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RIGIDITY OF MANIFOLDS WITH BOUNDARY UNDER A LOWER RICCI CURVATURE BOUND

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Abstract

We study Riemannian manifolds with boundary under a lower Ricci curvature bound, and a lower mean curvature bound for the boundary. We prove a volume comparison theorem of Bishop-Gromov type concerning the volumes of the metric neighborhoods of the boundaries. We conclude several rigidity theorems. As one of them, we obtain a volume growth rigidity theorem. We also show a splitting theorem of Cheeger-Gromoll type under the assumption of the existence of a single ray.

1. Introduction

In this paper, we study Riemannian manifolds with boundary under a lower Ricci curvature bound, and a lower mean curvature bound for the boundary. Heintze and Karcher in [18], and Kasue in [22] ([21]), have proved several comparison theorems for such manifolds with boundary. Furthermore, Kasue has proved rigidity theorems in [23], [24] for such manifolds with boundary (see also [25], [20]). These rigidity theorems state that if such manifolds satisfy suitable rigid conditions, then there exist diffeomorphisms preserving the Riemannian metrics between the manifolds and the model spaces. Other rigidity results have been also studied in [10] and [36], and so on.

In order to develop the geometry of such manifolds with boundary, we prove a volume comparison theorem of Bishop-Gromov type concerning the metric neighborhoods of the boundaries, and produce a volume growth rigidity theorem. We also prove a splitting theorem of Cheeger-Gromoll type under the assumption of the existence of a single ray emanating from the boundary. We obtain a lower bound for the smallest Dirichlet eigenvalues for the p-Laplacians. We also add a rigidity result to the list of the rigidity results obtained by Kasue in [24] on the smallest Dirichlet eigenvalues for the Laplacians.

The preceding rigidity results mentioned above have stated the existence of Riemannian isometries between manifolds with boundary and the model spaces. On the other hand, our rigidity results discussed below states the existence of isometries as metric spaces from a view point of metric geometry. These notions are equivalent to each other (see Subsection 2.3).

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1.1. Main results. For $\kappa \in \mathbb{R}$, we denote by M^n_κ the n-dimensional space form with constant curvature κ, and by g^n_κ the standard Riemannian metric on M^n_κ.

We say that $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the *ball-condition* if there exists a closed geodesic ball $B^n_{\kappa,\lambda}$ in M^n_κ with non-empty boundary $\partial B^n_{\kappa,\lambda}$ such that $\partial B^n_{\kappa,\lambda}$ has a constant mean curvature λ. We denote by $C_{\kappa,\lambda}$ the radius of $B^n_{\kappa,\lambda}$. We see that κ and λ satisfy the ball-condition if and only if either (1) $\kappa > 0$; (2) $\kappa = 0$ and $\lambda > 0$; or (3) $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$. Let $s_{\kappa,\lambda}(t)$ be a unique solution of the so-called Jacobi-equation

$$f''(t) + \kappa f(t) = 0$$

with intial conditions $f(0) = 1$ and $f'(0) = -\lambda$. We see that κ and λ satisfy the ball-condition if and only if the equation $s_{\kappa,\lambda}(t) = 0$ has a positive solution; in particular, $C_{\kappa,\lambda} = \inf\{t > 0 \mid s_{\kappa,\lambda}(t) = 0\}$.

We denote by $\mathbb{S}^{n-1}$ the $(n-1)$-dimensional standard unit sphere. Let ds^2_{n-1} be the canonical metric on $\mathbb{S}^{n-1}$. For an arbitrary pair of $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define an n-dimensional *model space $M^n_{\kappa,\lambda}$ with constant mean curvature boundary with Riemannian metric $g^n_{\kappa,\lambda}$* as follows: If $\kappa > 0$, then we put $(M^n_{\kappa,\lambda}, g^n_{\kappa,\lambda}) := (B^n_{\kappa,\lambda}, g^n_\kappa|_{B^n_{\kappa,\lambda}})$. If $\kappa \le 0$, then

