Harris, M.E.
Osaka J. Math.
56 (2019), 229–236

A BLOCK REFINEMENT OF THE GREEN-PUIG PARAMETERIZATION OF THE ISOMORPHISM TYPES OF INDECOMPOSABLE MODULES

Dedicated to the Memory of J.A. Green

Morton E. HARRIS

(Received July 31, 2017, revised October 23, 2017)

Abstract

Let p be a prime integer, let $\mathcal{O}$ be a commutative complete local Noetherian ring with an algebraically closed residue field k of charateristic p and let$G$ be a finite group. Let P be a p-subgroup of G and let X be an indecomposable $\mathcal{O}P$-module with vertex P. Let $\Lambda(G, P, X)$ denote a set of representatives for the isomorphism classes of indecomposable $\mathcal{O}G$-modules with vertex-source pair (P, X) (so that $\Lambda(G, P, X)$ is a finite set by the Green correspondence). As mentioned in [5, Notes on Section 26], L. Puig asserted that a defect multiplicity module determined by (P, X) can be used to obtain an extended parameterization of $\Lambda(G, P, X)$. In [5, Proposition 26.3], J. Thévenaz completed this program under the hypotheses that X is $\mathcal{O}$-free. Here we use the methods of proof of [5, Theorem 26.3] to show that the $\mathcal{O}$-free hypothesis on X is superfluous. (M. Linckelmann has also proved this, cf. [3]). Let B be a block of $\mathcal{O}G$. Then we obtain a corresponding paramaterization of the $(\mathcal{O}G)B$-modules in $\Lambda(G, P, X)$.

1. Introduction and Statements

Our notation and terminology are standard and tend to follow [5]. All rings have identities and are Noetherian and all modules over a ring are unitary and finitely generated left modules.

Let R be a ring. Then R-mod will denote the abelian category of finitely generated (left) R-modules. Let U, V be R-modules. Then $U|V$ in R-mod signifies that U is isomorphic in R-mod to a direct summand of V. Also if R has the unique decomposition property (cf.[1, p. 37]), then U is a component of V if $U|V$ and U is indecomposable in R-mod.

Throughout this paper, G is a finite group, p is a prime integer and $\mathcal{O}$ is a commutative complete local Noetherian ring with an algebraically closed residue field of characteristic p ([5, Assumption 2.1]).

The statements of the main results of this paper are given in this section. The required proofs are presented in Section 2.

Let P be a p-subgroup of G and let X be an indecomposable $\mathcal{O}P$-module with vertex P. Let E be an idempotent of $Z(\mathcal{O}G)$. Then, as in [2], $IT(G, P, X, E)$ will denote a set of representatives for the isomorphism types of indecomposable $(\mathcal{O}G)E$-modules with vertex-source pair (P, X). Clearly we may assume that the modules in $IT(G, P, X, E)$ are components of $E\mathrm{Ind}_P^G(X)$ in $\mathcal{O}G$-mod.

2010 Mathematics Subject Classification. 20C20.

Set $H = N_G(P, X)$. By [5, Proposition 20.8]:

(1.1) There is a natural bijection Pu_G^H, from $IT(G, P, X, 1)$ to $IT(H, P, X, 1)$ with inverse $\mathrm{Pu}_H^G\colon IT(H, P, X, 1) \to IT(G, P, X, 1)$.

Here $P \trianglelefteq H$ and $C_G(P) \trianglelefteq H$.

Clearly we may assume:

(1.2) The modules in $IT(H, P, X, 1)$ are components of $\mathrm{Ind}_P^H(X)$.

Set $A = \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_P^H(X))$ so that $A \cong \mathrm{Ind}_P^H(\mathrm{End}_{\mathcal{O}}(X))$ as interior H-algebras, $A^H = \mathrm{End}_{\mathcal{O}H}(\mathrm{Ind}_P^H(X))$ and $A^P = \mathrm{End}_{\mathcal{O}P}(\mathrm{Ind}_P^H(X))$. Also the points of A^H, $\mathcal{P}(A^H)$, biject with the isomorphism types of components of $\mathrm{Ind}_P^H(X)$ in $\mathcal{O}H$-mod and similarly for the points of A^P, $\mathcal{P}(A^P)$.

As X is an indecomposable $\mathcal{O}P$-module with vertex P, $\mathrm{End}_{\mathcal{O}}(X)$ is a primitive interior P-algebra with defect group P. Since $\mathrm{End}_{\mathcal{O}}(X)^P = \mathrm{End}_{\mathcal{O}P}(X)$ is a local algebra, $P_{\{\mathrm{Id}_X\}}$ is a defect of $\mathrm{End}_{\mathcal{O}}(X)$.

