \quad \text{for } j = 1, 2,$$

are biholomorphic mappings, after shrinking V_1 sufficiently small, if necessary. Thus we obtain a biholomorphic mapping

$$F := \Pi_2^{-1} \circ f \circ \Pi_1 : W_1 \to W_2$$

with

$$F(W_1 \cap \mathcal{E}) = W_2 \cap \mathcal{E} \quad \text{and} \quad F(W_1 \cap \partial\mathcal{E}) = W_2 \cap \partial\mathcal{E}.$$

As an immediate consequence of the main result of Alexander [1], F now extends to a holomorphic automorphism, denoted by the same letter F, of $\mathcal{E}$. Therefore

$$\varpi(F(\xi)) = f(\varpi(\xi)) \quad \text{for all } \xi \in \mathcal{E} \tag{3.1}$$

by analytic continuation. Let us represent F as $F = (F_0, F_1, \ldots, F_n)$ with respect to the coordinate system $(u, v) = (u, v_1, \ldots, v_n)$ in $\mathbb{C} \times \mathbb{C}^n = \mathbb{C}^{n+1}$ and assume that F has the form written in (2.4). Note that

$$\varpi^{-1}(\varpi(\xi)) = \{(u+4\pi\nu, v)\,;\, \nu \in \mathbb{Z}\} \quad \text{for any } \ \xi = (u, v) \in \mathcal{E}.$$

Then the equation (3.1) tells us the following fact: For any point $\xi = (u, v) \in \mathcal{E}$ and any integer ν, there exists an integer $n(\xi, \nu)$ such that

$$\begin{aligned} F_0(u+4\pi\nu, v) &= F_0(u, v) + 4\pi n(\xi, \nu); \\ F_i(u+4\pi\nu, v) &= F_i(u, v), \quad 1 \le i \le n. \end{aligned} \tag{3.2}$$

Since F is an automorphism of $\mathcal{E}$, the integer $n(\xi, \nu)$ is uniquely determined by (ξ, ν) and depends continuously on $\xi \in \mathcal{E}$ for each fixed $\nu \in \mathbb{Z}$. Consequently, $n(\xi, \nu)$ is independent on ξ; and so, we may write $n(\xi, \nu) = n(\nu)$. Also, it is clear that $n(\nu) = 0$ if and only if $\nu = 0$. Moreover, since $\mathcal{E}$ is a complete hyperbolic manifold in the sense of Kobayashi [7], the closure of the set $\{\xi \in \mathcal{E}\,;\, d_{\mathcal{E}}(p, \xi) < r\}$ is a compact subset of $\mathcal{E}$ for all $p \in \mathcal{E}$ and all $0 < r \in \mathbb{R}$. Thus

$$|n(\nu)| \to +\infty \quad \text{if and only if} \quad |\nu| \to +\infty. \tag{3.3}$$

Now we assert that F is an affine automorphism of $\mathcal{E}$. Indeed, this can be verified as follows. If we set

$$g(u, v) = \gamma_0 u + \sum_{j=1}^{n} \gamma_j v_j + \delta \quad \text{and} \quad h_i(u, v) = \alpha_{i0} u + \sum_{j=1}^{n} \alpha_{ij} v_j + \beta_i$$

for $0 \le i \le n$, then $F_i(u, v) = h_i(u, v)/g(u, v)$ and by (3.2)

$$\begin{aligned} \frac{4\pi\nu \cdot \alpha_{00} + h_0(u, v)}{4\pi\nu \cdot \gamma_0 + g(u, v)} &= \frac{h_0(u, v)}{g(u, v)} + 4\pi n(\nu); \\ \frac{4\pi\nu \cdot \alpha_{i0} + h_i(u, v)}{4\pi\nu \cdot \gamma_0 + g(u, v)} &= \frac{h_i(u, v)}{g(u, v)}, \quad 1 \le i \le n; \end{aligned} \tag{3.4}$$

for any point $(u, v) \in \mathcal{E}$ and any integer ν.

If $\gamma_0 \neq 0$, then it follows from (3.4) that

$$\alpha_{i0}/\gamma_0 = F_i(u, v) \quad \text{on } \mathcal{E} \ \text{ for } \ 1 \le i \le n;$$

which contradicts the fact that F is an automorphism of $\mathcal{E}$. Thus $\gamma_0 = 0$. In this case, it follows at once that $\alpha_{i0} = 0$ for each $i = 1, \ldots, n$. Therefore, F_i does not depend on the variable u; accordingly, it has the form $F_i(u, v) = F_i(v)$ for every $1 \le i \le n$. Next, consider the first equation in (3.4). If $\alpha_{00} = 0$, then F does not depend on the variable u. But this is absurd because F is an automorphism of $\mathcal{E}$. Thus $\alpha_{00} \neq 0$ and

$$\sum_{j=1}^{n} \gamma_j v_j + \delta = \alpha_{00} \cdot \nu / n(\nu) \quad \text{on } \ \mathcal{E},$$

where ν is any integer with $\nu \neq 0$. Clearly, this can only happen when $\gamma_j = 0$ for all $1 \le j \le n$; and hence, $g(u, v) = \delta$ on $\mathcal{E}$ and F reduces to an affine automorphism of $\mathcal{E}$, as asserted.

