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ANALYTIC EXTENSION OF JORGE-MEEKS TYPE MAXIMAL SURFACES IN LORENTZ-MINKOWSKI 3-SPACE

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Abstract

The Jorge-Meeks n-noid ($n \geq 2$) is a complete minimal surface of genus zero with n catenoidal ends in the Euclidean 3-space $\boldsymbol{R}^3$, which has $(2\pi/n)$-rotation symmetry with respect to its axis. In this paper, we show that the corresponding maximal surface f_n in Lorentz-Minkowski 3-space $\boldsymbol{R}^3_1$ has an analytic extension $\tilde{f}_n$ as a properly embedded zero mean curvature surface. The extension changes type into a time-like (minimal) surface.

Introduction

A number of zero mean curvature surfaces of mixed type in Lorentz-Minkowski 3-space $(\boldsymbol{R}^3_1; t, x, y)$ were found in [9], [5], [10], [8], [3], [1] and [2]. One of the main tools for the construction of such surfaces is based on the fact that fold singularities of space-like maximal surfaces have real analytic extensions to time-like minimal surfaces (cf. [5], [8], [7] and [2]). Some of the analytic extensions of such examples have neither singularities nor self-intersections. A typical such example is a space-like helicoid, which analytically extends to a time-like surface, and the entire surface coincides with the original helicoid as a minimal surface in $\boldsymbol{R}^3$. Also, the Scherk type surface

$$t(x, y) := \log \frac{\cosh y}{\cosh x} \tag{1}$$

gives an entire graph which changes type from a space-like maximal surface to a time-like zero mean curvature surface, as pointed out by Kobayashi [9]. Recently, it was shown in [3] that the space-like maximal analogues in $\boldsymbol{R}^3_1$ of the Schwarz D surfaces in $\boldsymbol{R}^3$ have analytic extensions as triply periodic embedded zero mean curvature surfaces. These examples caused the authors to be interested in space-like maximal analogues f_n $(n = 2, 3, \dots)$ of Jorge-Meeks minimal surfaces with n catenoidal ends. These surfaces have fold singularities, and have analytic extensions to time-like surfaces. We show in this paper that the analytic extension of f_n is a proper embedding.

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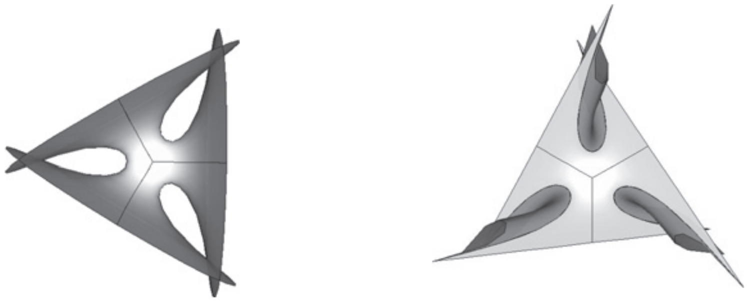

FIG. 1. The Jorge-Meeks trinoid in $\boldsymbol{R}^3$ and the analytic extension of f_3 (in the figure on the right-hand side, the time-like parts are indicated by black shading).

1. Preliminaries

We denote by $(\boldsymbol{R}^3_1; t, x, y)$ the Lorentz-Minkowski 3-space of signature $(-++)$ and denote the Riemann sphere by $S^2 := \boldsymbol{C} \cup \{\infty\}$.

DEFINITION 1.1. A pair (g, ω) consisting of a meromorphic function and a meromorphic 1-form defined on the Riemann sphere is called a *Weierstrass data on* S^2 if the metric

$$ds_E^2 := (1 + |g|^2)^2 |\omega|^2$$

has no zeros on S^2. A point where ds_E^2 diverges is called an *end* of ds_E^2.

We now fix a Weierstrass data (g, ω) on S^2 and let $\{p_1, \dots, p_n\}$ be the set of ends of ds_E^2. Then the real part of the map

$$F := \int_{z_0}^{z} (-2g, 1 + g^2, \mathrm{i}(1 - g^2))\omega \qquad \left(\mathrm{i} = \sqrt{-1}\right)$$

is a map

$$f_L = \mathrm{Re}(F),$$

into $\boldsymbol{R}^3_1$ which is defined on the universal cover of $S^2 \setminus \{p_1, \dots, p_n\}$. We call f_L the *maximal surface associated to* (g, ω), and F the *holomorphic lift* of f_L. If f_L is single-valued on $S^2 \setminus \{p_1, \dots, p_n\}$, then we say that f_L satisfies the *period condition*. The first fundamental form of f_L is given by

$$ds^2 = (1 - |g|^2)^2 |\omega|^2.$$

In particular, the singular set of f_L consists of the points where $|g| = 1$. In this situation, we set $F = (X_0, X_1, X_2)$. As pointed out in [11], the real part $f_E := \mathrm{Re}(F_E)$ of the holomorphic map $F_E := (X_1, X_2, \mathrm{i}X_0)$ gives a conformal minimal immersion into the Euclidean 3-space $\boldsymbol{R}^3$ defined on the universal cover of $S^2 \setminus \{p_1, \dots, p_n\}$ such that the first fundamental form of f_E coincides with $\pi^* ds_E^2$, where π is the covering projection. In particular, F_E (and also F) is an immersion. So the map f_L is a *maxface* in the sense of [11] (see also [4] and [2], in particular, a convenient definition of maxface which is equivalent to the original one is given in [2, Definition 2.7]). The minimal immersion f_E is called the *companion* of f_L.

We are interested in the maximal surface f_n associated to

$$g_n = z^{n-1}, \qquad \omega_n = \frac{\mathrm{i}dz}{(z^n-1)^2} \qquad (n = 2, 3, 4, \dots).$$

As pointed out in [11, Example 5.7], the companion of f_n is congruent to the well-known complete minimal surface with catenoidal ends, called a *Jorge-Meeks surface* (cf. [6]). In particular, the associated metric ds_E^2 is complete on

$$S^2 \setminus \{1, \zeta, \dots, \zeta^{n-1}\}, \qquad \text{where } \zeta := e^{2\pi\mathrm{i}/n}.$$

This means that (g_n, ω_n) is a Weierstrass data on S^2. It can be checked that f_n is single-valued on $S^2 \setminus \{1, \zeta, \dots, \zeta^{n-1}\}$, and the original Jorge-Meeks surface is as well. So f_n is a maxface, and we call $\{f_n\}_{n=2,3,\dots}$ the *Jorge-Meeks type maximal surfaces*. The singular set of f_n is the set $|z| = 1$, which consists of generic fold singularities in the sense of [2], that is, the image of the singular set consists of a union of non-degenerate null curves in $\boldsymbol{R}^3_1$.

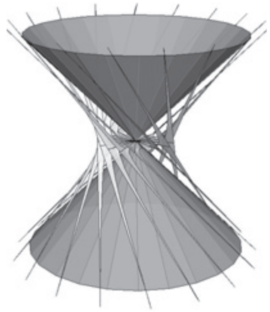

FIG. 1.1. Jorge-Meeks type maximal surface f_{17} (the light-cone is also shown).

We now observe that f_2 has a canonical analytic extension embedded in $\boldsymbol{R}^3_1$ (see Figure 1.2): By definition,

$$f_2 = \mathrm{Re}\left(\frac{\mathrm{i}}{z^2-1}, -\frac{\mathrm{i}z}{z^2-1}, \frac{1}{2}\log\frac{1-z}{1+z}\right).$$

If we set $f_2 = (x_0, x_1, x_2)$ and $z = re^{\mathrm{i}\theta}$, then

$$x_0 = \frac{r^2\sin 2\theta}{r^4 - 2r^2\cos 2\theta + 1}, \quad x_1 = -\frac{r\left(r^2+1\right)\sin\theta}{r^4 - 2r^2\cos 2\theta + 1},$$
$$x_2 = \frac{1}{4}\log\left(\frac{r^2 - 2r\cos\theta + 1}{r^2 + 2r\cos\theta + 1}\right).$$

In particular, it holds that

$$\frac{x_0}{x_1} = -\frac{2r\cos\theta}{r^2+1} = \tanh 2x_2.$$

Thus, the image of f_2 is a subset of the graph $t = x\tanh 2y$ (Figure 1.2, left), and it changes type on the set

$$S := \left\{\left(\pm\frac{\cosh 2y}{2}, y\right) ; y \in \boldsymbol{R}\right\},$$

in the xy-plane, and the connected domain with boundary S consists of the image of the orthogonal projection of f_2 into the xy-plane (cf. Figure 1.2, right). This means that the image of f_2 has an analytic extension that coincides with an entire zero mean curvature graph, like as in the case of the Scherk type surface (1) in the introduction.

Fig. 1.2. The analytic extension of f_2 and its orthogonal projection of the space-like part.

In [9], the maximal surface f_2 is called a *helicoid of the 2nd kind* and it was already pointed out that the function $t = x \tanh 2y$ is an entire solution to the maximal surface equation for graphs. The conjugate surface of f_2 (i.e. the imaginary part of the holomorphic lift of f_2) induces a singly periodic maxface, called the *hyperbolic catenoid* (see [2] for details).