Let

$$\begin{aligned}\mathcal{D}_P^H\colon \mathrm{End}_{\mathcal{O}}(X) \to \mathrm{Res}_P^H(A) &= \mathrm{Res}_P^H(\mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_P^H(X)))\\ &\cong \mathrm{Res}_P^H(\mathrm{Ind}_P^H(\mathrm{End}_{\mathcal{O}}(X)))\end{aligned}$$

denote the canonic embedding. Here $\{\mathrm{Id}_X\}$ is the unique point of $\mathrm{End}_{\mathcal{O}}(X)^P = \mathrm{End}_{\mathcal{O}P}(X)$ and $(1 \otimes_{\mathcal{O}} \mathrm{Id}_X \otimes_{\mathcal{O}} 1)\mathrm{Ind}_P^H(X) = 1 \otimes X$ so that $\mathcal{D}_P^H(\{\mathrm{Id}_X\}) = \gamma \in \mathcal{P}(A^P)$ by [5, Proposition 15.1] and P_γ is a local pointed group of A by [5, Proposition 15.1(d)].

Thus $N_G(P_\gamma) = N_G(P, X)$ since $j\mathrm{Ind}_P^H(X) \cong X$ in $\mathcal{O}P$-mod for any $j \in \gamma$ as on [5, p. 106].

Also $A = \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_P^H(X)) \cong \mathrm{Ind}_P^H(\mathrm{End}_{\mathcal{O}}(X))$ as interior H-algebras and $A^P = \mathrm{End}_{\mathcal{O}P}(\mathrm{Res}_P^H(\mathrm{Ind}_P^H(X)))$. Note that:

(1.3) $$\mathrm{Res}_P^H(\mathrm{Ind}_P^H(X)) \cong |H/P|X \text{ in } \mathcal{O}P\text{-mod}.$$

Thus:

(1.4) $A^P \cong Mat_{|H/P|}(\mathrm{End}_{\mathcal{O}P}(X))$ as $\mathcal{O}$-algebras; $A^P/J(A^P) \cong Mat_{|H/P|}(k)$ as k-algebras and $\mathcal{P}(A^P) = \gamma$.

Lemma 1.1. *Let W be a component of $Ind_P^H(X)$ in $\mathcal{O}H$-mod and let $e \in B\ell(\mathcal{O}H)$ be such that $eW = W$. Then:* (a) *(P, X) is the "unique" vertex-source pair of W;* (b) *Let $j \in \tau \in \mathcal{P}(A^H)$ be such that $jInd_P^H(X) \cong W$ in $\mathcal{O}H$-mod so that $jAj \cong End_{\mathcal{O}}(jInd_P^H(X))$ as interior H-algebras. Then P is a defect group of jAj and if $P_{\gamma'}$ is a defect of jAj and if $I\colon jAj \to A$ is the inclusion embedding of interior H-algebras, then $I(P_{\gamma'}) = P_\gamma$; and* (c) *$e^G$ is a defined block of G and $e^G Pu_H^G(W) = Pu_H^G(W)$.*

Corollary 1.2. *$IT(H, P, X, 1)$ bijects with $\mathcal{P}(A^H)$.*

Remark 1.3. By [4, Theorem 5.5.15], the map that associates each block e of $\mathcal{O}H$ to the H-conjugacy class $\mathcal{C}$ of blocks of $\mathcal{O}C_G(P)$ covered by e is a bijection. Moreover $\mathcal{C} = \{b \in B\ell(\mathcal{O}C_G(P)) \mid b^H = e\}$. Also $IT(H, P, X, e) \neq \phi$ for each $e \in B\ell(\mathcal{O}H)$ hence P is contained in a defect group of e by [2, Corollary 1.8]. Moreover e^G is defined for each $e \in B\ell(\mathcal{O}H)$ by [4, Theorem 5.3.5].

Also $A^H = \mathrm{End}_{\mathcal{O}H}(\mathrm{Ind}_P^H(X))$ is an interior $C_G(H)$-algebra where $C_G(H) \le C_G(P) \le H$ so that $C_G(H) = Z(H)$.

Set $\overline{H} = H/P$. Thus A^P is an $\overline{H}$-algebra and an interior $C_G(P)$-algebra. Since γ is the unique point of A^P, the maximal ideal m_γ of A^P such that $m_\gamma \cap \gamma = \phi$ is just $J(A^P)$. Hence the multiplicity algebra of P_γ, $S(\gamma) = A^P/J(A^P) \cong Mat_\mu(k)$ where μ is the multiple of γ as k-algebras and $S(\gamma) = \mathrm{End}_k(V_A(\gamma))$ where $V_A(\gamma)$ is a multiplicity module of P_γ on A. Clearly $S(\gamma)$ is an $\overline{H}$-algebra and [5, Theorem 19.1] yields:

Proposition 1.4. *There is a bijection between $\mathcal{P}(A^H)$ and $\{\delta \in \mathcal{P}(S(\gamma)^{\overline{H}}) \mid \overline{H}_\delta$ is a projective pointed group on $S(\gamma)\}$.*