$$(M^n_{\kappa,\lambda}, g^n_{\kappa,\lambda}) := \begin{cases} (B^n_{\kappa,\lambda}, g^n_\kappa|_{B^n_{\kappa,\lambda}}) & \text{if } \lambda > \sqrt{|\kappa|}, \\ (M^n_\kappa \setminus \operatorname{Int} B^n_{\kappa,-\lambda}, g^n_\kappa|_{M^n_\kappa \setminus \operatorname{Int} B^n_{\kappa,-\lambda}}) & \text{if } \lambda < -\sqrt{|\kappa|}, \\ ([0,\infty) \times \mathbb{S}^{n-1}, dt^2 + s^2_{\kappa,\lambda}(t) ds^2_{n-1}) & \text{if } |\lambda| = \sqrt{|\kappa|}, \\ ([t_{\kappa,\lambda},\infty) \times \mathbb{S}^{n-1}, dt^2 + s^2_{\kappa,0}(t) ds^2_{n-1}) & \text{if } |\lambda| < \sqrt{|\kappa|}, \end{cases}$$

where $t_{\kappa,\lambda}$ is the unique solution of the equation $s'_{\kappa,0}(t)/s_{\kappa,0}(t) = -\lambda$ under the assumptions $\kappa < 0$ and $|\lambda| < \sqrt{|\kappa|}$. We denote by $h^{n-1}_{\kappa,\lambda}$ the induced Riemannian metric on $\partial M^n_{\kappa,\lambda}$.

For $n \ge 2$, let M be an n-dimensional, connected Riemannian manifold with boundary with Riemannian metric g. The boundary ∂M is assumed to be smooth. We denote by h the induced Riemannian metric on ∂M. We say that M is *complete* if for the Riemannian distance d_M on M induced from the length structure determined by g, the metric space (M, d_M) is complete. We denote by Ric_g the Ricci curvature on M defined by g. For $K \in \mathbb{R}$, by $\mathrm{Ric}_M \ge K$, we mean that the infimum of Ric_g on the unit tangent bundle on the interior $\operatorname{Int} M$ of M is at least K. For $x \in \partial M$, we denote by H_x the mean curvature on ∂M at x in M. For $\lambda \in \mathbb{R}$, by $H_{\partial M} \ge \lambda$, we mean $\inf_{x\in\partial M} H_x \ge \lambda$. Let $\rho_{\partial M} : M \to \mathbb{R}$ be the distance function from ∂M defined as

$$\rho_{\partial M}(p) := d_M(p, \partial M).$$

The *inscribed radius of* M is defined as

$$D(M, \partial M) := \sup_{p\in M} \rho_{\partial M}(p).$$

For $r > 0$, we put $B_r(\partial M) := \{\, p \in M \mid \rho_{\partial M}(p) \le r \,\}$. We denote by vol_g the Riemannian volume on M induced from g.

One of the main results is the following volume comparison theorem:

Theorem 1.1. *For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, and for $n \geq 2$, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Then for all $r, R \in (0, \infty)$ with $r \leq R$, we have*

$$\frac{\mathrm{vol}_g B_R(\partial M)}{\mathrm{vol}_g B_r(\partial M)} \leq \frac{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_R(\partial M^n_{\kappa,\lambda})}{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_r(\partial M^n_{\kappa,\lambda})}.$$

Theorem 1.1 is an analogue to the Bishop-Gromov volume comparison theorem ([16], [17]). What happens in the equality case can be described by using the Jacobi fields along the geodesics perpendicular to the boundary (see Remark 4.10 and Proposition 5.3).

Remark 1.2. Theorem 1.1 is a relative volume comparison theorem. Under the same setting as in Theorem 1.1, Heintze and Karcher have proved in Theorem 2.1 in [18] that the absolute volume comparison inequality

$$\frac{\mathrm{vol}_g B_r(\partial M)}{\mathrm{vol}_h \partial M} \leq \frac{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_r(\partial M^n_{\kappa,\lambda})}{\mathrm{vol}_{h^{n-1}_{\kappa,\lambda}} \partial M^n_{\kappa,\lambda}}$$

holds for every $r > 0$. This inequality can be derived from Theorem 1.1. Similar volume comparison inequalities for submanifolds have been studied in [18].

Remark 1.3. Kasue has shown in Theorem A in [23] that if κ and λ satisfy the ball-condition, then $D(M, \partial M) \leq C_{\kappa,\lambda}$ (see Lemma 4.6); moreover, if there exists a point $p_0 \in M$ such that $\rho_{\partial M}(p_0) = C_{\kappa,\lambda}$, then M is isometric to $B^n_{\kappa,\lambda}$ (see Theorem 4.7).