2. Analytic extension of f_n

In this section, we will show that each f_n has a canonical analytic extension for $n \geq 3$ as well. By definition, the Jorge-Meeks type maximal surface $f_n = (x_0, x_1, x_2)$ and its holomorphic lift $F = (X_0, X_1, X_2)$ are given by

$$f_n = \mathrm{Re}(F), \qquad F = \int_0^z \alpha,$$

where we set

$$\begin{aligned} \alpha = \alpha(z) &= a(z)\,dz = \big(a_0(z), a_1(z), a_2(z)\big)\,dz \\ &:= \left(-\frac{2\mathrm{i}z^{n-1}}{(z^n-1)^2}, \frac{\mathrm{i}\left(1+z^{2n-2}\right)}{(z^n-1)^2}, -\frac{1-z^{2n-2}}{(z^n-1)^2}\right) dz. \end{aligned} \tag{2.1}$$

Using these expressions, we show the following:

Proposition 2.1. *Regarding f_n as a column vector-valued function, the image of f_n has the following two properties*:

$$f_n(\bar{z}) = S\,f_n(z), \qquad f_n(\zeta z) = R f_n(z),$$

where

$$S := \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad R := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\frac{2\pi}{n} & \sin\frac{2\pi}{n} \\ 0 & -\sin\frac{2\pi}{n} & \cos\frac{2\pi}{n} \end{pmatrix} \tag{2.2}$$

and

$$\zeta := e^{2\pi \mathrm{i}/n}.$$

Proof. Since

$$R = \frac{1}{2}\begin{pmatrix} 2 & 0 & 0 \\ 0 & \zeta + \zeta^{-1} & -\mathrm{i}\left(\zeta - \zeta^{-1}\right) \\ 0 & \mathrm{i}\left(\zeta - \zeta^{-1}\right) & \zeta + \zeta^{-1} \end{pmatrix},$$

the 1-form α in (2.1) satisfies

$$\overline{\alpha(\bar{z})} = S\,\alpha(z), \qquad \alpha(\zeta z) = R\alpha(z),$$

where α is considered as a column vector-valued 1-form. Since $F(0) = 0$, we have $\overline{F(\bar{z})} = S\,F(z)$ and $F(\zeta z) = RF(z)$. In particular, we have the relations $f_n(\bar{z}) = S\,f_n(z)$ and $f_n(\zeta z) = Rf_n(z)$. □

Lemma 2.2. *Up to a suitable translation in* $\boldsymbol{C}^3$ *by a vector in* $\mathrm{i}\boldsymbol{R}^3$, *the holomorphic lift* $F = (X_0, X_1, X_2)$ *of the Jorge-Meeks type maximal surface* f_n *has the following expression*:

$$X_0 = \frac{2\mathrm{i}}{n(z^n - 1)}, \tag{2.3}$$

$$X_1 = -\mathrm{i}\left(\frac{z(z^{n-2} + 1)}{n(z^n - 1)} + \frac{n-1}{n^2}\sum_{j=1}^{n-1}(\zeta^j - \zeta^{-j})\log(z - \zeta^j)\right), \tag{2.4}$$

$$X_2 = -\frac{z(z^{n-2} - 1)}{n(z^n - 1)} + \frac{n-1}{n^2}\sum_{j=0}^{n-1}(\zeta^j + \zeta^{-j})\log(z - \zeta^j). \tag{2.5}$$

Proof. The first identity (2.3) is obvious. To prove the second identity (2.4), we will show that differentiation of the right-hand side of (2.4) is equal to $a_1(z)$. Denoting the right-hand side of (2.4) by $\hat{X}_1$, we have that

$$\frac{d\hat{X}_1}{dz} - a_1(z) = \frac{d\hat{X}_1}{dz} - \frac{\mathrm{i}(1 + z^{2n-2})}{(z^n - 1)^2} = -\mathrm{i}\frac{(n-1)\varphi(z)}{n^2\,(z^n - 1)},$$

where we set

$$\varphi(z) := n\left(z^{n-2} - 1\right) + \sum_{j=1}^{n-1}\frac{(\zeta^j - \zeta^{-j})(z^n - 1)}{z - \zeta^j}.$$

For $z = \zeta^k$ $(k = 0, 1, 2, \ldots, n-1)$,

$$\begin{aligned} \varphi(\zeta^k) &= n\left(\zeta^{-2k} - 1\right) + \sum_{j=1}^{n-1}(\zeta^j - \zeta^{-j})\left.\frac{z^n - 1}{z - \zeta^j}\right|_{z=\zeta^k} \\ &= n\left(\zeta^{-2k} - 1\right) + \left(\zeta^k - \zeta^{-k}\right)n\zeta^{-k} \\ &= 0. \end{aligned} \tag{2.6}$$

Here we have used the following identity:

$$\left.\frac{z^n-1}{z-\zeta^j}\right|_{z=\zeta^k} = \begin{cases} 0 & \text{if } j \neq k, \\ \left.\dfrac{d}{dz}(z^n-1)\right|_{z=\zeta^k} = n\zeta^{-k} & \text{if } j = k. \end{cases}$$

The equality (2.6) means that the number of zeros for $\varphi(z)$ is at least n. However, $\varphi(z)$ is a polynomial in z of degree at most $n-1$. So we conclude that φ vanishes identically, and hence $d\hat{X}_1/dz - a_1(z) = 0$.

Similarly, denoting the right-hand side of (2.5) by $\hat{X}_2$, we have

$$\frac{d\hat{X}_2}{dz} - a_2(z) = \frac{d\hat{X}_2}{dz} + \frac{1-z^{2n-2}}{(z^n-1)^2} = -\frac{(n-1)\psi(z)}{n^2\,(z^n-1)},$$

where $\psi(z)$ is a polynomial of degree at most $n-1$ given by

$$\psi(z) := n\left(z^{n-2}+1\right) - \sum_{j=0}^{n-1} \frac{(\zeta^{-j}+\zeta^j)(z^n-1)}{z-\zeta^j}.$$

It can be easily checked that $\psi(\zeta^k) = 0$ for each $k = 0, 1, 2, \ldots, n-1$. These prove that $d\hat{X}_2/dz - a_2(z) = 0$, and thus (2.5) is verified. □

Using Lemma 2.2, we obtain an integration-free formula of f_n as follows.

Proposition 2.3. *The Jorge-Meeks type maximal surface $f_n = (x_0, x_1, x_2)$ has the following expressions*:

$$x_0 = \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)}, \tag{2.7}$$

$$\begin{aligned} x_1 = &-\frac{(r^{2n-1}+r)\sin\theta + (r^{n+1}+r^{n-1})\sin(n-1)\theta}{n(r^{2n}-2r^n\cos n\theta+1)} \\ &+ \frac{n-1}{n^2}\sum_{j=1}^{n-1} \log\left(r^2 - 2r\cos\left(\theta - \frac{2\pi j}{n}\right) + 1\right)\sin\frac{2\pi j}{n}, \end{aligned} \tag{2.8}$$

$$\begin{aligned} x_2 = &\frac{-(r^{2n-1}+r)\cos\theta + (r^{n+1}+r^{n-1})\cos(n-1)\theta}{n(r^{2n}-2r^n\cos n\theta+1)} \\ &+ \frac{n-1}{n^2}\sum_{j=0}^{n-1} \log\left(r^2 - 2r\cos\left(\theta - \frac{2\pi j}{n}\right) + 1\right)\cos\frac{2\pi j}{n}, \end{aligned} \tag{2.9}$$

where $z = re^{i\theta}$.

Proof. Since

$$\begin{aligned} x_0 = \operatorname{Re} X_0 &= -\frac{2\operatorname{Im}(\bar{z}^n - 1)}{n(z^n-1)(\bar{z}^n-1)} \\ &= -\frac{2\operatorname{Im}(r^n e^{-in\theta})}{n(r^{2n}-2r^n\cos n\theta+1)} = \frac{2r^n\sin n\theta}{n(r^{2n}-2r^n\cos n\theta+1)}, \end{aligned}$$

the first identity (2.7) is obtained. Similarly, one can easily check that

$$\operatorname{Re}\left(-\frac{iz\left(z^{n-2}+1\right)}{n\,(z^n-1)}\right) = -\frac{\left(r^{2n-1}+r\right)\sin\theta + \left(r^{n+1}+r^{n-1}\right)\sin(n-1)\theta}{n\,(r^{2n}-2r^n\cos n\theta+1)},$$

$$\operatorname{Re}\left(-\frac{z\left(z^{n-2}-1\right)}{n\left(z^n-1\right)}\right)=\frac{-(r^{2n-1}+r)\cos\theta+(r^{n+1}+r^{n-1})\cos(n-1)\theta}{n(r^{2n}-2r^n\cos n\theta+1)}.$$

On the other hand,

$$\begin{aligned}\operatorname{Im}\sum_{j=1}^{n-1}(\zeta^j-\zeta^{-j})\log(z-\zeta^j)&=\sum_{j=1}^{n-1}2\sin\frac{2\pi j}{n}\log|z-\zeta^j|\\&=\sum_{j=1}^{n-1}\sin\frac{2\pi j}{n}\log|z-\zeta^j|^2=\sum_{j=1}^{n-1}\sin\frac{2\pi j}{n}\log((z-\zeta^j)(\bar{z}-\zeta^{-j}))\\&=\sum_{j=1}^{n-1}\log\left(r^2-2r\cos\left(\theta-\frac{2\pi j}{n}\right)+1\right)\sin\frac{2\pi j}{n},\end{aligned}$$

which proves (2.8). Similarly, we have (2.9). □

The following assertion is an immediate consequence of Proposition 2.3.