Let $\alpha\colon H \to A^\times = \mathrm{End}_{\mathcal{O}}(\mathrm{Ind}_P^H(X))^\times$ denote the group homomorphism that expresses the fact that A is an interior H-algebra. (Thus, if $h \in H$ and $v \in \mathrm{Ind}_P^H(X)$, $\alpha(h)v = hv$). Consequently α induces an H-algebra homorphism $\alpha\colon \mathcal{O}H \to A$, an $\overline{H}$-group homorphism $\alpha\colon C_G(P) \to (A^P)^\times$ and an $\overline{H}$-algebra homomorphism $\alpha\colon \mathcal{O}C_G(P) \to A^P$.

Let T be a transversal of $Z(P)$ in $C_G(P)$ with $1 \in T$ so that $C_G(P) = \bigcup_{t\in T}(tZ(P))$ and $PC_G(P) = \bigcup_{t\in T}(tP)$ are disjoint. Also let $\pi\colon A^P \to S(\gamma) = A^P/J(A^P)$ denote the canonic $\overline{H}$-algebra epimorphism. Thus $\pi\circ\alpha\colon C_G(P) \to S(\gamma)^\times$ is an $\overline{H}$-group homorphism with $Z(P) \le \mathrm{Ker}(\pi\circ\alpha)$ by [5, p. 104] and $\pi\circ\alpha$ induces an $\overline{H}$-algebra homorphism $\pi\circ\alpha\colon \mathcal{O}C_G(P) \to S(\gamma)$. As above $S(\gamma) = \mathrm{End}_k(V_A(\gamma))$ where $V_A(\gamma)$ is a multiplicity module of P_γ on A.

Let $\mathcal{H} = \{(\overline{s}, \hbar) \in (S(\gamma)^\times \times \overline{H}) \mid \overline{s}\sigma\overline{s}^{-1} = {}^{\hbar}\sigma$ for all $\sigma \in S(\gamma)\}$ and set $\mathcal{N} = \{((\pi\circ\alpha)(t), \overline{t}) \mid t \in T\}$. Let $t \in T$ and $a \in A^P$. Then, ${}^{\overline{t}}a = \alpha(t)a\alpha(t^{-1})$, and so ${}^{\overline{t}}\pi(a) = (\pi\circ\alpha)(t)\pi(a)(\pi\circ\alpha)(t^{-1})$. Thus $\mathcal{N} \subseteq \mathcal{H} \subseteq S(\gamma)^\times \times \overline{H}$.

Proposition 1.5. (a) *$\mathcal{H} \le S(\gamma)^\times \times \overline{H}$ and with $\overline{H}$ acting by conjugation on $\overline{H}$, $\overline{H}$ acts diagonally on $\mathcal{H}$;* (b) *$\mathcal{N} \trianglelefteq \mathcal{H}$ and $\mathcal{N}$ is $\overline{H}$-invariant; and* (c) *If $(\overline{s}, \hbar)$ and $(\overline{u}, \overline{x}) \in \mathcal{H}$, then ${}^{\overline{x}}(\overline{s}, \hbar) = (\overline{u}, \overline{x})(\overline{s}, \hbar)(\overline{u}, \overline{x})^{-1}$.*

Moreover, we have the following diagram in the category of groups:

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & k^\times & \xrightarrow{\phi} & \mathcal{H} & \xrightarrow{\eta} & \overline{H} & \longrightarrow & 1 \\
 & & & & \rho\big\downarrow & & & & \\
 & & & & S(\gamma)^\times & & & &
\end{array} \tag{1.5}
$$

where $\phi(u) = (u\,\mathrm{Id}_{S(\gamma)}, 1_{\overline{H}})$, for all $u \in k^\times$, η is the projection on the second component and ρ is the projection on the first component.

Clearly:

(1.6) The row in (1.5) is exact and letting $\overline{H}$ act trivially on $k^\times$, all of the group homomorphisms in (1.5) are $\overline{H}$-homorphisms.

Clearly $k\mathcal{H}$ is an $\overline{H}$-algebra.

Lemma 1.6. $Z(k\mathcal{H}) = (k\mathcal{H})^{\overline{H}}$.

Let $\omega\colon C_G(P) \to \mathcal{N}$ be such that $tz \mapsto ((\pi\circ\alpha)(t), \overline{t})$ for all $t \in T$ and $z \in Z(\mathcal{P})$. Clearly ω is an $\overline{H}$-group epimorphism with $Z(P) = \mathrm{Ker}(\omega)$ and ω induces an $\overline{H}$-group isomorphism $\overline{\omega}\colon \overline{C_G(P)} \to \mathcal{N}$ and an $\overline{H}$-algebra homomorphism $\omega\colon \mathcal{O}C_G(P) \to k\mathcal{N}$ and an $\overline{H}$-algebra isomorphism $\overline{\omega}\colon k\overline{C_G(P)} \to k\mathcal{N}$.