Remark 1.4. It has been recently shown in [28] that if M is an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq 0$ and $H_{\partial M} \geq \lambda > 0$, then $D(M, \partial M) \leq C_{0,\lambda}$; moreover, if ∂M is compact, then M is compact, and $D(M, \partial M) = C_{0,\lambda}$ if and only if M is isometric to $B^n_{0,\lambda}$. It has been recently proved in [27] that for $\kappa < 0$ and $\lambda > \sqrt{|\kappa|}$, if M is an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$, then $D(M, \partial M) \leq C_{\kappa,\lambda}$; moreover, if ∂M is compact, then $D(M, \partial M) = C_{\kappa,\lambda}$ if and only if M is isometric to $B^n_{\kappa,\lambda}$. A similar result has been proved in [27] for manifolds with boundary under a lower Bakry-Émery Ricci curvature bound. It has been also recently stated in [14] that if $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition, and if M is an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$, then $D(M, \partial M) \leq C_{\kappa,\lambda}$; moreover, if ∂M is compact, then M is compact, and $D(M, \partial M) = C_{\kappa,\lambda}$ if and only if M is isometric to $B^n_{\kappa,\lambda}$.

Remark 1.5. We prove Theorem 1.1 by using a geometric study of the cut locus for the boundary, and a comparison result for the Jacobi fields along geodesics perpendicular to the boundary.

For metric measure spaces, Strum [35], and Ohta [31], [32] have independently introduced the so-called measure contraction property that is equivalent to a lower Ricci curvature bound for manifolds without boundary. We prove a measure contraction inequality for manifolds with boundary (see Proposition 8.4). Using our measure contraction inequality, we give another proof of Theorem 1.1.

For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, if κ and λ satisfy the ball-condition, then we put $\bar{C}_{\kappa,\lambda} := C_{\kappa,\lambda}$; otherwise, $\bar{C}_{\kappa,\lambda} := \infty$. We define a function $\bar{s}_{\kappa,\lambda} : [0,\infty) \to \mathbb{R}$ by

$$\bar{s}_{\kappa,\lambda}(t) := \begin{cases} s_{\kappa,\lambda}(t) & \text{if } t < \bar{C}_{\kappa,\lambda}, \\ 0 & \text{if } t \geq \bar{C}_{\kappa,\lambda}, \end{cases}$$

and define a function $f_{n,\kappa,\lambda} : [0,\infty) \to \mathbb{R}$ by

$$f_{n,\kappa,\lambda}(t) := \int_0^t \bar{s}^{n-1}_{\kappa,\lambda}(u)\, du.$$

For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, we define $[0, \bar{C}_{\kappa,\lambda}) \times_{\kappa,\lambda} \partial M$ as the warped product $([0, \bar{C}_{\kappa,\lambda}) \times \partial M, dt^2 + s^2_{\kappa,\lambda}(t)h)$ with Riemannian metric $dt^2 + s^2_{\kappa,\lambda}(t)h$, and we put $d_{\kappa,\lambda} := d_{[0,\bar{C}_{\kappa,\lambda})\times_{\kappa,\lambda}\partial M}$.

Theorem 1.1 yields the following volume growth rigidity theorem:

Theorem 1.6. *For $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$, and for $n \geq 2$, let M be an n-dimensional Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Let h denote the induced Riemannian metric on ∂M. If*

$$\liminf_{r\to\infty} \frac{\mathrm{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \geq \mathrm{vol}_h \partial M,$$

then the metric space (M, d_M) is isometric to $([0, \bar{C}_{\kappa,\lambda}) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Moreover, if κ and λ satisfy the ball-condition, then (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$.

Remark 1.7. Under the same setting as in Theorem 1.6, by Theorem 1.1, we always have the following (see Proposition 5.1):

$$\limsup_{r\to\infty} \frac{\mathrm{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \leq \mathrm{vol}_h \partial M.$$

Theorem 1.6 is certainly concerned with a rigidity phenomenon.

1.2. Splitting theorems. Kasue in Theorem C in [23] has proved the following splitting theorem. For $\kappa \leq 0$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \sqrt{|\kappa|}$. If M is noncompact and ∂M is compact, then (M, d_M) is isometric to $([0,\infty) \times_{\kappa,\sqrt{|\kappa|}} \partial M, d_{\kappa,\sqrt{|\kappa|}})$. The same result has been proved by Croke and Kleiner in Theorem 2 in [9].