Corollary 2.4. *f_n satisfies the identity*

$$f_n(1/r,\theta)=f_n(r,\theta)\qquad(r>0,\ 0\le\theta<2\pi). \tag{2.10}$$

Since $f_n(r,\theta)$ is invariant under the symmetry $r\mapsto 1/r$, the singular set $\{|z|=1\}$ of f_n coincides with the fixed point set under the symmetry. We remark that the set $\{|z|=1\}$ consists of non-degenerate fold singularities as in [2]. So, it is natural to introduce a new variable u by

$$u:=\frac{r+r^{-1}}{2}, \tag{2.11}$$

which is invariant under the symmetry $r\mapsto 1/r$. We set

$$\bar{D}_1^*:=\{z\in\boldsymbol{C}\,;\,0<|z|\le 1\}.$$

By Corollary 2.4, $f_n(\bar{D}_1^*\setminus\{1,\zeta,\dots,\zeta^{n-1}\})$ coincides with the whole image of f_n. To obtain the analytic extension of f_n, we define an analytic map

$$\iota:\bar{D}_1^*\ni z=re^{\mathrm{i}\theta}\mapsto\left(\frac{r+r^{-1}}{2},\theta\right)\in\boldsymbol{R}\times\boldsymbol{R}/2\pi\boldsymbol{Z}.$$

The image of the map ι is given by

$$\hat{\Omega}_n:=\{(u,\theta)\in\boldsymbol{R}\times\boldsymbol{R}/2\pi\boldsymbol{Z}\,;\,u\ge 1\}.$$

The map ι is bijective, whose inverse is given by

$$\iota^{-1}:\boldsymbol{R}\times\boldsymbol{R}/2\pi\boldsymbol{Z}\ni(u,\theta)\mapsto\left(u-\sqrt{u^2-1},\theta\right)\in\bar{D}_1^*.$$

Using the Chebyshev polynomials, the formulas (2.7)–(2.9) can be rewritten in terms of (u,θ) as follows (see the appendix for the definition and basic properties of the Chebyshev polynomials).

Corollary 2.5. *By setting $\tilde{f}_n=f_n\circ\iota^{-1}$ and $\tilde{f}_n=(\tilde{x}_0,\tilde{x}_1,\tilde{x}_2)$, it holds that*

$$\tilde{x}_0 = \frac{\sin n\theta}{n(T_n(u) - \cos n\theta)}, \tag{2.12}$$

$$\tilde{x}_1 = -\frac{T_{n-1}(u)\sin\theta + u\sin(n-1)\theta}{n(T_n(u) - \cos n\theta)} + \frac{n-1}{n^2}\sum_{j=1}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\sin\frac{2\pi j}{n}, \tag{2.13}$$

$$\tilde{x}_2 = \frac{-T_{n-1}(u)\cos\theta + u\cos(n-1)\theta}{n(T_n(u) - \cos n\theta)} + \frac{n-1}{n^2}\sum_{j=0}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\cos\frac{2\pi j}{n}, \tag{2.14}$$

where $T_n(u)$, $T_{n-1}(u)$ denote the first Chebyshev polynomials in the variable u of degree n, $n-1$, respectively.

Proof. Since

$$x_0 = \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n\cos n\theta + 1)} = \frac{\sin n\theta}{n\left(\frac{1}{2}(r^n + r^{-n}) - \cos n\theta\right)},$$

(A.2) in the appendix yields (2.12). Similarly, the first terms of (2.8) and (2.9) are the same as the first terms of (2.13) and (2.14), respectively. On the other hand,

$$\begin{aligned}
&\sum_{j=1}^{n-1}\log\left(r^2 - 2r\cos\left(\theta - \frac{2\pi j}{n}\right) + 1\right)\sin\frac{2\pi j}{n}\\
&\quad= \sum_{j=1}^{n-1}\log\left(2r\left(\frac{r+r^{-1}}{2} - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\right)\sin\frac{2\pi j}{n}\\
&\quad= \sum_{j=1}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\sin\frac{2\pi j}{n} + \log 2r\sum_{j=1}^{n-1}\sin\frac{2\pi j}{n}.
\end{aligned}$$

Then we have (2.13) because

$$\sum_{j=1}^{n-1}\sin\frac{2\pi j}{n} = \operatorname{Im}\sum_{j=0}^{n}\zeta^j = 0.$$

Similarly, we have (2.14). □

If we consider $\tilde{f}_n$ instead of f_n, the origin $z = 0$ in the source space of f_n does not lie in that of $\tilde{f}_n$. To indicate what the origin in the old complex coordinate z becomes in the new real coordinates (u, θ), we attach a new point p_∞ to $\hat{\Omega}_n$ as the 'point at infinity', and extend the map ι so that

$$\iota(0) = p_\infty.$$

Hence we have a one-to-one correspondence between $\{|z| \le 1\}$ and $\hat{\Omega}_n \cup \{p_\infty\}$. In particular, $\hat{\Omega}_n \cup \{p_\infty\}$ can be considered as an analytic 2-manifold. We prove the following:

Proposition 2.6. *The map $\tilde{f}_n : \hat{\Omega}_n \cup \{p_\infty\} \to \boldsymbol{R}^3_1$ can be analytically extended to the* *domain*

$$\Omega_n := \left\{(u,\theta) \in \boldsymbol{R} \times \boldsymbol{R}/2\pi\boldsymbol{Z}\,;\, u > \max_{j=0,\dots,n-1}\left[\cos\left(\theta - \frac{2\pi j}{n}\right)\right]\right\} \cup \{p_\infty\}. \tag{2.15}$$

Proof. In fact, (2.12)–(2.14) are meaningful if

$$T_n(u) - \cos n\theta > 0 \text{ and } u > \cos\left(\theta - \frac{2\pi j}{n}\right) \qquad (j = 0, 1, \dots, n-1). \tag{2.16}$$

Moreover, $T_n(u) - \cos n\theta$ is factorized as (cf. Lemma A.1 in the appendix)

$$T_n(u) - \cos n\theta = 2^{n-1} \prod_{j=0}^{n-1} \left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right). \tag{2.17}$$

So the condition (2.16) reduces to

$$u > \cos\left(\theta - \frac{2\pi j}{n}\right) \qquad (j = 0, 1, \dots, n-1).$$

Thus, the components $\tilde{x}_0, \tilde{x}_1$ and $\tilde{x}_2$ of $\tilde{f}_n$ given in (2.12), (2.13) and (2.14) can be extended to Ω_n. □

By Proposition 2.6, we may assume that the map $\tilde{f}_n$ is defined in Ω_n. From now on, we call this newly obtained analytic map

$$\tilde{f}_n \colon \Omega_n \to \boldsymbol{R}^3_1$$

the *analytic extension* of f_n.

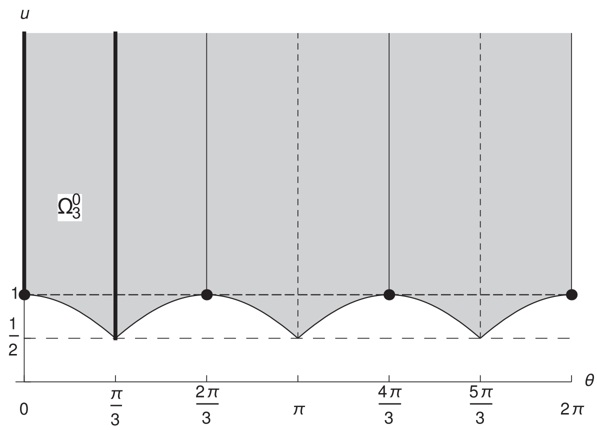

Fig. 2.1. The domain Ω_n and the fundamental domain Ω_n^0 for $n = 3$, where the region of $u > 1$ is the space-like part and the regions of $u < 1$ are the time-like parts.

Proposition 2.1 and the real analyticity of $\tilde{f}_n$ imply that

$$\tilde{f}_n(u, -\theta) = S\,\tilde{f}_n(u,\theta), \quad \tilde{f}_n\left(u, \theta + \frac{2\pi}{n}\right) = R\tilde{f}_n(u,\theta), \tag{2.18}$$

where S and R are the matrices given in (2.2), and $\tilde{f}_n$ is considered as a column vector-valued function. Let G be the finite isometry group of $\boldsymbol{R}_1^3$ generated by S and R. The subset

$$\Omega_n^0 = \left\{(u,\theta)\,;\, u > \cos\theta,\ 0 \le \theta \le \frac{\pi}{n}\right\} \tag{2.19}$$

of Ω_n is called the *fundamental domain* of $\tilde{f}_n$. The equation (2.12) yields the following proposition.

Proposition 2.7. *The whole image of $\tilde{f}_n$ can be generated by $\tilde{f}_n(\Omega_n^0)$ via the action of G, where Ω_n^0 is the fundamental domain as in* (2.19).

3. Properness of $\tilde{f}_n$

Firstly, we prepare some inequalities which will be necessary for proving that $\tilde{f}_n$ is a proper mapping.

By the definition (2.15) of Ω_n, we have the following two inequalities on Ω_n (cf. (2.16), (2.17))

$$T_n(u) > \cos n\theta, \tag{3.1}$$

and

$$u > \cos\frac{\pi}{n} \qquad \text{on } \Omega_n, \tag{3.2}$$

since the function $\max_{j=0,\dots,n-1}\left[\cos\left(\theta - 2\pi j/n\right)\right]$ has a minimum value $\cos(\pi/n)$.