Clearly $\rho\colon \mathcal{H} \to S(\gamma)^\times$ induces an $\overline{H}$-algebra homorphism $\rho\colon k\mathcal{H} \to S(\gamma)$.

Since T is a transversal of P in $PC_G(P)$, we may extend T to a transversal S of P in H.

For $t \in T$, set $\Sigma(t) = ((\pi \circ \alpha)(t), \overline{t})$ and for $v \in S - T$, set $\Sigma(v) = (\overline{s}_v, \overline{v})$ for any element $\overline{s}_v$ of $\eta^{-1}(v)$.

Clearly the ideal $\overline{I}$ of $k\mathcal{H}$ generated by $\{\phi(u) - u(1_{S(\gamma)}, 1_{\overline{H}}) \mid u \in k^\times\}$ is contained in the kernel of $\rho\colon k\mathcal{H} \to S(\gamma)$. Then $k\mathcal{H}/\overline{I} \cong \bigoplus_{s\in S}(k\Sigma(s))$ as k-vector spaces and $k\mathcal{H}/\overline{I}$ is a twisted group algebra over $\overline{H}$ of dimension $|\overline{H}|$ denoted by $k_\#\mathcal{H}$ and ρ induces an $\overline{H}$-algebra homorphism $\overline{\rho}\colon k_\#\mathcal{H} \to S(\gamma)$. Consequently $V_A(\gamma)$ is via $\overline{\rho}$ a $k_\#\mathcal{H}$-module.

In our situation, [5, Lemma 26.1] directly yields:

(1.7) The multiplicity module $V_A(\gamma)$ is isomorphic to the regular (left) module $k_\#\mathcal{H}$ in $k_\#\mathcal{H}$-mod.

At this point, (1.7) and the discussion on [5, p. 157] (applying [5, Example 13.5, Lemma 12.4, and Corollary 17.8]) yields our generalization of [5, Proposition 26.3] which has been independently obtained by M. Linckelmann in [3, Proposition 5.7.6]:

Theorem 1.7. *There are bijections between:*

(a) $IT(G, P, X, 1)$*;*
(b) $IT(H, P, X, 1)$*;*
(c) $\mathcal{P}(A^H)$*;*
(d) $\{\delta \in \mathcal{P}(S(\gamma)^H) \mid \overline{H}_\delta$ *is projective*$\}$*; and*
(e) *The isomorphism classes of components of the* $k_\#\mathcal{H}$*-module* $k_\#\mathcal{H}$.

Remark 1.8. As remarked in [5, Notes to Section 26], the idea of using the defect multiplicity module as a third invariant for the parameterization of indecomposable modules is an idea of L. Puig. Theorem 1.7 generalizes [5, Theorem 26.3] where X is an $\mathcal{O}$-lattice to arbitrary pairs (P, X) of G where X is any indecomposable $\mathcal{O}P$-module with vertex P. Our proof of Theorem 1.7 essentially follows the proof of [5, Theorem 26.3]. (If X is also an $\mathcal{O}$-lattice, then so is any module in $IT(G, P, X, 1)$).

Next we proceed to present our block version of Theorem 1.7

Lemma 1.9. (a) *Let* $x \in C_G(P)$*. Then* $(\pi \circ \alpha)(x) = (\rho \circ \omega)(x)$*; and* (b) *Let* $e \in B\ell(\mathcal{O}H)$ *so that* $e = \sum_{\delta\in\mathcal{C}} \delta$ *where* $\mathcal{C}$ *is an orbit of* H *on* $B\ell(OC_G(P))$*. Then* $0 \neq \omega(e) \in (k\mathcal{N})^{\overline{H}} \leq (k\mathcal{H})^{\overline{H}} = Z(k\mathcal{H})$*. Thus* $\omega(e)$ *is an idempotent of* $Z(k\mathcal{H})$*.*

For the remainder of this article, fix $B \in B\ell(\mathcal{O}G)$ and set $N = N_G(P)$.

Lemma 1.10. *The following two conditions are equivalent:*
(a) *P is contained in a defect group of B; and*
(b) $IT(G, P, X, B) \neq \phi$*.*

Let $\mathcal{B}(B, H) = \{e \in B\ell(\mathcal{O}H) \mid Br_P(B)\overline{e} = \overline{e}\}$ so that $\mathcal{B}(B, H) = \{e \in B\ell(\mathcal{O}H) \mid e^G = B\}$ by [4, Theorem 5.3.5]. Set $E = \sum_{e\in\mathcal{B}(B,H)} e$. Thus we may assume:

(1.8) $IT(H, P, X, E) = \bigcup_{e\in\mathcal{B}(B,H)} IT(H, P, X, e)$ is disjoint and $IT(H, P, X, e) \neq \phi$ for each $e \in \mathcal{B}(B, H)$. (since $P \trianglelefteq H$).