Let us express the affine automorphism F of $\mathcal{E}$ as in Fact C in Section 2. Then, if we write $B = \sqrt{k}\widetilde{B}$ with $\widetilde{B} \in U(n)$, it follows from (3.1) that

$$f\left(v, e^{iu/2}\right) = \left(\widetilde{B}\sqrt{k}v + b, e^{-\langle \widetilde{B}\sqrt{k}v, b\rangle - (1/2)\|b\|^2} e^{(a/2)i}\left(e^{iu/2}\right)^k\right) \tag{3.5}$$

for all $(u, v) \in \mathcal{E}$. Moreover, since f is a single-valued holomorphic mapping defined on D, the positive real number k has to be an integer. With the notation as in Fact A, let us now introduce an automorphism g of D defined by

$$g = \varphi_b \circ \varphi_{e^{(a/2)i}E_1} \circ \varphi_{\widetilde{B}},$$

where $a \in \mathbb{R}$, $b \in \mathbb{C}^n$, $\widetilde{B} \in U(n)$ are the same objects appearing in (3.5) and E_1 denotes the identity matrix of degree one. Then the equation (3.5) can be rewritten as

$$f(z, w) = g \circ \rho_k \circ L_{1,1}(z, w) \quad \text{for all} \quad (z, w) \in D^*,$$

since $L_{1,1} = \mathrm{id}_D$. Therefore we conclude that $f = g \circ \rho_k \circ L_{1,1}$ on D by analytic continuation; thereby the proof of the assertion (II) of Theorem 1 is completed in Case 1.

CASE 2. $(\mu_1, \mu_2) \neq (1, 1)$: In this case, putting $D = D_{n,1}(1)$, let us consider the biholomorphic mapping $L_{\mu_j,1} : D_{n,1}(\mu_j) \to D$ for $j = 1, 2$ defined in the Introduction. Then the composite mapping

$$\tilde{f} := L_{\mu_2,1} \circ f \circ L_{\mu_1,1}^{-1} : D \to D$$

is a proper holomorphic self-mapping of D. Hence, $\tilde{f}$ can be written in the form

$$\tilde{f}(z, w) = \left(\widetilde{B}\sqrt{k}z + b, e^{-\langle \widetilde{B}\sqrt{k}z, b\rangle - (1/2)\|b\|^2} e^{(a/2)i} w^k\right) \quad \text{on} \quad D$$

as in (3.5); accordingly,

$$\begin{aligned} f(z, w) &= \left(\widetilde{B}\sqrt{k}z^* + b^*, e^{-\mu_2\langle \widetilde{B}\sqrt{k}z^*, b^*\rangle - (\mu_2/2)\|b^*\|^2} e^{(a/2)i} w^k\right) \\ &= g \circ \rho_k \circ L_{\mu_1,\mu_2}(z, w) \quad \text{on} \quad D_{n,1}(\mu_1), \end{aligned}$$

where we have put $z^* = \sqrt{\mu_1/\mu_2}\, z$, $b^* = b/\sqrt{\mu_2}$ and

$$g = \varphi_{b^*} \circ \varphi_{e^{(a/2)i}E_1} \circ \varphi_{\tilde{B}} \in \mathrm{Aut}(D_{n,1}(\mu_2)).$$

Therefore we have proved the assertion (II) of Theorem 1 in Case 2; thereby the proof of Theorem 1 is now completed. □

REMARK 2. In [9], we proved that Δ_D is just the degeneracy set for the Kobayashi pseudodistance d_D of D without using any information on $\mathrm{Aut}(D)$. In fact, this comes from the fact that d_D is identically zero on $\Delta_D \cong \mathbb{C}^n$ and that there exists a strictly plurisubharmonic function u on D^* with $0 < u(\zeta) < 1$ for all $\zeta \in D^*$, which implies the hyperbolicity of D at every point $p \in D^*$ by a result of Sibony [14]. Let f be an arbitrary element in $\mathrm{Aut}(D)$. Then f preserves D^* and it is uniquely determined by the restriction $f|_{D^*} \in \mathrm{Aut}(D^*)$. Therefore, our proof here of the assertion (II) of Theorem 1 based on the explicit description of $\mathrm{Aut}(\mathcal{E})$ of the universal covering space $\mathcal{E}$ of D^* gives an alternative proof of Kim-Ninh-Yamamori [6; Theorem 10] in the case where D is the Fock-Bargmann-Hartogs domain $D_{n,1}(\mu)$ in $\mathbb{C}^n \times \mathbb{C}$.

3.2. A geometric interpretation of Theorem 1. Throughout this subsection, we use the same terminology and notation in the Introduction. Then, just by the definition of the compact-open topology, it is easily seen that the action-mapping $\Phi : \mathrm{Aut}(D_2)\times P(D_1, D_2) \to P(D_1, D_2)$ is continuous. Moreover, $\mathrm{Aut}(D_2)$ acts freely on $P(D_1, D_2)$, since any proper holomorphic mapping $p : D_1 \to D_2$ is surjective. We have now two cases to consider:

CASE 1. $m_1 = 1$: In this case, we have $m_2 = 1$ by the assertion (II) of Theorem 1. Now, putting $(n, 1) = (n_j, m_j)$ for $j = 1, 2$, we assert the following:

(A.1) Every orbit $P_k = \mathrm{Aut}(D_2) \cdot p_k$ is open and closed in $P(D_1, D_2)$, and the topological space P_k in the topology induced from that of $P(D_1, D_2)$ is homeomorphic to the connected Lie group $\mathrm{Aut}(D_2)$. In particular, P_k is the connected component of $P(D_1, D_2)$ containing the point p_k;

(A.2) $P_1 = B(D_1, D_2)$ and $P(D_1, D_2)$ can be decomposed into the connected components P_k: $P(D_1, D_2) = \bigcup_{k=1}^{\infty} P_k$; and

(A.3) $P(D_1, D_2)$ is closed in $C(D_1, \mathbb{C}^N)$.