Lemma 3.1. *On the fundamental domain Ω_n^0, it holds that*

$$u - \cos\left(\theta - \frac{2\pi j}{n}\right) \ge 2\sin^2\frac{\pi}{n} \qquad (j = 2,\dots,n-1). \tag{3.3}$$

Proof. Since $u > \cos\theta$ and $0 \le \theta \le \pi/n$ on Ω_n^0,

$$\begin{aligned} u - \cos\left(\theta - \frac{2\pi j}{n}\right) &> \cos\theta - \cos\left(\theta - \frac{2\pi j}{n}\right) = 2\sin\left(\frac{\pi j}{n} - \theta\right)\sin\frac{\pi j}{n} \\ &\ge 2\sin\frac{\pi(j-1)}{n}\sin\frac{\pi j}{n} \ge 2\sin^2\frac{\pi}{n} \end{aligned}$$

for $2 \le j \le n-1$, proving (3.3). □

Using these, we prove the following assertion:

Proposition 3.2. *The analytic extension $\tilde{f}_n : \Omega_n \to \boldsymbol{R}_1^3$ is a proper mapping.*

Proof. By Proposition 2.1, it is sufficient to show that the restriction of $\tilde{f}_n$ to Ω_n^0 is a proper mapping. We set

$$C := \overline{\Omega_n^0} \setminus \Omega_n^0 = \left\{(\cos\theta,\theta)\,;\, 0 \le \theta \le \frac{\pi}{n}\right\}.$$

Consider a sequence $\{(u_k,\theta_k)\}_{k=1,2,\dots}$ in Ω_n^0 such that

$$\lim_{k\to\infty}(u_k,\theta_k) = (\cos\theta_\infty,\theta_\infty) \in C \qquad \left(0 \le \theta_\infty \le \frac{\pi}{n}\right).$$

It is sufficient to show that the sequence $\{\tilde{f}_n(u_k,\theta_k)\}$ is unbounded in $\boldsymbol{R}_1^3$.

Case 1: We consider the case that $0 \le \theta_\infty < \pi/n$. The sequence $\{(u_k, \theta_k)\}$ is bounded since it converges. So we can take positive numbers u_0 and δ such that $u_k < u_0$ and $\theta_k < (\pi/n) - \delta$ for all k, that is,

$$(u_k, \theta_k) \in \Omega_{\delta,u_0} := \left\{(u,\theta) \in \Omega_n^0\,;\ u \le u_0,\ \theta \le \frac{\pi}{n} - \delta\right\}.$$

We now set (cf. (2.14))

$$\begin{aligned} \tilde{x}_2 &= \tilde{x}_{2,a} + \tilde{x}_{2,b}, \\ \tilde{x}_{2,a} &:= \frac{-T_{n-1}(u)\cos\theta + u\cos(n-1)\theta}{n(T_n(u) - \cos n\theta)}, \\ \tilde{x}_{2,b} &:= \frac{n-1}{n^2}\sum_{j=0}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\cos\frac{2\pi j}{n}. \end{aligned}$$

Since the numerator of $\tilde{x}_{2,a}$ satisfies (cf. (A.1) in the appendix)

$$-T_{n-1}(u)\cos\theta + u\cos(n-1)\theta\Big|_{u=\cos\theta} = -\cos(n-1)\theta\cos\theta + \cos\theta\cos(n-1)\theta = 0,$$

there exists a real analytic function $\varphi(u,\theta)$ such that

$$-T_{n-1}(u)\cos\theta + u\cos(n-1)\theta = (u - \cos\theta)\varphi(u,\theta).$$

Since

$$\begin{aligned} u - \cos\left(\theta - \frac{2\pi}{n}\right) &\ge \cos\theta - \cos\left(\theta - \frac{2\pi}{n}\right) \\ &= 2\sin\left(\frac{\pi}{n} - \theta\right)\sin\frac{\pi}{n} > 2\sin\delta\sin\frac{\pi}{n} \end{aligned} \tag{3.4}$$

holds on Ω_{δ,u_0}, (2.17) and (3.3) in Lemma 3.1 yield that there exist a real analytic function $\psi(u,\theta)$ and a positive number ε such that

$$T_n(u) - \cos n\theta = (u - \cos\theta)\psi(u,\theta), \qquad \psi(u,\theta) \ge \varepsilon > 0 \quad \text{on} \quad \Omega_{\delta,u_0}.$$

Thus $\tilde{x}_{2,a} = \varphi(u,\theta)/n\psi(u,\theta)$ is bounded on Ω_{δ,u_0}.

Since (3.3) in Lemma 3.1 and (3.4) imply that

$$\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right) \qquad (j = 1, 2, \ldots, n-1)$$

is bounded on Ω_{δ,u_0}, we can write

$$\tilde{x}_{2,b} = \frac{n-1}{n^2}\log(u - \cos\theta) + \beta(u,\theta),$$

where $\beta(u,\theta)$ is a real analytic function bounded on Ω_{δ,u_0}. Thus, $\tilde{x}_2(u_k,\theta_k) \to -\infty$ as $k \to \infty$.

Case 2: We next consider the case that $\theta_\infty = \pi/n$. In other words, we suppose the sequence $\{(u_k,\theta_k)\}$ converges to $(\cos(\pi/n), \pi/n)$. In this case, we seek to prove

$$\lim_{k\to\infty}\tilde{x}_1(u_k,\theta_k) = -\infty. \tag{3.5}$$

We may assume $\{(u_k,\theta_k)\} \subset \Omega_n^0 \cap \{u \le u_0\}$ for some constant u_0. We set (cf. (2.13))

$$\begin{aligned}
\tilde{x}_1 &= \tilde{x}_{1,a} + \tilde{x}_{1,b},\\
\tilde{x}_{1,a} &:= -\frac{T_{n-1}(u)\sin\theta + u\sin(n-1)\theta}{n(T_n(u) - \cos n\theta)},\\
\tilde{x}_{1,b} &:= \frac{n-1}{n^2}\sum_{j=1}^{n-1}\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\sin\frac{2\pi j}{n}.
\end{aligned}$$

Let $(u,\theta) \in \Omega_n^0$. Then $u > \cos\theta$ and $\cos\theta \in (\cos(\pi/n), 1) \subset [\cos(\pi/n), \infty)$. So both u and $\cos\theta$ belong to $[\cos(\pi/n), \infty)$. Since T_{n-1} is monotone increasing on $[\cos(\pi/n), \infty)(\subset [\cos(\pi/(n-1)), \infty))$ (cf. (A.1) and Proposition A.5 in the appendix), it holds that

$$T_{n-1}(u) > T_{n-1}(\cos\theta) = \cos(n-1)\theta \qquad \text{on} \quad \Omega_n^0.$$

Noticing this, we have

$$T_{n-1}(u)\sin\theta + u\sin(n-1)\theta \geq \cos(n-1)\theta\sin\theta + \cos\theta\sin(n-1)\theta = \sin n\theta \geq 0$$

on Ω_n^0. By (3.1), the inequality $\tilde{x}_{1,a} \leq 0$ holds on Ω_n^0. By (3.3) in Lemma 3.1,

$$\log\left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right) \qquad (j = 2, \ldots, n-1)$$

is bounded on $\Omega_n^0 \cap \{u \leq u_0\}$, and we can write

$$\tilde{x}_1 = \tilde{x}_{1,a} + \tilde{x}_{1,b} \leq \tilde{x}_{1,b} = \hat{\beta}(u,\theta) + \log\left(u - \cos\left(\theta - \frac{2\pi}{n}\right)\right)\sin\frac{2\pi}{n}$$

on Ω_n^0, where $\hat{\beta}(u,\theta)$ is a real analytic function bounded on $\Omega_n^0 \cap \{u \leq u_0\}$. Since the right-hand side tends to $-\infty$ as $(u,\theta) \to (\cos(\pi/n), \pi/n)$, (3.5) holds. □

4. Immersedness of $\tilde{f}_n$

Proposition 4.1. *The analytic extension $\tilde{f}_n \colon \Omega_n \to \boldsymbol{R}_1^3$ is an immersion.*

Proof. In this proof, f and $\tilde{f}$ denote f_n and $\tilde{f}_n$, respectively, for notational simplicity. Since $\partial/\partial z = (1/2z)(r\partial/\partial r - \mathrm{i}\partial/\partial\theta)$ for $z = re^{\mathrm{i}\theta}$, we have

$$\alpha = dF = F_z\,dz = (F + \bar{F})_z\,dz = 2f_z\,dz = \frac{1}{z}(rf_r - \mathrm{i}f_\theta)\,dz,$$

where $\alpha = a(z)\,dz$ is as given in (2.1), that is,

$$za(z) = rf_r - \mathrm{i}f_\theta. \tag{4.1}$$

On the other hand, we have (cf. (2.1))

$$za(z) = \left(\frac{-2\mathrm{i}r^n e^{\mathrm{i}n\theta}}{(r^n e^{\mathrm{i}n\theta} - 1)^2}, \frac{\mathrm{i}re^{\mathrm{i}\theta}\left(1 + r^{2n-2}e^{\mathrm{i}(2n-2)\theta}\right)}{(r^n e^{\mathrm{i}n\theta} - 1)^2}, -\frac{re^{\mathrm{i}\theta}\left(1 - r^{2n-2}e^{\mathrm{i}(2n-2)\theta}\right)}{(r^n e^{\mathrm{i}n\theta} - 1)^2}\right). \tag{4.2}$$