From [2, Theorem 1.6(c)] we have:

(1.9) Pu_G^H induces a bijection from $IT(G, P, X, B)$ to $IT(H, P, X, E)$.

Fix $e \in \mathcal{B}(B, H)$.

Here $\alpha\colon \mathcal{O}H \to A$ such that $\alpha(u)w = uw$ for all $u \in \mathcal{O}H$ and $w \in \mathrm{Ind}_P^H(X)$ induces the $\mathcal{O}$-algebra homomorphism $\alpha\colon Z(\mathcal{O}H) \to A^H$. Also $IT(H, P, X, e) \neq \phi$ and we may assume that each module in $IT(H, P, X, e)$ is a component of $e\mathrm{Ind}_P^H(X)$. Thus $\alpha(e)$ is an idempotent of A^H.

Set $C = \alpha(e)A\alpha(e)$ so that C is an interior H-subalgebra of A, $C \cong \mathrm{End}_{\mathcal{O}}(e\mathrm{Ind}_P^H(X))$ as interior H-algebras and the inclusion map $I\colon C \to A$ is an embedding of interior H-algebras.

Lemma 1.1(a) implies that each component of $e\mathrm{Ind}_P^H(X)$ has (P, X) as its "unique" vertex-source pair and $IT(H, P, X, e)$ bijects with $\mathcal{P}\mathrm{End}_{\mathcal{O}H}(e\mathrm{Ind}_P^H)$ where $\mathrm{End}_{\mathcal{O}H}(e\mathrm{Ind}_P^H(X)) \cong \mathrm{End}_{\mathcal{O}}(e\mathrm{Ind}_P^H(X))^H \cong C^H$ as $\mathcal{O}$-algebras.

Thus [5, Proposition 15.1 (a)] implies that C^P has a unique point γ' such that $I_*(P_{\gamma'}) = P_\gamma$ and $P_{\gamma'}$ is a local pointed group on C. Thus $H = N_H(P_\gamma) = N_H(P_{\gamma'})$ and if $M_{\gamma'}$ denotes the maximal ideal of C^P such that $M_{\gamma'} \cap \gamma' = \phi$ and if $\pi'\colon C^P \to S(\gamma') = C^P/M_{\gamma'}$ denotes the canonic $\overline{H}$-algebra epimorphism, then I induces an $\overline{H}$-algebra embedding $\overline{I}(\gamma')\colon S(\gamma') \to S(\gamma)$ such that the following diagram commutes:

(1.10)
$$\begin{array}{ccc} C^P & \xrightarrow{I^P} & A^P \\ {\scriptstyle \pi'}\downarrow & & {\scriptstyle \pi}\downarrow \\ S(\gamma') & \xrightarrow{\overline{I}(\gamma')} & S(\gamma). \end{array}$$

Since $e \in (\mathcal{O}C_G(P))^H$, $\alpha(e) \in A^H \leq A^P$ so that $0 \neq (\pi \circ \alpha)(e) \in S(\gamma)^{\overline{H}}$ and $(\pi \circ \alpha)(e)$ is an idempotent of $S(\gamma)^H$ as $\mathrm{Ker}(\pi) = J(A^P)$.

Note that $I(\mathrm{Id}_C) = \alpha(e) \in A^H$.

Set $\mathcal{H}' = \{(\overline{s}', \hbar) \in S(\gamma')^\times \times \overline{H} \mid \overline{s}'\sigma'(\overline{s}')^{-1} = {}^{\hbar}\sigma' \text{ for all } \sigma' \in S(\gamma')\}$.

From [5, Theorem 19.1], we have:

Proposition 1.11. *There are bijections between*
(a) $IT(H, P, X, e)$*;*
(b) $\mathcal{P}(C^H)$*; and*
(c) $\{\delta \in \mathcal{P}(S(\gamma')^{\overline{H}}) \mid \overline{H}_\delta$ *is projective*$\}$.

Moreover we have the following diagram of $\overline{H}$-group homorphisms:

(1.11)
$$\begin{array}{ccccccccc} 1 & \longrightarrow & k^\times & \xrightarrow{\phi'} & \mathcal{H}' & \xrightarrow{\eta'} & \overline{H} & \longrightarrow & 1 \\ & & & & {\scriptstyle \rho'}\downarrow & & & & \\ & & & & S(\gamma')^\times & & & & \end{array}$$

where $\overline{H}$ acts trivially on $k^\times$ and diagonally on $\mathcal{H}'$, $\phi'(\lambda) = (\lambda 1_{S(\gamma)}, \overline{1})$ for all $\lambda \in k^\times$, η', ρ' are the projections on the second and first component, respectively, and the row is exact.