In [23], the proof of the splitting theorem is based on the original proof of the Cheeger-Gromoll splitting theorem in [8]. For a ray γ on M, let b_γ be the busemann function on M for γ. The key points in [23] are to show the existence of a ray γ on M such that for all $t \geq 0$ we have $\rho_{\partial M}(\gamma(t)) = t$, and the subharmonicity of the function $b_\gamma - \rho_{\partial M}$ in a distribution sense, and to apply an analytic maximal principle (see [15]). In [9], the splitting theorem has been proved by using the Calabi maximal principle ([4]) similarly to the elementary proof of the Cheeger-Gromoll splitting theorem developed by Eschenburg and Heintze in [11]. It seems that the proof in [9] relies on the compactness of ∂M.

Let M be a connected complete Riemannian manifold with boundary. For $x \in \partial M$, we denote by u_x the unit inner normal vector at x. Let $\gamma_x : [0, T) \to M$ be the geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) = u_x$. We define a function $\tau : \partial M \to \mathbb{R} \cup \{\infty\}$ by

$$\tau(x) := \sup\{\, t > 0 \mid \rho_{\partial M}(\gamma_x(t)) = t \,\}.$$

We point out that the following splitting theorem holds for the case where the boundary is not necessarily compact.

Theorem 1.8. *For $n \geq 2$ and $\kappa \leq 0$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \sqrt{|\kappa|}$. Assume that for some $x \in \partial M$, we have $\tau(x) = \infty$. Then (M, d_M) is isometric to $([0,\infty) \times_{\kappa, \sqrt{|\kappa|}} \partial M, d_{\kappa, \sqrt{|\kappa|}})$.*

Theorem 1.8 can be proved by a similar way to that of the proof of the splitting theorem in [23]. We give a proof of Theorem 1.8 in which we use the Calabi maximal principle. Our proof can be regarded as an elementary proof of the splitting theorem in [23].

REMARK 1.9. In Theorem 1.8, if ∂M is noncompact, then we can not replace the assumption of τ with that of the existence of a single ray orthogonally emanating from the boundary. For instance, we put

$$M := \{(p, q) \in \mathbb{R}^2 \mid p < 0, p^2 + q^2 \leq 1\} \cup \{(p, q) \in \mathbb{R}^2 \mid p \geq 0, |q| \leq 1\}.$$

Observe that M is a 2-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M = 0$ and $H_{\partial M} \geq 0$. For all $x \in \partial M$, we have $\tau(x) = 1$. The geodesic $\gamma_{(-1,0)}$ is a ray in M. On the other hand, M is not isometric to the standard product $[0, \infty) \times \partial M$.

1.3. Eigenvalues. Let M be a Riemannian manifold with boundary with Riemannian metric g. For $p \in [1, \infty)$, the $(1, p)$-*Sobolev space* $W_0^{1,p}(M)$ *on M with compact support* is defined as the completion of the set of all smooth functions on M whose support is compact and contained in $\mathrm{Int}\, M$ with respect to the standard $(1, p)$-Sobolev norm. Let $\|\cdot\|$ denote the standard norm induced from g, and div the divergence with respect to g. For $p \in [1, \infty)$, the *p-Laplacian* $\Delta_p f$ for $f \in W_0^{1,p}(M)$ is defined as

$$\Delta_p f := -\operatorname{div}\left(\|\nabla f\|^{p-2}\, \nabla f\right),$$

where the equality holds in a weak sense on $W_0^{1,p}(M)$. A real number λ is said to be a *p-Dirichlet eigenvalue* for Δ_p on M if we have a non-zero function f in $W_0^{1,p}(M)$ such that $\Delta_p f = \lambda |f|^{p-2} f$ holds on $\mathrm{Int}\, M$ in a weak sense on $W_0^{1,p}(M)$. For $p \in [1, \infty)$, the *Rayleigh quotient $R_p(f)$ for $f \in W_0^{1,p}(M)$* is defined as

$$R_p(f) := \frac{\int_M \|\nabla f\|^p \, d\,\mathrm{vol}_g}{\int_M |f|^p \, d\,\mathrm{vol}_g}.$$

We put $\mu_{1,p}(M) := \inf_f R_p(f)$, where the infimum is taken over all non-zero functions in $W_0^{1,p}(M)$. The value $\mu_{1,2}(M)$ is equal to the infimum of the spectrum of Δ_2 on M. If M is compact, and if $p \in (1, \infty)$, then $\mu_{1,p}(M)$ is equal to the infimum of the set of all p-Dirichlet eigenvalues for Δ_p on M.