We define by $\xi^{jk} := {}^t(\xi^j, \xi^k)$ $(j < k)$ for $\xi = (\xi^0, \xi^1, \xi^2) \in \boldsymbol{C}^3$, and here ${}^t(*)$ means transposition. Then

$$2ir\det(f_r^{jk}, f_\theta^{jk}) = \det(rf_r^{jk} - \mathrm{i}f_\theta^{jk}, rf_r^{jk} + \mathrm{i}f_\theta^{jk}) = \det(za^{jk}, \overline{za^{jk}}).$$

Using this and (4.2), one can arrive at

$$\det(f_r^{01}, f_\theta^{01}) = \frac{2r^{n-2}\left(r^{2n} - r^2\right)\sin(n-1)\theta}{(r^{2n} - 2r^n\cos n\theta + 1)^2}, \tag{4.3}$$

$$\det(f_r^{02}, f_\theta^{02}) = -\frac{2r^{n-2}\left(r^{2n} - r^2\right)\cos(n-1)\theta}{(r^{2n} - 2r^n\cos n\theta + 1)^2}. \tag{4.4}$$

Since we set $u = (r + r^{-1})/2$, we have

$$f_u = \frac{2r^2}{r^2 - 1} f_r. \tag{4.5}$$

The equality (4.3) is equivalent to

$$\begin{aligned}\det(f_u^{01}, f_\theta^{01}) &= \frac{4r^n\left(r^{2n} - r^2\right)\sin(n-1)\theta}{(r^2 - 1)\,(r^{2n} - 2r^n\cos n\theta + 1)^2} \\ &= \frac{4\left(r^{n-1} - r^{1-n}\right)\sin(n-1)\theta}{(r - r^{-1})\,(r^n + r^{-n} - 2\cos n\theta)^2} = \frac{U_{n-2}(u)\sin(n-1)\theta}{(T_n(u) - \cos n\theta)^2},\end{aligned}$$

where $U_{n-2}(u)$ denotes the second Chebyshev polynomial of degree $n-2$. (See (A.2), (A.4) and (A.5) in the appendix.) Similarly, by (4.4), we have

$$\det(f_u^{02}, f_\theta^{02}) = -\frac{U_{n-2}(u)\cos(n-1)\theta}{(T_n(u) - \cos n\theta)^2}.$$

By the real analyticity, the identities

$$\begin{aligned}\det(\tilde{f}_u^{01}, \tilde{f}_\theta^{01}) &= \frac{U_{n-2}(u)\sin(n-1)\theta}{(T_n(u) - \cos n\theta)^2}, \\ \det(\tilde{f}_u^{02}, \tilde{f}_\theta^{02}) &= -\frac{U_{n-2}(u)\cos(n-1)\theta}{(T_n(u) - \cos n\theta)^2}\end{aligned} \tag{4.6}$$

hold on Ω_n. Hence, it cannot occur that $\det(\tilde{f}_u^{01}, \tilde{f}_\theta^{01})$ and $\det(\tilde{f}_u^{02}, \tilde{f}_\theta^{02})$ vanish simultaneously, since $U_{n-2}(u) > 0$ by (3.2) (cf. Corollary A.4 in the appendix). We conclude that $\tilde{f}_n$ is an immersion. □

5. Embeddedness of $\tilde{f}_n$

5.1. Outline. We show that $\tilde{f}_n = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)\colon \Omega_n \to (\boldsymbol{R}_1^3; t, x, y)$ is an embedding. The set

$$\tilde{x}_0^{-1}(h) \quad (\subset \Omega_n)$$

is called the *contour-line* of height $t = h$, and

$$\Lambda_h := \tilde{f}_n(\tilde{x}_0^{-1}(h)) = \tilde{f}_n(\Omega_n) \cap \{t = h\}$$

is called the *level curve set* of height $t = h$. To show the embeddedness of $\tilde{f}_n$, it is sufficient to show that $\tilde{f}_n\colon \tilde{x}_0^{-1}(h) \to \Lambda_h$ is injective at each height h.

Since (cf. (2.7) or (2.12))

$$\tilde{x}_0 = \frac{2r^n \sin n\theta}{n(r^{2n} - 2r^n \cos n\theta + 1)} = \frac{\sin n\theta}{n(T_n(u) - \cos n\theta)}, \tag{5.1}$$

the contour-line of height $h = 0$ is given by

$$\tilde{x}_0^{-1}(0) = \bigcup_{k=0}^{2n-1} \left\{ (u,\theta) \in \Omega_n \,;\, \theta = \frac{k}{n}\pi \right\} \cup \{p_\infty\}. \tag{5.2}$$

The figure of the contour-line $\tilde{x}_0^{-1}(0)$ and its image (i.e. the level curve set of height $h = 0$) are indicated in Figure 5.1, where p_∞ corresponds to the origin of the level curve.

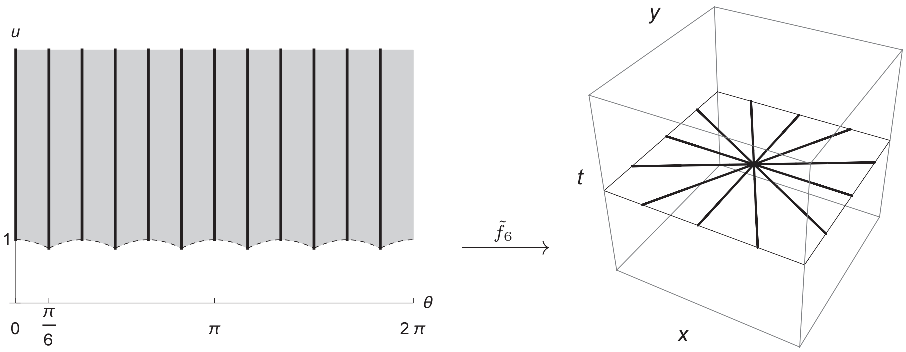

FIG. 5.1. Contour-lines $\tilde{x}_0^{-1}(0)$ and the level curve set Λ_0 at height $h = 0$ in the case $n = 6$.

On the other hand, if $h \neq 0$, we have

$$\tilde{x}_0^{-1}(h) = \left\{ (u,\theta) \in \Omega_n \,;\, T_n(u) = \cos n\theta + \frac{1}{nh} \sin n\theta \right\}.$$

The following assertion is immediately obtained.

Proposition 5.1. *The function $\tilde{x}_0$ (cf. (5.1)) is non-negative valued on Ω_n^0, where Ω_n^0 is the fundamental domain given by* (2.19)*.*

If $h < 0$, $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$ is an empty set. So we consider the case $h > 0$. Let Λ_h^0 be the level curve set of the image $\tilde{f}_n(\Omega_n^0)$ of the fundamental domain Ω_0 for $h > 0$, that is,

$$\Lambda_h^0 := \tilde{f}_n(\Omega_n^0) \cap \{t = h\} = \tilde{f}_n\left(\tilde{x}_0^{-1}(h) \cap \Omega_n^0\right) \qquad (h > 0).$$

As a consequence of Proposition 2.7, we obtain the following:

Corollary 5.2. *For each $h > 0$,*

$$\Lambda_h = \bigcup_{k=0}^{n-1} R^k \Lambda_h^0, \quad \textit{and} \quad \Lambda_{-h} = S\Lambda_h$$

hold, where R and S are the matrices defined in (2.2) (*cf.* (2.18))*.*

Corollary 5.2 implies that we should seek to prove that

(1) the map $\tilde{f}_n$ restricted to $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$, i.e., $\tilde{f}_n \colon \tilde{x}_0^{-1}(h) \cap \Omega_n^0 \to \Lambda_h^0$ is injective,
(2) $\bigcup_{k=0}^{n-1} R^k \Lambda_h^0$ is a disjoint union.

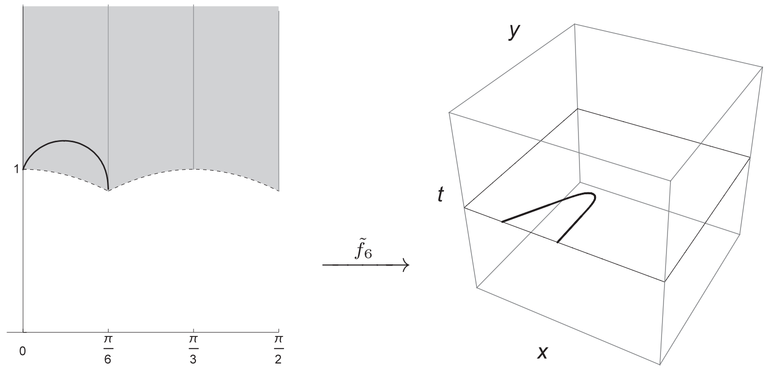

FIG.5.2. Contour-line $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$ and the level curve set Λ_h^0 for $h = 0.01$ in the case $n = 6$.

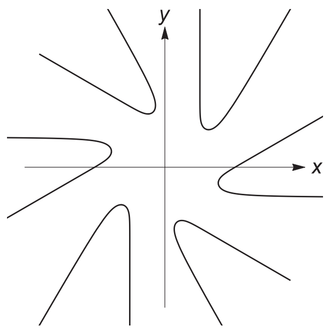

FIG.5.3. The level curve set Λ_h of $\tilde{f}_6(\Omega_6)$ for $h = 1$.

5.2. Contour-lines in Ω_n^0. Now we investigate $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$. As mentioned above, we suppose $h > 0$.