Since $\overline{I}(\gamma')\colon S(\gamma') \to S(\gamma)$ is an $\overline{H}$-algebra embedding, $\overline{I}(\gamma')$ induces the $\overline{H}$-algebra isomorphism $\overline{J}(\gamma')\colon S(\gamma') \to ((\pi\circ\alpha)(e))S(\gamma)((\pi\circ\alpha)(e))$. Note that $S(\gamma') = \mathrm{End}_k(V_C(\gamma'))$ where $V_C(\gamma')$ is the multiplicity module of $P_{\gamma'}$ on C.

From [5, Proposition 15.4] and the fact that $\overline{J}(\gamma')(\mathrm{Id}_{S(\gamma')}) = (\pi \circ \alpha)(e)$ we have:

(1.12) There is an isomorphism of the short exact sequence of groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & k^{\times} & \xrightarrow{\phi} & \mathcal{H} & \xrightarrow{\eta} & \overline{H} & \longrightarrow & 1 \\
 & & {\scriptstyle i}\big\downarrow & & {\scriptstyle f}\big\downarrow & & {\scriptstyle j}\big\downarrow & & \\
1 & \longrightarrow & k^{\times} & \xrightarrow{\phi'} & \mathcal{H}' & \xrightarrow{\eta'} & \overline{H} & \longrightarrow & 1
\end{array}$$

where i and j are identity maps and $f((\overline{s},\hbar)) = (\overline{J}(\gamma')^{-1}((\pi \circ \alpha)e)\overline{s}((\pi \circ \alpha(e))),\hbar)$ for all $(\overline{s},\hbar) \in \mathcal{H}$.

Thus f induces k-algebra isomorphisms f: $k\mathcal{H} \to k\mathcal{H}'$ and $f_{\#}$: $k_{\#}\mathcal{H} \to k_{\#}\mathcal{H}'$. Also ρ: $\mathcal{H} \to S(\gamma)^{\times}$ and ρ': $\mathcal{H}' \to S(\gamma')^{\times}$ induce k-algebra homomorphisms ρ: $k\mathcal{H} \to S(\gamma) = \mathrm{End}_k(V_A(\gamma))$, $\rho_{\#}$: $k_{\#}\mathcal{H} \to S(\gamma)$, ρ': $k\mathcal{H}' \to S(\gamma') = \mathrm{End}_k(V_C(\gamma'))$ and $\rho'_{\#}$: $k_{\#}\mathcal{H}' \to S(\gamma')$. Thus:

(1.13) $\rho'_{\#}$: $k_{\#}\mathcal{H}' \to S(\gamma') = \mathrm{End}_k(V_C(\gamma'))$ describes $V_C(\gamma')$ as a $k_{\#}\mathcal{H}'$-module; and

(1.14) $\rho'_{\#} \circ f_{\#}$: $k_{\#}\mathcal{H} \to S(\gamma')$ describes $V_C(\gamma')$ as a $k_{\#}\mathcal{H}$-module.

Since $(\pi\circ\alpha)(e) \in S(\gamma)^{\overline{H}}$ and if $(\overline{s},\hbar) \in \mathcal{H}$, then $\overline{s}((\pi\circ\alpha)(e))\overline{s}^{-1} = {}^{\hbar}((\pi\circ\alpha)(e)) = (\pi\circ\alpha)(e)$. Thus

(1.15) $(\pi \circ \alpha)(e)V_A(\gamma)$ is a $k_{\#}\mathcal{H}$-submodule of $V_A(\gamma)$.

By [5, Lemma 12.4], there is an interior $\mathcal{H}$-algebra isomorphism $\mathcal{A}$: $((\pi\circ\alpha)(e))S(\gamma)((\pi\circ\alpha)(e)) \to \mathrm{End}_k(((\pi\circ\alpha)(e))V_A(\gamma))$. Consequently $\mathcal{A}\circ\overline{J}(\gamma')\circ\rho'\circ f$: $\mathcal{H} \to \mathrm{End}_k((\pi\circ\alpha)(e)V_A(\gamma))^{\times}$ is a group homomorphism that induces the k-algebra homomorphism $\mathcal{A} \circ \overline{J}(\gamma') \circ \rho'_{\#} \circ f_{\#}$: $k_{\#}\mathcal{H} \to \mathrm{End}_k(((\pi \circ \alpha)(e))V_A(\gamma))$.