Once the assertion (A.1) has been shown, (A.2) is an immediate consequence of (II) in Theorem 1. So, we first verify the assertion (A.1).

To prove the closedness of P_k in (A.1), let us consider an arbitrary sequence $\{q_\nu\}_{\nu=1}^{\infty}$ in P_k converging to a point $q \in P(D_1, D_2)$. It then follows from Theorem 1 that there exist some $g \in \mathrm{Aut}(D_2)$ and $p_\ell = \rho_\ell \circ L_{\mu_1,\mu_2} \in P(D_1, D_2)$ such that $q = g \cdot p_\ell$. Let $\{g_\nu\}_{\nu=1}^{\infty}$ be a sequence in $\mathrm{Aut}(D_2)$ such that $q_\nu = g_\nu \cdot p_k$ for $\nu = 1, 2, \ldots$. According to Fact A in Section 2, we can write

$$g = \varphi_v \circ \varphi_B \circ \varphi_A, \quad g_\nu = \varphi_{v_\nu} \circ \varphi_{B_\nu} \circ \varphi_{A_\nu} \quad \text{for } \nu = 1, 2, \ldots, \tag{3.6}$$

where $v, v_\nu \in \mathbb{C}^n$, $B, B_\nu \in U(1)$ and $A, A_\nu \in U(n)$. Thanks to the compactness of $U(n) \times U(1)$, one may assume that $\{(A_\nu, B_\nu)\}_{\nu=1}^{\infty}$ converges to some element $(\widetilde{A}, \widetilde{B})$ of $U(n) \times U(1)$. Moreover, since $\lim_{\nu\to\infty} q_\nu = q$, we have

$$\lim_{\nu\to\infty}(v_\nu, 0) = \lim_{\nu\to\infty} q_\nu(0) = q(0) = (v, 0) \tag{3.7}$$

for the origin $0 = (0, 0) \in D_1 \subset \mathbb{C}^n \times \mathbb{C}$. Consequently, we have

$$\lim_{\nu\to\infty} g_\nu = \varphi_v \circ \varphi_{\widetilde{B}} \circ \varphi_{\widetilde{A}} =: \tilde{g} \in \mathrm{Aut}(D_2); \tag{3.8}$$

and hence, $\tilde{g} \cdot p_k = \lim_{\nu\to\infty} g_\nu \cdot p_k = g \cdot p_\ell$ or

$$\left(\tilde{z} + v, e^{-\mu_2\langle\tilde{z},v\rangle-(\mu_2/2)\|v\|^2}\widetilde{B}w^k\right) = \left(z^* + v, e^{-\mu_2\langle z^*,v\rangle-(\mu_2/2)\|v\|^2}Bw^\ell\right) \tag{3.9}$$

for any point $(z, w) \in D_1$, where we have put

$$\tilde{z} = \sqrt{k}\sqrt{\mu_1/\mu_2}\,\widetilde{A}z \quad \text{and} \quad z^* = \sqrt{\ell}\sqrt{\mu_1/\mu_2}\,Az.$$

Clearly, the equation (3.9) assures us that $k = \ell$; thereby $q = g \cdot p_k \in P_k$. Therefore we have proved that P_k is closed in $P(D_1, D_2)$, as desired.

To prove the openness of P_k in (A.1), note that $P(D_1, D_2) = \bigcup_{k=1}^{\infty} P_k$ by Theorem 1. Moreover, by the same argument as in the preceding paragraph, it can be checked that $P_k \cap P_\ell = \emptyset$ unless $k = \ell$. So, putting $P_k^c = \bigcup_{\ell\neq k} P_\ell$, the complement of P_k in $P(D_1, D_2)$, we would like to prove that P_k^c is closed in $P(D_1, D_2)$. For this purpose, choose an arbitrary sequence $\{q_\nu\}_{\nu=1}^{\infty}$ in P_k^c converging to a point $q \in P(D_1, D_2)$. Express q, q_ν as

$$q = g \cdot p_\ell, \;\; q_\nu = g_\nu \cdot p_{n(\nu)} \quad \text{for } \nu = 1, 2, \ldots,$$

where $g, g_\nu \in \mathrm{Aut}(D_2)$ and $n(\nu) \in \mathbb{N}$ with $n(\nu) \neq k$ for all ν. Write again g, g_ν as in (3.6) and repeat the same argument as above. Then it can be seen that $\{(v_\nu, A_\nu, B_\nu)\}_{\nu=1}^\infty$ converges to some element $(v, \widetilde{A}, \widetilde{B}) \in \mathbb{C}^n \times U(n) \times U(1)$, after taking a subsequence, if necessary. Recall that $\{q_\nu\}_{\nu=1}^\infty$ converges to q in $P(D_1, D_2)$. Then

$$\begin{aligned}\lim_{\nu\to\infty} &\left(\tilde{z}_\nu + v_\nu, e^{-\mu_2\langle \tilde{z}_\nu, v_\nu\rangle - (\mu_2/2)\|v_\nu\|^2} B_\nu w^{n(\nu)}\right) \\ &\qquad\qquad = \left(z^* + v, e^{-\mu_2\langle z^*, v\rangle - (\mu_2/2)\|v\|^2} B w^\ell\right)\end{aligned}$$

uniformly on any compact set in D_1, where we have put

$$\tilde{z}_\nu = \sqrt{n(\nu)}\sqrt{\mu_1/\mu_2}\, A_\nu z \quad \text{and} \quad z^* = \sqrt{\ell}\sqrt{\mu_1/\mu_2}\, A z.$$