Proposition 5.3. (1) *Given $h > 0$ and $0 < \theta < \pi/n$, the equation $\tilde{x}_0(u, \theta) = h$ is uniquely solved for $u \in (\cos(\pi/n), \infty)$. Indeed, it determines the implicit function $u = u(h, \theta)$ defined on $A := \{(h, \theta)\,;\, h > 0,\ 0 < \theta < \pi/n\}$ satisfying $\cos\theta < u(h, \theta)$. Moreover, the following hold*:

(i) *For a fixed $\theta_0 \in (0, \pi/n)$, the function $h \mapsto u(h, \theta_0)$ is monotone decreasing, and*

$$\lim_{h \searrow 0} u(h, \theta_0) = \infty.$$

(ii) *For a fixed $h_0 > 0$,*

$$\lim_{\theta \searrow 0} u(h_0, \theta) = 1, \qquad \lim_{\theta \nearrow \pi/n} u(h_0, \theta) = \cos(\pi/n).$$

(2) *The derivative $(\tilde{x}_0)_u$ is given by*

$$(\tilde{x}_0)_u = -\frac{U_{n-1}(u) \sin n\theta}{(T_n(u) - \cos n\theta)^2}.$$

Proof. (1): The equation $\tilde{x}_0(u, \theta) = h(> 0)$ is equivalent to

$$T_n(u) = \cos n\theta + \frac{1}{nh} \sin n\theta,$$

and

$$-1 < \cos n\theta < \cos n\theta + \frac{1}{nh}\sin n\theta$$

holds on $A = \{(h, \theta)\,;\, h > 0,\ 0 < \theta < \pi/n\}$. On the other hand, the Chebyshev polynomial $T_n(u)$ is monotone increasing on the interval $[\cos(\pi/n), \infty)$ (see Proposition A.5 in the appendix), and hence it has the inverse function

$$T_n^{-1}\colon [-1, \infty) \to [\cos(\pi/n), \infty),$$

which is monotone increasing. Thus,

$$u(h, \theta) := T_n^{-1}\left(\cos n\theta + \frac{1}{nh}\sin n\theta\right) \quad \text{on } A \tag{5.3}$$

is well-defined and the desired one. Obviously,

$$\cos\theta = T_n^{-1}(\cos n\theta) < T_n^{-1}\left(\cos n\theta + \frac{1}{nh}\sin n\theta\right) = u(h, \theta)$$

holds on A.

Since T_n^{-1} is monotone increasing on $[-1, \infty)$, the formula (5.3) immediately implies the assertions (i) and (ii).

(2): This can be determined directly from (2.12). □

Hereafter, we set (cf. (5.3))

$$u_h(\theta) := u(h, \theta)$$

which can be considered as a function of θ fixing h. Proposition 5.3 implies that the contour-line $\tilde{x}_0^{-1}(h) \cap \Omega_n^0$ satisfies

$$\tilde{x}_0^{-1}(h) \cap \Omega_n^0 = \left\{(u, \theta) \in \Omega_n^0\,;\, u = u_h(\theta)\right\} = \left\{(u_h(\theta), \theta) \in \Omega_n^0\,;\, 0 < \theta < \pi/n\right\}. \tag{5.4}$$

The level curve set Λ_h^0, i.e., $\tilde{f}_n(\tilde{x}_0^{-1}(h) \cap \Omega_n^0)$ is given by

$$\Lambda_h^0 = \{(h, \tilde{x}_1(u_h(\theta), \theta), \tilde{x}_2(u_h(\theta), \theta))\,;\, 0 < \theta < \pi/n\}\,.$$

We show the following properties of the level curve set Λ_h^0.

Lemma 5.4.

(1) $\tilde{x}_1(u_h(\theta), \theta)$ *is a monotone decreasing function of* $\theta \in (0, \pi/n)$*, whose value is less than* $-h$*,*

(2) $\tilde{x}_2(u_h(\theta), \theta)$ *attains a maximum at* $\theta = \dfrac{\pi}{2(n-1)} \in (0, \pi/n)$.

Proof. (1): By (4.6) and Proposition 5.3 (2), we have

$$\begin{aligned}\frac{d}{d\theta}\tilde{x}_1(u_h(\theta), \theta) &= \frac{\partial \tilde{x}_1}{\partial u}\frac{du_h}{d\theta} + \frac{\partial \tilde{x}_1}{\partial \theta} = -\frac{\partial \tilde{x}_1}{\partial u}\frac{\frac{\partial \tilde{x}_0}{\partial \theta}}{\frac{\partial \tilde{x}_0}{\partial u}} + \frac{\partial \tilde{x}_1}{\partial \theta} \\ &= \frac{1}{(\tilde{x}_0)_u}\det(\tilde{f}_u^{01}, \tilde{f}_\theta^{01}) = -\frac{U_{n-2}(u_h(\theta))}{U_{n-1}(u_h(\theta))}\frac{\sin(n-1)\theta}{\sin n\theta},\end{aligned} \tag{5.5}$$

which is negative for $0 < \theta < \pi/n$ (cf. Corollary A.4 in the appendix). Hence, $\tilde{x}_1(u_h(\theta), \theta)$ is a monotone decreasing function of θ. Next, according to (2.13), we set

$$\tilde{x}_1(u_h(\theta),\theta) = \tilde{x}_{1,a}(\theta) + \tilde{x}_{1,b}(\theta),$$

where

$$\begin{aligned}\tilde{x}_{1,a}(\theta) &= -h\left(T_{n-1}(u_h(\theta))\frac{\sin\theta}{\sin n\theta} + u_h(\theta)\frac{\sin(n-1)\theta}{\sin n\theta}\right),\\ \tilde{x}_{1,b}(\theta) &= \frac{n-1}{n^2}\sum_{j=1}^{n-1}\log\left(u_h(\theta) - \cos\left(\theta - \frac{2\pi j}{n}\right)\right)\sin\frac{2\pi j}{n}.\end{aligned}$$

These satisfy

$$\lim_{\theta\searrow 0}\tilde{x}_{1,a}(\theta) = -h\left(T_{n-1}(1)\frac{1}{n} + 1\frac{n-1}{n}\right) = -h\left(\frac{1}{n} + \frac{n-1}{n}\right) = -h,$$

because of part (ii) of item (1) in Proposition 5.3. Moreover,

$$\lim_{\theta\searrow 0}\tilde{x}_{1,b}(\theta) = \frac{n-1}{n^2}\sum_{j=1}^{n-1}\log\left(1 - \cos\left(0 - \frac{2\pi j}{n}\right)\right)\sin\frac{2\pi j}{n} = 0$$

holds, since the terms in the summation cancel for each pair $(j, n-j)$. Therefore

$$\lim_{\theta\searrow 0}\tilde{x}_1(u_h(\theta),\theta) = -h + 0 = -h.$$

Since the function $\theta \mapsto \tilde{x}_1(u_h(\theta),\theta)$ is monotone decreasing, we conclude that $\tilde{x}_1(u_h(\theta),\theta) < -h$ for all $\theta \in (0,\pi/n)$.

(2): Similarly to (5.5), we have

$$\begin{aligned}\frac{d}{d\theta}\tilde{x}_2(u_h(\theta),\theta) &= \frac{\partial\tilde{x}_2}{\partial u}\frac{du_h}{d\theta} + \frac{\partial\tilde{x}_2}{\partial\theta} = -\frac{\partial\tilde{x}_2}{\partial u}\frac{\frac{\partial\tilde{x}_0}{\partial\theta}}{\frac{\partial\tilde{x}_0}{\partial u}} + \frac{\partial\tilde{x}_2}{\partial\theta}\\ &= \frac{1}{(\tilde{x}_0)_u}\det(\tilde{f}^{02}_u, \tilde{f}^{02}_\theta) = \frac{U_{n-2}(u_h(\theta))}{U_{n-1}(u_h(\theta))}\frac{\cos(n-1)\theta}{\sin n\theta},\end{aligned} \tag{5.6}$$

which is

$$\begin{cases}\text{positive if } \ 0 < \theta < \frac{\pi}{2(n-1)},\\ \text{zero if } \ \theta = \frac{\pi}{2(n-1)},\\ \text{negative if } \ \frac{\pi}{2(n-1)} < \theta < \frac{\pi}{n}.\end{cases}$$

This proves the assertion (2). □

Proposition 5.5. *The restriction of the map $\tilde{f}_n$ given by*

$$\tilde{f}_n\colon \tilde{x}_0^{-1}(h)\cap\Omega_n^0 \ni (u_h(\theta),\theta) \mapsto (h, \tilde{x}_1(u_h(\theta),\theta), \tilde{x}_2(u_h(\theta))) \in \Lambda_h^0 \tag{5.7}$$

is injective.