From the above, $\overline{J}(\gamma')$: $S(\gamma') \to ((\pi\circ\alpha)(e))S(\gamma)((\pi\circ\alpha)(e))$ is an $\overline{H}$-algebra isomorphism of simple k-algebras and $((\pi \circ \alpha)(e)V_A(\gamma))$ is an irreducble module for $((\pi \circ \alpha)(e))S(\gamma)((\pi \circ \alpha)(e))$. Thus there is a k-module isomorphism j': $V_C(\gamma') \to (\pi \circ \alpha)(e)V_A(\gamma)$ such that if $v' \in V_C(\gamma')$ and $\sigma' \in S(\gamma')$, then $j'(\sigma' v') = ((\mathcal{A} \circ \overline{J}(\gamma'))(\sigma'))j'(v')$.

Here $\mathcal{A} \circ \overline{J}(\gamma') \circ \rho' \circ f$: $\mathcal{H} \to \mathrm{End}_k((\pi \circ \alpha)(e)V_A(\gamma))^{\times}$ is a group homomorphism that induces the k-algebra homomorphism:

$$\mathcal{A} \circ \overline{J}(\gamma') \circ \rho'_{\#} \circ f_{\#}\colon k_{\#}\mathcal{H} \to \mathrm{End}_k(((\pi \circ \alpha)(e))V_A(\gamma)).$$

Thus if $(\overline{s},\hbar) \in \mathcal{H}$ and $v' \in V_C(\gamma')$, then

$$\begin{aligned}
j'(\rho' \circ f(\overline{s},\hbar))(v') &= (\mathcal{A} \circ \overline{J}(\gamma'))((\rho' \circ f)(\overline{s},\hbar))j'(v') \\
&= \mathcal{A} \circ \overline{J}(\gamma')(\overline{J}(\gamma')^{-1}((\pi \circ \alpha)(e))\overline{s}((\pi \circ \alpha)(e)))j'(v') \\
&= \mathcal{A}((\pi \circ \alpha)(e)\overline{s}(\pi \circ \alpha)(e))j'(v') = \overline{s}j'(v').
\end{aligned}$$

We have proved:

Theorem 1.12. *There are bijections between:*

(a) *$IT(H,P,X,e)$;*
(b) *$\mathcal{P}(C^H)$;*
(c) *$\{\delta \in \mathcal{P}(S(\gamma)^{\overline{H}}) \mid \overline{H}_{\delta}$ is a projective pointed group on $S(\gamma')\}$;*
(d) *The isomorphism classes of projective components of the $k_{\#}\mathcal{H}'$-module $V_C(\gamma')$; and*

(e) *The isomorphism classes of components of the $k_{\#}\mathcal{H}$-module $\omega(e)(k_{\#}\mathcal{H})$.*

Corollary 1.13. *There is a bijection between*

(a) *$IT(G, P, X, B)$; and*

(b) *The isomorphism classes of components of the $k_{\#}\mathcal{H}$-module $\omega(E)k_{\#}\mathcal{H}$.*

2. Required Proofs

2.1. A Proof of Lemma 1.1. Assume the situation of Lemma 1.1. Here $\mathrm{Res}^H_P(W) \cong sX$ in $\mathcal{O}P$-mod for some positive integer s. Let (R, U) be a vertex-source pair of W so that $W \mid \mathrm{Ind}^H_R(U)$ in $\mathcal{O}H$-mod. Let T be a set of representatives for the (P, R)-double cosets in H with $1 \in T$. Then $\mathrm{Res}^H_P(\mathrm{Ind}^H_R(U)) \cong \bigoplus_{t\in T}(\mathrm{Ind}^P_{P\cap({}^tR)})(\mathrm{Res}^{{}^tR}_{P\cap({}^tR)}({}_tU))$ in $\mathcal{O}P$-mod. Thus there is a $t \in T$ such that $X \mid \mathrm{Ind}^P_{P\cap({}^tR)}(\mathrm{Res}^{{}^tR}_{P\cap({}^tR)}({}_tU))$ in $\mathcal{O}P$-mod. Whence $P \le {}^tR$, $P \le R$ and since $W \mid \mathrm{Ind}^H_P(X)$ in $\mathcal{O}H$-mod, $R = P$ and (a) holds. Now [2, Corollary 1.7] implies (c): Since $\mathcal{P}(A^P) = \gamma$ by (1.4), (b) holds and we are done.

2.2. A Proof of Proposition 1.5. Assume the situation of Proposition 1.5. Clearly $\mathcal{H} \le S(\gamma)^\times \times \overline{H}$ and $\mathcal{N} \le \mathcal{H}$. Let $\overline{x} \in \overline{H}$ and $(\overline{s}, \hbar) \in S(\gamma)^\times \times \overline{H}$. Then ${}^{\overline{x}}(\overline{s}, \hbar) = ({}^{\overline{x}}\overline{s}, \overline{x}\hbar\overline{x}^{-1})$. Since $\overline{s}\sigma\overline{s}^{-1} = {}^{\hbar}\sigma$ for all $\sigma \in S(\gamma)$, ${}^{\overline{x}}\overline{s}{}^{\overline{x}}\sigma({}^{\overline{x}}\overline{s})^{-1} = {}^{\overline{x}\hbar}\sigma = {}^{\overline{x}\hbar\overline{x}^{-1}}({}^{\overline{x}}\sigma)$ for all $\sigma \in S(\gamma)$. Thus (a) is proved. Let $t \in T$. Then ${}^{\overline{x}}((\pi \circ \alpha)(t), \overline{t}) = ((\pi \circ \alpha)({}^{\overline{x}}t), \overline{x}\overline{t}\overline{x}^{-1})$ and $\mathcal{N}$ is $\overline{H}$ invariant. Also let $(\overline{s}, \hbar) \in \mathcal{H}$. Then