Due to the volume estimate obtained by Kasue in [25], we obtain the following:

Theorem 1.10. *For $\kappa \in \mathbb{R}$, $\lambda \in \mathbb{R}$ and $D \in (0, \bar{C}_{\kappa,\lambda}]$, and for $n \geq 2$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that* $\mathrm{Ric}_M \geq (n-1)\kappa$, *$H_{\partial M} \geq \lambda$ and $D(M, \partial M) \leq D$. Suppose ∂M is compact. Then for all $p \in (1, \infty)$, we have*

$$\mu_{1,p}(M) \geq (\, p\, C(n, \kappa, \lambda, D)\,)^{-p},$$

where $C(n, \kappa, \lambda, D)$ is a positive constant defined by

$$C(n, \kappa, \lambda, D) := \sup_{t \in [0,D)} \frac{\int_t^D s_{\kappa,\lambda}^{n-1}(s)\, ds}{s_{\kappa,\lambda}^{n-1}(t)}.$$

REMARK 1.11. In Theorem 1.10, since ∂M is compact, $D(M, \partial M)$ is finite if and only if M is compact (see Lemma 3.4). We see that $C(n, \kappa, \lambda, \infty)$ is finite if and only if $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$; in this case, $C(n, \kappa, \lambda, D) = ((n-1)\lambda)^{-1} \left(1 - e^{-(n-1)\lambda D}\right)$; in particular, $(2\, C(n, \kappa, \lambda, \infty))^{-2} = ((n-1)\lambda/2)^2$.

REMARK 1.12. For compact manifolds with boundary of non-negative Ricci curvature, similar lower bounds for $\mu_{1,p}$ to that in Theorem 1.10 have been obtained in [26], in [37] and in [38].

We recall the works of Kasue in [24] for compact manifolds with boundary. Let $n \geq 2$, $\kappa, \lambda \in \mathbb{R}$ and $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$. Kasue has proved in Theorem 2.1 in [24] that there exists a positive constant $\mu_{n,\kappa,\lambda,D}$ such that for every n-dimensional, connected compact Riemannian manifold M with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$, $H_{\partial M} \geq \lambda$ and $D(M, \partial M) \leq D$, we have $\mu_{1,2}(M) \geq \mu_{n,\kappa,\lambda,D}$; moreover, in some extremal case, the equality holds if and only if M is isometric to some model space. The extremal case happens only if κ and λ satisfy the ball-condition or the condition that the equation $s'_{\kappa,\lambda}(t) = 0$ has a positive solution. Note that the equation $s'_{\kappa,\lambda}(t) = 0$ has a positive solution if and only if either (1) $\kappa = 0$ and $\lambda = 0$; (2) $\kappa < 0$ and $\lambda \in (0, \sqrt{|\kappa|})$; or (3) $\kappa > 0$ and $\lambda \in (-\infty, 0)$. Let

$$\bar{\mu}_{n,\kappa,\lambda,D} := \left(4 \sup_{t \in (0,D)} \int_t^D s_{\kappa,\lambda}^{n-1}(s)\, ds \int_0^t s_{\kappa,\lambda}^{1-n}(s)\, ds \right)^{-1}.$$

It has been shown in Lemma 1.3 in [24] that $\mu_{n,\kappa,\lambda,D} > \bar{\mu}_{n,\kappa,\lambda,D}$. Therefore, for every n-dimensional, connected compact Riemannian manifold M with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$, $H_{\partial M} \geq \lambda$ and $D(M, \partial M) \leq D$, we have $\mu_{1,2}(M) > \bar{\mu}_{n,\kappa,\lambda,D}$. This estimate for $\mu_{1,2}$ is better than that in Theorem 1.10.