It then follows at once that $\lim_{\nu\to\infty} \sqrt{n(\nu)}\, A_\nu z = \sqrt{\ell}\, A z$ uniformly on any compact set in $\mathbb{C}^n$ and so $\lim_{\nu\to\infty} n(\nu) = \ell$. Hence, $\ell = n(\nu) \neq k$ for all $\nu \geq \nu_o$ with some $\nu_o \in \mathbb{N}$; and consequently, $q = g \cdot p_\ell \in P_k^c$. Therefore we have shown that P_k^c is closed in $P(D_1, D_2)$; proving the openness of P_k in $P(D_1, D_2)$.

Next, for every $k \in \mathbb{N}$, we would like to prove that the mapping

$$\Psi : \mathrm{Aut}(D_2) \to P_k \quad \text{defined by} \quad \Psi(f) = f \cdot p_k$$

for $f \in \mathrm{Aut}(D_2)$ is a homeomorphism from $\mathrm{Aut}(D_2)$ onto the subspace P_k of $P(D_1, D_2)$. Since $\mathrm{Aut}(D_2)$ acts freely on $P(D_1, D_2)$ as mentioned above, Ψ is an injective continuous mapping from $\mathrm{Aut}(D_2)$ onto P_k. Thus, in order to show that the inverse mapping $\Psi^{-1} : P_k \to \mathrm{Aut}(D_2)$ of Ψ is also continuous, it suffices to prove that $\Psi : \mathrm{Aut}(D_2) \to P_k$ is a closed mapping. To this end, take a closed subset S of $\mathrm{Aut}(D_2)$ arbitrarily and consider a sequence $\{q_\nu\}_{\nu=1}^\infty$ in $\Psi(S)$ converging to some point $q \in P_k$. Let g be an element of $\mathrm{Aut}(D_2)$ and $\{g_\nu\}_{\nu=1}^\infty$ a sequence in S such that $q = g \cdot p_k$ and $q_\nu = g_\nu \cdot p_k$ for $\nu = 1, 2, \ldots$. By the same reasoning as above, one may assume that $\{g_\nu\}_{\nu=1}^\infty$ converges to some element $\tilde{g} \in \mathrm{Aut}(D_2)$. Then, since S is a closed subset of $\mathrm{Aut}(D_2)$, it follows that $\tilde{g} \in S$ and $q = \tilde{g} \cdot p_k \in \Psi(S)$; proving that Ψ is, in fact, a closed mapping. As a result, we have shown that Ψ gives a homeomorphism from $\mathrm{Aut}(D_2)$ onto P_k; completing the proof of the assertion (A.1).

Finally we wish to prove the assertion (A.3). For this, take an arbitrary sequence $\{q_\nu\}_{\nu=1}^\infty$ in $P(D_1, D_2)$ converging to a point $q \in C(D_1, \mathbb{C}^N)$. Express

$$q_\nu = g_\nu \cdot p_{n(\nu)}, \;\; g_\nu = \varphi_{v_\nu} \circ \varphi_{B_\nu} \circ \varphi_{A_\nu} \quad \text{for } \nu = 1, 2, \ldots,$$

as before. Also, represent q as $q = (q^1, q^2)$ with respect to the coordinate system (z, w) in $\mathbb{C}^n \times \mathbb{C}$. Then we have $\lim_{\nu\to\infty} v_\nu = q^1(0)$ as in (3.7). Hence, putting $v := q^1(0) \in \mathbb{C}^n$ for simplicity, one may assume that $\{g_\nu\}_{\nu=1}^\infty$ converges to an element $\tilde{g} \in \mathrm{Aut}(D_2)$ defined in (3.8). Moreover, since $\{q_\nu\}_{\nu=1}^\infty$ converges to q in the compact-open topology, we have that

$$\lim_{\nu\to\infty} \left(\tilde{z}_\nu + v_\nu, e^{-\mu_2\langle \tilde{z}_\nu, v_\nu\rangle - (\mu_2/2)\|v_\nu\|^2} B_\nu w^{n(\nu)}\right) = q(z, w)$$

uniformly on any compact set in D_1, where $\tilde{z}_\nu = \sqrt{n(\nu)}\sqrt{\mu_1/\mu_2}\, A_\nu z$ for all ν. It then follows that

$$\lim_{\nu\to\infty} \sqrt{n(\nu)}\sqrt{\mu_1/\mu_2}\, A_\nu z = q^1(z, w) - v$$

uniformly on any compact set in D_1; and hence,

$$\lim_{\nu\to\infty} n(\nu) \;=\; (\mu_2/\mu_1)\|q^1(z^o,0)-v\|^2$$

for any $z^o \in \mathbb{C}^n$ with $\|z^o\| = 1$. Since $n(\nu) \in \mathbb{N}$ for all ν, this says that there exists a large $\nu_o \in \mathbb{N}$ such that $n(\nu) = n(\nu_o)$ for all $\nu \geq \nu_o$. Therefore we conclude that

$$q = \lim_{\nu\to\infty} q_\nu = \tilde{g}\cdot p_{n(\nu_o)} \;\in\; P(D_1,D_2)$$

by (A.2); proving the assertion (A.3).