Proof. The equation (5.4) and Lemma 5.4 (1) imply that the above correspondence (5.7) gives a regular curve without self-intersection. □

5.3. Level curve sets. Firstly, we deal with the level curve set Λ_0 of height $h = 0$ (cf. Figure 5.1).

Proposition 5.6. *The map $\tilde{f}_n$ restricted to $\tilde{x}_0^{-1}(0)$ is injective.*

To prove the assertion, we prepare the following lemma:

Lemma 5.7.

$$(\tilde{x}_1)_u = \frac{\sin(2n-1)\theta + 2U_{n-2}(u)\sin(n-1)\theta + U_{2n-2}(u)\sin\theta}{2(T_n(u) - \cos n\theta)^2}, \tag{5.8}$$

$$(\tilde{x}_2)_u = \frac{-\cos(2n-1)\theta - 2U_{n-2}(u)\cos(n-1)\theta + U_{2n-2}(u)\cos\theta}{2(T_n(u) - \cos n\theta)^2}. \tag{5.9}$$

Proof. By (4.1), (4.2) and (4.5), we obtain that

$$\begin{aligned}
\frac{(\tilde{x}_1)_u}{2} &= \frac{r}{r^2-1}\operatorname{Re}(za_1(z)) \\
&= \frac{2(r^{3n} - r^{n+2})\sin(n-1)\theta + \left(r^2-1\right)r^{2n}\sin(2n-1)\theta + \left(r^{4n} - r^2\right)\sin\theta}{(r^2-1)\left(r^{2n} - 2r^n\cos n\theta + 1\right)^2}, \\
\frac{(\tilde{x}_2)_u}{2} &= \frac{r}{r^2-1}\operatorname{Re}(za_2(z)) \\
&= -\frac{2(r^{3n} - r^{n+2})\cos(n-1)\theta + \left(r^2-1\right)r^{2n}\cos(2n-1)\theta + \left(r^2 - r^{4n}\right)\cos\theta}{(r^2-1)\left(r^{2n} - 2r^n\cos n\theta + 1\right)^2}.
\end{aligned}$$

So $(r + r^{-1})/2 = u$ proves (5.8) and (5.9). □

Proof of Proposition 5.6. Recall the equality (5.2) which asserts that

$$\tilde{x}_0^{-1}(0) = \bigcup_{k=0}^{2n-1} B_k \cup \{p_\infty\},$$

where

$$B_k := \left\{(u,\theta) \in \Omega_n\,;\, \theta = \frac{k}{n}\pi\right\}.$$

Consider the map

$$\tilde{f}_n|_{B_k} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2)|_{\theta = k\pi/n} = (0, \tilde{x}_1(u, k\pi/n), \tilde{x}_2(u, k\pi/n)).$$

It follows from (5.8), (5.9) and (A.6) that

$$\frac{d}{du}\left(\tilde{f}_n|_{B_k}\right) = V(u)\left(0, \sin\frac{k}{n}\pi, \cos\frac{k}{n}\pi\right),$$

where

$$V(u) := \frac{U_{n-2}(u)}{T_n(u) - (-1)^k}.$$

This implies that $\tilde{f}_n|_{B_k}$ parametrizes a straight half-line with the velocity $V(u)$. If k is even, $\tilde{f}_n|_{B_k}$ is defined on the interval $(1,\infty)$ and $V(u)$ is positive on $(1,\infty)$. If k is odd, $\tilde{f}_n|_{B_k}$ is defined on the interval $(\cos(\pi/n),\infty)$ and $V(u)$ is positive on $(\cos(\pi/n),\infty)$. Hence, for any

k, the map $\tilde{f}_n|_{B_k}$ is injective. Moreover, the monotonicity of $\tilde{f}_n|_{B_k}$ and the equality

$$\lim_{u\to\infty} \tilde{f}_n|_{B_k}(u) = \tilde{f}_n(p_\infty) = (0,0,0)$$

imply that the point p_∞ is the unique inverse-image of $(0,0,0)$. Therefore we conclude the map $\tilde{f}_n\colon \tilde{x}_0^{-1}(0) \to \boldsymbol{R}^3_1$ is injective. □

We next consider the case where the height h is not equal to 0.

For a fixed h, let P_h denote a plane in $(\boldsymbol{R}^3_1; t, x, y)$ defined by the equation $t = h$, with coordinate system (x, y).

Proposition 5.8. *For any fixed $h > 0$, the level curve set Λ^0_h of height h lies in the region*

$$D_h := \{(x, y)\,;\ x < -h,\ x\cos(2\pi/n) - y\sin(2\pi/n) + h > 0\} \subset P_h.$$

Proof. We parametrize Λ^0_h so that (cf. Proposition 5.5)

$$(x_h(\theta), y_h(\theta)) := (\tilde{x}_1(u_h(\theta), \theta), \tilde{x}_2(u_h(\theta), \theta)) \qquad (0 < \theta < \pi/n).$$

We have already shown that $x_h(\theta) < -h$ (cf. Lemma 5.4 (1)). It remains to show

$$\varphi_h(\theta) := x_h(\theta)\cos(2\pi/n) - y_h(\theta)\sin(2\pi/n) + h > 0$$

for $0 < \theta < \pi/n$. Using (5.5), (5.6), we have

$$\frac{d}{d\theta}\varphi_h(\theta) = -\frac{U_{n-2}(u_h(\theta))}{U_{n-1}(u_h(\theta))\sin n\theta}\sin\left((n-1)\theta + \frac{2\pi}{n}\right).$$

This implies that $\varphi_h(\theta)$ has a minimum at

$$\theta_0 = \frac{n-2}{(n-1)n}\pi.$$

Hence, we have only to prove that

$$\varphi_h(\theta_0) > 0. \tag{5.10}$$

Indeed,

$$\Phi(h) := \varphi_h(\theta_0) = \tilde{x}_1(u_h(\theta_0), \theta_0)\cos(2\pi/n) - \tilde{x}_2(u_h(\theta_0), \theta_0)\sin(2\pi/n) + h$$

satisfies

$$\lim_{h\searrow 0}\Phi(h) = 0\cdot\cos(2\pi/n) - 0\cdot\sin(2\pi/n) + 0 = 0, \tag{5.11}$$

because of part (i) of item (1) in Proposition 5.3. Moreover, a straightforward computation using Proposition 5.3 (2), (5.8) and (5.9) leads us to

$$\frac{d\Phi}{dh}(h) = \frac{1 + U_{2n-2}(u_h(\theta_0))}{2U_{n-1}(u_h(\theta_0))} + 1 = \frac{1 + U_{2n-2}(u_h(\theta_0)) + 2U_{n-1}(u_h(\theta_0))}{2U_{n-1}(u_h(\theta_0))}.$$

We wish to know the sign of $d\Phi/dh$. Note that $u_h(\theta_0) \in (\cos\theta_0, \infty)$ for $h \in (0, \infty)$. For this purpose, we set

$$\Upsilon(u) := \frac{1 + U_{2n-2}(u) + 2U_{n-1}(u)}{2U_{n-1}(u)} \qquad \text{for } u \in (\cos\theta_0, \infty).$$

Then, it is obvious $\Upsilon(u) > 0$ for $u \in [1,\infty)$ (cf. Proposition A.3 in the appendix). For $u \in (\cos\theta_0, 1)$, it is also obvious that the denominator of $\Upsilon(u)$ is positive. Since there exists a unique $\alpha \in (0,\theta_0)$ with $u = \cos\alpha$, the numerator is computed as

$$1 + U_{2n-2}(\cos\alpha) + 2U_{n-1}(\cos\alpha) = \frac{\sin\alpha + \sin(2n-1)\alpha + 2\sin n\alpha}{\sin\alpha}$$
$$= \frac{2\sin n\alpha\cos(n-1)\alpha + 2\sin n\alpha}{\sin\alpha} = \frac{2\sin n\alpha}{\sin\alpha}(\cos(n-1)\alpha + 1).$$

So the numerator is positive because $0 < \alpha < \theta_0 = \frac{n-2}{(n-1)n}\pi$.

Thus, $\Upsilon(u) > 0$ for all $u \in (\cos\theta_0, \infty)$. Hence we obtain

$$\frac{d\Phi}{dh}(h) = \Upsilon(u_h(\theta_0)) > 0 \qquad \text{for } h \in (0,\infty). \tag{5.12}$$

It follows from (5.11) and (5.12) that $\Phi(h) > 0$ for all $h \in (0,\infty)$, that is, $\varphi_h(\theta_0) > 0$ for all $h \in (0,\infty)$. We have now proved (5.10). □

We are in a position to complete a proof of the embeddedness of $\tilde{f}_n$.

Theorem 5.9. *For any integer $n \geq 2$, the analytic extension $\tilde{f}_n : \Omega_n \to \boldsymbol{R}^3_1$ is a proper embedding.*

Proof. The assertion for $n = 2$ is trivial, as stated in Section 1. We have already proved that $\tilde{f}_n$ is a proper immersion (cf. Propositions 3.2 and 4.1). So it is sufficient to show that $\tilde{f}_n$ is injective for each $n \geq 3$. For this purpose, we will show that $\tilde{f}_n$ restricted to each contour-line $\tilde{x}_0^{-1}(h)$ is injective. We have already done this for $h = 0$ in Proposition 5.6. For $h \neq 0$, it suffices to show Λ_h^0 never intersects the other $R^k\Lambda_h^0$ ($k = 1, 2, \ldots, n-1$), since we have already seen $\tilde{f}_n \colon \tilde{x}_0^{-1}(h) \cap \Omega_n^0 \to \Lambda_h^0 \subset D_h$ is injective (cf. Proposition 5.5). In fact, the region D_h of Proposition 5.8 does not intersect the other $R^k(D_h)$ ($k = 1, 2, \ldots, n-1$) (see Figures 5.4 and 5.5), thus, Λ_h^0 never intersects the other $R^k\Lambda_h^0$. Therefore, we conclude that $\tilde{f}_n \colon \Omega_n \to \boldsymbol{R}^3_1$ is an injective proper immersion, i.e., a proper embedding. □

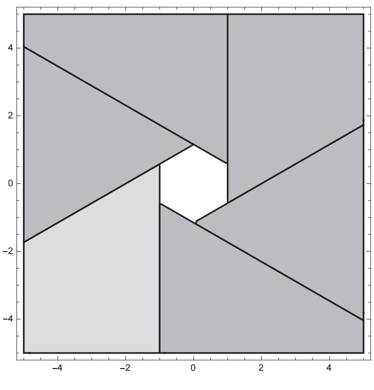

FIG. 5.4.
$\bigcup_{k=0}^{n-1} R^k(D_h)$
($n = 6$, $h = 1$)