$$\begin{aligned}(\overline{s}, \hbar)((\pi \circ \alpha)(t), \overline{t})(\overline{s}^{-1}, \hbar^{-1}) &= (\overline{s}(\pi \circ \alpha)(t)\overline{s}^{-1}, \hbar\overline{t}\hbar^{-1}) \\ &= ({}^{\hbar}((\pi \circ \alpha)(t)), \overline{(hth^{-1})}) \\ &= ((\pi \circ \alpha)({}^{h}t), \overline{hth^{-1}})\end{aligned}$$

and so $\mathcal{N} \trianglelefteq \mathcal{H}$. Let $(\overline{s}, \hbar)$ and $(\overline{u}, \overline{x}) \in \mathcal{H}$ as in (c). Then ${}^{\overline{x}}(\overline{s}, \hbar) = ({}^{\overline{x}}\overline{s}, {}^{\overline{x}}\hbar) = (\overline{usu}^{-1}, \overline{x}\hbar\overline{x}^{-1})$ and we are done.

A Proof of Lemma 1.6. Let $\Gamma = \sum_{i=1}^n k_i(\overline{s}_i, \hbar_i)$ where $k_i \in k^\times$ and $(\overline{s}_i, \hbar_i) \in \mathcal{H}$ for all $1 \le i \le n$. Let $(\overline{r}, \overline{x}) \in \mathcal{H}$. Then

$$\begin{aligned}(\overline{r}, \overline{x})\Gamma(\overline{r}^{-1}, \overline{x}^{-1}) &= \sum_{i=1}^n k_i(\overline{rs_i}\overline{r}^{-1}, \overline{x}\hbar_i\overline{x}^{-1}) \\ &= \sum_{i=1}^n k_i^{\overline{x}}(\overline{s_i}, \hbar_i) \\ &= {}^{\overline{x}}\Gamma.\end{aligned}$$

Thus $\Gamma \in Z(k\mathcal{H})$ if and only if $\Gamma \in (k\mathcal{H})^{\overline{H}}$ and we are done.

A Proof of Lemma 1.9. Let $x \in C_G(P)$ so that $x = tz$ for a unique $t \in T$ and $z \in Z(P)$. Then

$$(\rho \circ \omega)(x) = \rho((\pi \circ \alpha)(t), \overline{t}) = (\pi \circ \alpha)(t) = (\pi \circ \alpha)(tz)$$

since $Z(P) \le \mathrm{Ker}(\pi \circ \alpha)$ and (a) follows. As in (b), clearly $\omega(e) \in (k\mathcal{N})^{\overline{H}} \le (k\mathcal{H})^{\overline{H}} = Z(k\mathcal{H})$ by Lemma 1.6. Let $-\colon \mathcal{O} \to k$ denote the canonic $\mathcal{O}$-algebra epimorphism and let $-\colon \mathcal{O}C_G(P) \to kC_G(P)$ denote the canonic $\overline{H}$-algebra epimorphism induced by $-$. Here

$\overline{\omega}(\overline{e}) = \omega(e)$. As $\overline{e}$ is an idempotent of $kC_G(P)$ and $\overline{\omega}\colon k\overline{C_G(P)} \to k\mathcal{N}$ is an $\overline{H}$-algebra isomorphism (b) also follows.

A Proof of Lemma 1.10. Clearly [1, III, Corollary 6.8] yields (b) implies (a). Since [2, Corollary 1.8] implies the converse, we are done.

References

[1] W. Feit: The representation theory of finite groups, North-Holland, Amsterdam, 1982.

[2] M.E. Harris: *A block refinement of the Green–Puig correspondences in finite group modular representation theory*, Int. J. Algebra **5** (2011), 315–323.

[3] M. Linckelmann: private communication of notes for a forthcoming book.

[4] H. Nagao and Y. Tsushima: Representations of finite groups, Academic Press, San Diego, 1989.

[5] J. Thévenaz: G-algebras and modular representation theory, Oxford University Press, New York, 1995.

Department of Mathematics
Statistics and Computer Science (M/C 249)
University of Illinois at Chicago
851 South Morgan Street
Chicago, IL 60607–7045
USA
e-mail: meharris@uic.edu