Case 2. $m_1 \geq 2$: In this case, as an immediate consequence of the assertion (I) of Theorem 1, we obtain that

$$(n_1,m_1) = (n_2,m_2) \quad \text{and} \quad P(D_1,D_2) = B(D_1,D_2) = \mathrm{Aut}(D_2)\cdot L_{\mu_1,\mu_2}.$$

Moreover, with exactly the same argument as in Case 1 above, it can be checked easily that $P(D_1,D_2)$ is closed in $C(D_1,\mathbb{C}^N)$ and the mapping

$$\Psi : \mathrm{Aut}(D_2) \to P(D_1,D_2) \quad \text{defined by} \quad \Psi(f) = f\cdot L_{\mu_1,\mu_2}$$

for $f \in \mathrm{Aut}(D_2)$ induces a homeomorphism from $\mathrm{Aut}(D_2)$ onto $P(D_1,D_2)$.

4. Proof of Theorem 2

Throughout this section, we use the following notation: For the given Fock-Bargmann-Hartogs domains

$$D_j \;=\; D_{n_j,m_j}(\mu_j) \;=\; \left\{(z_j,w_j) \in \mathbb{C}^{n_j}\times\mathbb{C}^{m_j}\,;\, \|w_j\|^2 < e^{-\mu_j\|z_j\|^2}\right\}$$

for $j = 1,2$, we denote by

$$\begin{aligned}
&d_{D_1\times D_2} \;\text{ the Kobayashi pseudodistance of } D_1\times D_2;\\
&\Delta_{D_1\times D_2} \;\text{ the degeneracy set for } d_{D_1\times D_2}\text{; and set}\\
&(D_1\times D_2)^* \;=\; D_1\times D_2 \setminus \Delta_{D_1\times D_2}.
\end{aligned}$$

Thus, $d_{D_1\times D_2}$ induces a true distance on $(D_1\times D_2)^*$ and $(D_1\times D_2)^*$ is hyperbolic in the sense of Kobayashi. Also, we often set

$$\begin{aligned}
&(k_1,k_2,k_3,k_4) = (n_1,m_1,n_2,m_2),\; N_1 = n_1+m_1,\; N_2 = n_2+m_2,\\
&N = N_1+N_2 \;\text{ and }\; \zeta = (\zeta_1,\dots,\zeta_N) = (\zeta^1,\zeta^2,\zeta^3,\zeta^4) = (z_1,w_1,z_2,w_2).
\end{aligned}$$

The proof of Theorem 2 will be divided into several steps as follows:

Step 1. $\mathrm{Aut}^o(D_1\times D_2)$ *can be canonically identified with the product Lie group* $\mathrm{Aut}(D_1)\times\mathrm{Aut}(D_2)$: Since $\mathrm{Aut}(D_j)$ is a connected Lie group for $j = 1,2$, it is obvious that $\mathrm{Aut}(D_1)\times\mathrm{Aut}(D_2) \subset \mathrm{Aut}^o(D_1\times D_2)$. Therefore, to prove Step 1, it suffices to show the opposite inclusion. For this, recall that

$$d_{D_1\times D_2}((p_1,p_2),(q_1,q_2)) \;=\; \max\left\{d_{D_j}(p_j,q_j)\,;\, j=1,2\right\}$$

for $p_j,q_j \in D_j$, $j = 1,2$, and Δ_{D_j} is just the degeneracy set for d_{D_j} for $j = 1,2$. Then it is easily checked that

$$\Delta_{D_1\times D_2} = (D_1 \times \Delta_{D_2}) \cup (\Delta_{D_1} \times D_2) \quad \text{and} \quad (D_1 \times D_2)^* = D_1^* \times D_2^*.$$

Since $d_{D_1\times D_2}$ is invariant under the action of $\mathrm{Aut}(D_1 \times D_2)$, we therefore have

$$\varphi(\Delta_{D_1\times D_2}) = \Delta_{D_1\times D_2},\ \varphi(D_1^* \times D_2^*) = D_1^* \times D_2^* \quad \text{for all } \varphi \in \mathrm{Aut}(D_1 \times D_2).$$

Thus, the natural restriction mapping $\Phi : \mathrm{Aut}(D_1 \times D_2) \to \mathrm{Aut}(D_1^* \times D_2^*)$ gives an injective continuous homomorphism from $\mathrm{Aut}(D_1 \times D_2)$ into $\mathrm{Aut}(D_1^* \times D_2^*)$. In particular, we have $\Phi(\mathrm{Aut}^o(D_1 \times D_2)) \subset \mathrm{Aut}^o(D_1^* \times D_2^*)$.