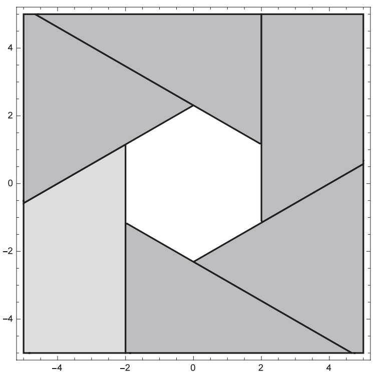

FIG. 5.5.
$\bigcup_{k=0}^{n-1} R^k(D_h)$
($n = 6$, $h = 2$)

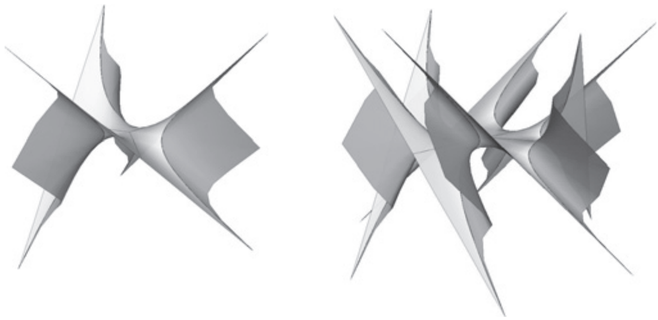

FIG. 5.6. Images of $\tilde{f}_3$ and $\tilde{f}_6$ (the time-like parts are indicated by black shading).

Appendix A. Some properties of Chebyshev polynomials

The *first Chebyshev polynomial* $T_n(x)$ $(n = 1, 2, \dots)$ is, by definition, the polynomial of degree n such that

$$T_n(\cos\theta) = \cos n\theta. \tag{A.1}$$

It holds that

$$T_n(u) = \frac{r^n + r^{-n}}{2} \qquad \left(u := \frac{r + r^{-1}}{2}\right). \tag{A.2}$$

Lemma A.1. *The following identity holds*:

$$\Psi_n(u) := T_n(u) - \cos n\theta = 2^{n-1} \prod_{j=0}^{n-1} \left(u - \cos\left(\theta - \frac{2\pi j}{n}\right)\right). \tag{A.3}$$

Proof. By (A.1), we have

$$\Psi_n\left(\cos\left(\theta - \frac{2\pi j}{n}\right)\right) = \cos\left(n\left(\theta - \frac{2\pi j}{n}\right)\right) - \cos n\theta = 0.$$

Since $\Psi_n(u)$ is a polynomial in u of degree n and the highest coefficient of $T_n(u)$ is equal to 2^{n-1}, we obtain the assertion. □

The *second Chebyshev polynomial* $U_n(x)$ $(n = 1, 2, \dots)$ is, by definition, the polynomial of degree n such that

$$\sin(n + 1)\theta = \sin\theta\, U_n(\cos\theta). \tag{A.4}$$

It holds that

$$U_{n-1}(u) = \frac{r^n - r^{-n}}{r - r^{-1}} \qquad \left(u = \frac{r + r^{-1}}{2}\right). \tag{A.5}$$

The first and the second Chebyshev polynomials are related as follows:

$$\frac{d}{dx} T_n(x) = nU_{n-1}(x).$$

Proposition A.2. *For $m \geq 1$, it holds that*

$$U_{2m}(x) - 1 = 2T_{m+1}(x)U_{m-1}(x). \tag{A.6}$$

Proof. It is sufficient to show the identity for $x = \cos\theta$ ($\theta \in [0, 2\pi)$). Then

$$\begin{aligned} U_{2m}(\cos\theta) - 1 &= \frac{\sin(2m+1)\theta}{\sin\theta} - 1 = \frac{\sin(2m+1)\theta - \sin\theta}{\sin\theta} \\ &= \frac{2\cos(m+1)\theta \sin m\theta}{\sin\theta} = 2T_{m+1}(\cos\theta)U_{m-1}(\cos\theta). \end{aligned}$$

□

Proposition A.3. *Let n be an integer greater than 2 (resp. $n = 2$). Then $y = U_{n-1}(x)$ is monotone increasing on the interval $\{x\,;\,\cos(\pi/(n-1)) \le x < \infty\}$ and the range is $\{y\,;\,-1 \le y < \infty\}$ (resp. $\{y\,;\,-2 \le y < \infty\}$). Furthermore, $U_{n-1}(\cos(\pi/n)) = 0$ and $U_{n-1}(1) = n$ hold.*

Corollary A.4. *For arbitrary $m \le n-1$,*

$$U_m(x) > 0 \textit{ for } \cos(\pi/n) < x < \infty.$$

Proposition A.5. *Let n be an integer greater than or equal to 2. Then $y = T_n(x)$ is monotone increasing on the interval $\{x\,;\,\cos(\pi/n) \le x < \infty\}$ and the range is $\{y\,;\,-1 \le y < \infty\}$. Furthermore, $T_n(\cos(\pi/(2n))) = 0$ and $T_n(1) = 1$ hold.*

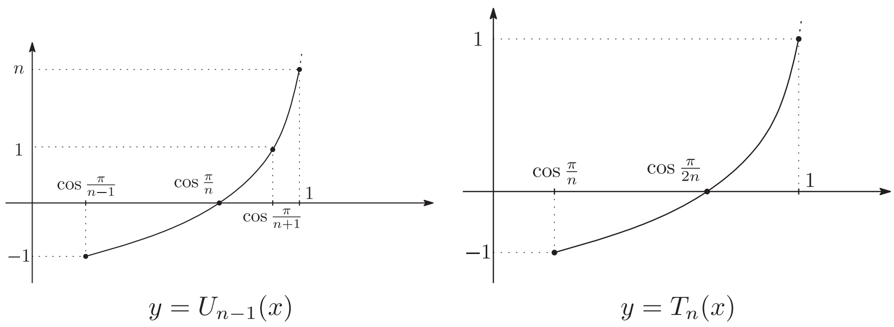

FIG. A.1. Chebyshev polynomials are monotone increasing on the interval toward the right.

NOTE ADDED IN PROOF. After submitting the paper, the authors wrote another paper

Entire zero-mean curvature graphs of mixed type in Lorentz-Minkowski 3-space,

published in Quart. J. Math. 67 (2016), 801–837 (doi:10.1093/qmath/haw038), in which a class of maximal surfaces called 'Kobayashi surfaces' is introduced. Kobayashi surfaces admit analytic extensions, some of which are not only properly embedded but also expressed as entire zero-mean curvature graphs of mixed type without singularities. The Jorge-Meeks type surfaces f_n investigated in the present paper are Kobayashi surfaces. However, the results of the above mentioned paper do not imply the embeddedness of the analytic extension $\tilde{f}_n$ of f_n. In fact, the method used in the above paper cannot apply to the case here, since $\tilde{f}_n$ ($n \ge 3$) are not expressed as entire graphs over the space-like plane.

References

[1] S. Fujimori, Y.W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada and S.-D. Yang: *Zero mean curvature surfaces in Lorentz-Minkowski* 3*-space which change type across a light-like line*, Osaka J. Math., **52** (2015), 285–297; Erratum: Osaka J. Math., **53** (2016), 289–292.

[2] S. Fujimori, Y.W. Kim, S.-E. Koh, W. Rossman, H. Shin, M. Umehara, K. Yamada and S.-D. Yang: *Zero mean curvature surfaces in Lorentz-Minkowski* 3*-space and* 2*-dimensional fluid mechanics*, Math. J. Okayama Univ., **57** (2015), 173–200.

[3] S. Fujimori, W. Rossman, M. Umehara, K. Yamada and S.-D. Yang: *Embedded triply periodic zero mean curvature surfaces of mixed type in Lorentz-Minkowski* 3*-space*, Michigan Math. J., **63** (2014), 189–207.

[4] S. Fujimori, K. Saji, M. Umehara and K. Yamada: *Singularities of maximal surfaces*, Math. Z., **259** (2008), 827–848.

[5] C. Gu: *The extremal surfaces in the* 3*-dimensional Minkowski space*, Acta Math. Sinica, **1** (1985), 173–180.

[6] L.P. Jorge and W.H. Meeks, III: *The topology of complete minimal surfaces of finite total Gaussian curvature*, Topology, **22** (1983), 203–221.

[7] Y.W. Kim, S.-E Koh, H. Shin and S.-D. Yang: *Spacelike maximal surfaces, timelike minimal surfaces, and Björling representation formulae*, J. Korean Math. Soc., **48** (2011), 1083–1100.

[8] V.A. Klyachin: *Zero mean curvature surfaces of mixed type in Minkowski space*, Izvestiya Math., **67** (2003), 209–224.

[9] O. Kobayashi: *Maximal surfaces in the* 3*-dimensional Minkowski space* $\mathbb{L}^3$, Tokyo J. Math., **6** (1983), 297–309.

[10] V. Sergienko and V.G. Tkachev: *Doubly periodic maximal surfaces with singularities*, Proceedings on Analysis and Geometry (Russian) (Novosibirsk Akademgorodok, 1999), 571–584, Izdat. Ross. Akad. Nauk Sib. Otd. Inst. Mat., Novosibirsk, 2000.

[11] M. Umehara and K. Yamada: *Maximal surfaces with singularities in Minkowski space*, Hokkaido Math. J., **35** (2006), 13–40.

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