Choose an element f in $\mathrm{Aut}^o(D_1 \times D_2)$ arbitrarily and set

$$f^* = \Phi(f) \in \mathrm{Aut}^o(D_1^* \times D_2^*).$$

Since D_1^* and D_2^* are hyperbolic, we have $\mathrm{Aut}^o(D_1^* \times D_2^*) = \mathrm{Aut}^o(D_1^*) \times \mathrm{Aut}^o(D_2^*)$ by the fact (‡) in the Introduction. Accordingly, one can find automorphisms φ_1, φ_2 of D_1^*, D_2^*, respectively, such that $f^* = \varphi_1 \times \varphi_2$. This implies that, if we write

$$f(\zeta) = (f_1(\zeta), f_2(\zeta)) \quad \text{for } \zeta = (z_1, w_1, z_2, w_2) \in D_1 \times D_2,$$

where f_j is a mapping from $D_1 \times D_2$ into D_j for $j = 1, 2$, and set

$$f_j^* = f_j|_{D_1^*\times D_2^*} \quad \text{for } j = 1, 2,$$

then $f_j^* = \varphi_j$ for $j = 1, 2$; and hence, f_1^* (resp. f_2^*) depends only on $(z_1, w_1) \in D_1^*$ (resp. on $(z_2, w_2) \in D_2^*$). Thus, f_1, f_2 must be of the form

$$f_1(\zeta) = f_1(z_1, w_1),\ f_2(\zeta) = f_2(z_2, w_2) \quad \text{on } D_1 \times D_2 \tag{4.1}$$

by analytic continuation. Exactly the same conclusion as in (4.1) remains valid for the inverse mapping f^{-1} of f. Hence, we conclude that $f_j \in \mathrm{Aut}(D_j)$ for $j = 1, 2$ and $f = f_1 \times f_2 \in \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)$; proving the opposite inclusion. Therefore we have shown the assertion in Step 1.

Step 2. *Every element f in $\mathrm{Aut}(D_1 \times D_2)$ can be written in the form $f = L \circ g$, where $g \in \mathrm{Aut}^o(D_1 \times D_2)$ and L is a linear automorphism of $D_1 \times D_2$, that is, it is the restriction to $D_1 \times D_2$ of some non-singular linear transformation of $\mathbb{C}^N$*: First of all, notice that the automorphism group $\mathrm{Aut}(D_1 \times D_2)$ has the structure of a Lie group with respect to the compact-open topology, because its identity component $\mathrm{Aut}^o(D_1 \times D_2)$ is a Lie group by Step 1. Now, let $T(D_1 \times D_2) \cong T^N$ be the subgroup of $\mathrm{Aut}(D_1 \times D_2)$ introduced in Section 2 for the Reinhardt domain $D_1 \times D_2$ in $\mathbb{C}^N$. Then, for the given element $f \in \mathrm{Aut}(D_1 \times D_2)$, $f^{-1}T(D_1 \times D_2)f$ as well as $T(D_1 \times D_2)$ is a maximal torus in $\mathrm{Aut}^o(D_1 \times D_2)$ (cf. [13; Section 4]). Hence, by the well-known conjugacy theorem for maximal tori in a connected Lie group, there exists an element g in $\mathrm{Aut}^o(D_1 \times D_2)$ such that

$$gf^{-1}T(D_1 \times D_2)fg^{-1} = T(D_1 \times D_2).$$

Consequently, by Fact B in Section 2, $L := f \circ g^{-1}$ is an algebraic automorphism of $D_1 \times D_2$ and $f = L \circ g$. Moreover, since $D_1 \times D_2$ containes the origin 0 of $\mathbb{C}^N$, L has to be of the form

$$L(\zeta) = (\alpha_1\zeta_{\sigma(1)}, \ldots, \alpha_N\zeta_{\sigma(N)}) \quad \text{for } \zeta = (\zeta_i) \in D_1 \times D_2 \tag{4.2}$$

by (2.2), where $(\alpha_i) \in (\mathbb{C}^*)^N$ and σ is a permutation of $\{1, \ldots, N\}$; proving the assertion in

Step 2.

STEP 3. *Analysis of L*: In order to prove Theorem 2, we would like to investigate the linear automorphism L of $D_1 \times D_2$ in Step 2 more closely. To this end, represent L as

$$L(\zeta) \;=\; \left(L^1(\zeta), L^2(\zeta), L^3(\zeta), L^4(\zeta)\right) \quad \text{for } \; \zeta \in D_1 \times D_2$$

with respect to the coordinate system $\zeta = (\zeta^1, \zeta^2, \zeta^3, \zeta^4)$ in $\mathbb{C}^N$. Note that the coordinate subspace $\mathbb{C}^{n_1} \times \{0\} \times \mathbb{C}^{n_2} \times \{0\}$ is contained in $D_1 \times D_2$; while $D_1 \times D_2$ is bounded in the (w_1, w_2)-direction. Hence

$$L^j(z_1, 0, z_2, 0) = 0 \quad \text{for all } \; (z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}, \;\; j = 2, 4;$$

which implies that L can be expressed as

$$L(\zeta) \;=\; \begin{pmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ 0 & M_{22} & 0 & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ 0 & M_{42} & 0 & M_{44} \end{pmatrix} \begin{pmatrix} z_1 \\ w_1 \\ z_2 \\ w_2 \end{pmatrix} \quad \text{on } \; \mathbb{C}^N, \tag{4.3}$$

where M_{ij} is a certain $k_i \times k_j$ matrix for $1 \le i, j \le 4$. Moreover, by (4.2) the $N \times N$ matrix $M := \left(M_{ij}\right)_{1 \le i, j \le 4}$ has the following property (like as a permutation matrix):

($\star$) Every row and every column of M contain exactly one nonzero entry.

Take now an arbitrary point $\zeta(w_1^o) = (0, w_1^o, 0, 0) \in D_1 \times D_2$ with $w_1^o \neq 0$. Then, by (4.3), we have

$$L(\zeta(w_1^o)) \;=\; \left(M_{12} w_1^o, M_{22} w_1^o, M_{32} w_1^o, M_{42} w_1^o\right).$$

Assume that $M_{22} w_1^o \neq 0$ and $M_{42} w_1^o \neq 0$. It then follows from Step 1 and (2.1) that

$$\begin{aligned} &\dim_{\mathbb{R}} \mathrm{Aut}^o(D_1 \times D_2) \cdot L(\zeta(w_1^o)) \\ &\quad = \; \dim_{\mathbb{R}} \left[\mathrm{Aut}(D_1) \cdot (M_{12} w_1^o, M_{22} w_1^o) \times \mathrm{Aut}(D_2) \cdot (M_{32} w_1^o, M_{42} w_1^o)\right] \\ &\quad = \; (2N_1 - 1) + (2N_2 - 1) = 2(N - 1); \;\; \text{and} \\ &\dim_{\mathbb{R}} L\left(\mathrm{Aut}^o(D_1 \times D_2) \cdot \zeta(w_1^o)\right) \\ &\quad = \; \dim_{\mathbb{R}} \mathrm{Aut}^o(D_1 \times D_2) \cdot \zeta(w_1^o) \; = \; 2(N_1 + n_2) - 1, \end{aligned}$$

since $L \in \mathrm{Aut}(D_1 \times D_2)$. Consequently, we arrive at a contradiction:

$$2(N - 1) = 2(N_1 + n_2) - 1 \quad \text{or} \quad 2m_2 = 1,$$

since $L^{-1} \mathrm{Aut}^o(D_1 \times D_2) L = \mathrm{Aut}^o(D_1 \times D_2)$. Thus $\|M_{22} w_1^o\| \|M_{42} w_1^o\| = 0$ for all $w_1^o \in \mathbb{C}^{m_1}$ with $0 < \|w_1^o\| < 1$; and hence, $M_{22} = 0$ or $M_{42} = 0$. Replacing $\zeta(w_1^o)$ by a point $\zeta(w_2^o) = (0, 0, 0, w_2^o) \in D_1 \times D_2$ with $w_2^o \neq 0$ and repeating the same argument as above, we obtain that $M_{24} = 0$ or $M_{44} = 0$. Therefore, since M is non-singular, we now have two possibilities as follows:

$$\text{Case (a): } \; M_{22} = 0, \; M_{44} = 0; \qquad \text{Case (b): } \; M_{24} = 0, \; M_{42} = 0.$$

In Case (a), we wish to prove that $(k_1, k_2) = (k_3, k_4)$ and M has the form

$$M = \begin{pmatrix} 0 & 0 & \sqrt{\mu_2/\mu_1}\widetilde{M}_{13} & 0 \\ 0 & 0 & 0 & M_{24} \\ \sqrt{\mu_1/\mu_2}\widetilde{M}_{31} & 0 & 0 & 0 \\ 0 & M_{42} & 0 & 0 \end{pmatrix},$$

where $\widetilde{M}_{13}$, $\widetilde{M}_{31} \in U(k_1)$ and M_{24}, $M_{42} \in U(k_2)$. For this, recall that M is a non-singular $N \times N$ matrix having the property ($\star$). Then, in Case (a) one can check that $k_2 = k_4$ and

$$\det M_{24} \neq 0,\ \det M_{42} \neq 0 \quad \text{and} \quad M_{14} = 0,\ M_{34} = 0,\ M_{12} = 0,\ M_{32} = 0.$$

Therefore, L has the form

$$L(z_1, w_1, z_2, w_2) = (M_{11}z_1 + M_{13}z_2, M_{24}w_2, M_{31}z_1 + M_{33}z_2, M_{42}w_1)$$

for $(z_1, w_1, z_2, w_2) \in \mathbb{C}^N$.

Notice that $\partial D_1 \times \partial D_2$ is the subset of $\partial(D_1 \times D_2)$ consisting of all non-smooth boundary points of $D_1 \times D_2$. Then the linear automorphism L of $D_1 \times D_2$ maps $\partial D_1 \times \partial D_2$ onto itself. Thus

$$\|M_{24}w_2\|^2 = e^{-\mu_1\|M_{11}z_1+M_{13}z_2\|^2},\quad \|M_{42}w_1\|^2 = e^{-\mu_2\|M_{31}z_1+M_{33}z_2\|^2} \tag{4.4}$$

whenever $\|w_1\|^2 = e^{-\mu_1\|z_1\|^2}$, $\|w_2\|^2 = e^{-\mu_2\|z_2\|^2}$. Taking the points $(0, w_1, 0, w_2)$ with $\|w_1\| = \|w_2\| = 1$, we therefore have

$$\|M_{24}w_2\| = 1,\ \|M_{42}w_1\| = 1; \text{ and hence,}\ \ M_{24},\ M_{42} \in U(m_1).$$

Together with (4.4), this implies that

$$\mu_2\|z_2\|^2 = \mu_1\|M_{11}z_1 + M_{13}z_2\|^2,\quad \mu_1\|z_1\|^2 = \mu_2\|M_{31}z_1 + M_{33}z_2\|^2 \tag{4.5}$$

for any boundary point $(z_1, w_1, z_2, w_2) \in \partial D_1 \times \partial D_2$. Notice that these equations hold for arbitrary elements $(z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ because one can always find elements $(w_1, w_2) \in \mathbb{C}^{m_1} \times \mathbb{C}^{m_2}$ in such a way that $(z_1, w_1) \in \partial D_1$ and $(z_2, w_2) \in \partial D_2$. Thus, considering the special case where

$$(z_1, z_2) = (z_1, 0) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2} \quad (\text{resp. } (z_1, z_2) = (0, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2})$$

in (4.5), we obtain that $M_{11} = 0$ (resp. $M_{33} = 0$). In particular, since M is a non-singular $N \times N$ matrix and since $k_2 = k_4$ as shown above, it follows that $k_1 = k_3$. Moreover, we now have by (4.5) that

$$\mu_2\|z_2\|^2 = \mu_1\|M_{13}z_2\|^2,\ \mu_1\|z_1\|^2 = \mu_2\|M_{31}z_1\|^2 \quad \text{for all } (z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}.$$

Therefore, $\sqrt{\mu_1/\mu_2}M_{13}$ and $\sqrt{\mu_2/\mu_1}M_{31}$ are unitary matrices, as desired.

In Case (b), we assert that M has the form

$$M = \begin{pmatrix} M_{11} & 0 & 0 & 0 \\ 0 & M_{22} & 0 & 0 \\ 0 & 0 & M_{33} & 0 \\ 0 & 0 & 0 & M_{44} \end{pmatrix},$$

where M_{ii} $(1 \leq i \leq 4)$ are all unitary matrices. Indeed, in Case (b) we have $\det M_{22} \neq 0$ and $\det M_{44} \neq 0$, since M is non-singular. Then $M_{12} = 0$, $M_{32} = 0$ and $M_{14} = 0$, $M_{34} = 0$ by

($\star$). Hence, L has the form

$$L(z_1, w_1, z_2, w_2) \ = \ (M_{11}z_1 + M_{13}z_2, M_{22}w_1, M_{31}z_1 + M_{33}z_2, M_{44}w_2)$$

for $(z_1, w_1, z_2, w_2) \in \mathbb{C}^N$. So, by the same reasoning as in Case (a), we have

$$\|M_{22}w_1\|^2 = e^{-\mu_1\|M_{11}z_1+M_{13}z_2\|^2}, \quad \|M_{44}w_2\|^2 = e^{-\mu_2\|M_{31}z_1+M_{33}z_2\|^2}$$

whenever $\|w_1\|^2 = e^{-\mu_1\|z_1\|^2}$, $\|w_2\|^2 = e^{-\mu_2\|z_2\|^2}$. Therefore, M_{22} and M_{44} are unitary matrices; and hence,

$$\|z_1\|^2 = \|M_{11}z_1 + M_{13}z_2\|^2, \quad \|z_2\|^2 = \|M_{31}z_1 + M_{33}z_2\|^2$$

for all $(z_1, z_2) \in \mathbb{C}^{n_1} \times \mathbb{C}^{n_2}$ as in Case (a). Accordingly, we conclude that

$$M_{13} = 0, \ M_{11} \in U(n_1) \quad \text{and} \quad M_{31} = 0, \ M_{33} \in U(n_2);$$

as asserted.

Step 4. *Completion of the proof*: Using the result obtained above, we shall complete the proof of Theorem 2. For this purpose, choose an element f of $\mathrm{Aut}(D_1 \times D_2)$ arbitrarily. Then, by Step 2, f can be written in the form

$$f = L \circ g \quad \text{with some element } \ g \in \mathrm{Aut}^o(D_1 \times D_2).$$

Consider the case where $(n_1, m_1) = (n_2, m_2)$. We then have two Cases (a) and (b) as in Step 3. In Case (a), let us define the linear transformation T of $\mathbb{C}^N$ by setting

$$T(z_1, w_1, z_2, w_2) \ = \ \left(\widetilde{M}_{31}z_1, M_{42}w_1, \widetilde{M}_{13}z_2, M_{24}w_2\right) \quad \text{on } \mathbb{C}^N,$$

where $\widetilde{M}_{31}$, M_{42}, $\widetilde{M}_{13}$ and M_{24} are the unitary matrices appearing in Case (a) of Step 3. Then T can be regarded as an element of $\mathrm{Aut}(D_1)\times\mathrm{Aut}(D_2)$ by Fact A and the linear automorphism L of $D_1\times D_2$ can be expressed as $L = I\circ T$, where I is the involutive automorphism of $D_1\times D_2$ defined in the statement of Theorem 2. Therefore we have

$$f = I \circ (T \circ g) \quad \text{with } \ T \circ g \in \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)$$

by Step 1.

In Case (b), the linear automorphism L of $D_1 \times D_2$ has the form

$$L(z_1, w_1, z_2, w_2) \ = \ (M_{11}z_1, M_{22}w_1, M_{13}z_2, M_{24}w_2) \quad \text{on } \ D_1 \times D_2.$$

Thus, $L \in \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)$ and $f = L \circ g \ \in \ \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)$ by Step 1.

As a result, we have shown that

$$\mathrm{Aut}(D_1 \times D_2) \ \subset \ (\mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)) \ \cup \ \{I \circ f\,;\ f \in \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2)\}$$

in any case. The opposite inclusion is now obvious; thereby we have completed the proof of the assertion (I) of Theorem 2.

Finally, consider the case where $(n_1, m_1) \neq (n_2, m_2)$. Then, only the Case (b) occurs in Step 3. Hence

$$\mathrm{Aut}(D_1 \times D_2) \ = \ \mathrm{Aut}(D_1) \times \mathrm{Aut}(D_2);$$

proving the assertion (II) of Theorem 2.

Therefore the proof of Theorem 2 is now completed. □

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