Let $n \geq 2$, $\kappa < 0$ and $\lambda = \sqrt{|\kappa|}$. The model space $M^n_{\kappa,\lambda}$ is non-compact. For $t \in [0, \infty)$, we put $\phi_{n,\kappa,\lambda}(t) := t\, e^{\frac{(n-1)\lambda t}{2}}$. The smooth function $\phi_{n,\kappa,\lambda} \circ \rho_{\partial M^n_{\kappa,\lambda}}$ on $M^n_{\kappa,\lambda}$ satisfies $R_2(\phi_{n,\kappa,\lambda} \circ \rho_{\partial M^n_{\kappa,\lambda}}) = ((n-1)\lambda/2)^2$; hence, $\mu_{1,2}(M^n_{\kappa,\lambda}) \leq ((n-1)\lambda/2)^2$. Notice that the value $(2\, C(n, \kappa, \lambda, \infty))^{-2}$ in Theorem 1.10 is equal to $((n-1)\lambda/2)^2$ (see Remark 1.11). Theorem 1.10 implies $\mu_{1,2}(M^n_{\kappa,\lambda}) = ((n-1)\lambda/2)^2$. Let $D \in (0, \infty)$. As mentioned above, we have already known in [24] that for every n-dimensional, connected compact Riemannian manifold M with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$, $H_{\partial M} \geq \lambda$ and $D(M, \partial M) \leq D$, we have $\mu_{1,2}(M) > \bar{\mu}_{n,\kappa,\lambda,D}$. The value $\bar{\mu}_{n,\kappa,\lambda,D}$ is equal to $((n-1)\lambda/2)^2 \left(1 - e^{-(n-1)\lambda D/2}\right)^{-2}$, and tends to $\mu_{1,2}(M^n_{\kappa,\lambda})$ as $D \to \infty$.

By using Theorem 1.10 and the splitting theorem in [23], we add the following result for not necessarily compact manifolds with boundary to the list of the rigidity results obtained in [24].

Theorem 1.13. *Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. For $n \geq 2$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Then for all $p \in (1, \infty)$, we have*

$$\mu_{1,p}(M) \geq \left(\frac{(n-1)\lambda}{p}\right)^p;$$

if the equality holds for some $p \in (1, \infty)$, then (M, d_M) is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$*; moreover, if $p = 2$, then the equality holds if and only if (M, d_M) is isometric to* $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$.

REMARK 1.14. In Theorem 1.13, the author does not know whether in the case of $p \neq 2$ the value $\mu_{1,p}([0,\infty) \times_{\kappa,\lambda} \partial M)$ is equal to $((n-1)\lambda/p)^p$.

Cheeger and Colding in Theorem 2.11 in [7] have proved the segment inequality for complete Riemannian manifolds under a lower Ricci curvature bound. They have mentioned that their segment inequality gives a lower bound for the smallest Dirichlet eigenvalue for the Laplacian on a closed ball.

Based on the proof of Theorem 1.1, we prove a segment inequality of Cheeger-Colding type for manifolds with boundary (see Proposition 7.2). Using our segment inequality, we obtain a lower bound for $\mu_{1,p}$ smaller than the lower bound in Theorem 1.10 (see Proposition 7.4).

1.4. Organization. In Section , we prepare some notations and recall the basic facts on Riemannian manifolds with boundary.

In Section , for a connected complete Riemannian manifold with boundary, we study the basic properties of the cut locus for the boundary. The basic properties seem to be well-known, however, they has not been summarized in any literature. For the sake of the readers, we discuss them in order to prove our results.

In Section 4, by using the study of the cut locus for the boundary in Section , we prove Theorem 1.1.

In Section 5, we prove Theorem 1.6. The rigidity follows from the study in the equality case in Theorem 1.1.

In Section 6, we prove Theorem 1.8.

In Section 7, we prove Theorems 1.10 and 1.13. We also prove a segment inequality (see Proposition 7.2). After that, we show the Poincaré inequality (see Lemma 7.3), and we conclude Proposition 7.4.

In Section 8, we prove a measure contraction inequality (see Proposition 8.4). We also give another proof of Theorem 1.1.

2. Preliminaries

We refer to [3] for the basics of metric geometry, and to [34] for the basics of Riemannian manifolds with boundary.

2.1. Metric spaces. Let (X, d_X) be a metric space. For $r > 0$ and $A \subset X$, we denote by $U_r(A)$ the open r-neighborhood of A in X, and by $B_r(A)$ the closed one.

For a metric space (X, d_X), the length metric $\bar{d}_X$ is defined as follows: For two points $x_1, x_2 \in X$, we put $\bar{d}_X(x_1, x_2)$ to the infimum of the length of curves connecting x_1 and x_2 with respect to d_X. A metric space (X, d_X) is said to be a *length space* if $d_X = \bar{d}_X$.

Let (X, d_X) be a metric space. For an interval $I \subset \mathbb{R}$, let $\gamma : I \to X$ be a curve. We say that γ is a *normal minimal geodesic* if for all $s, t \in I$, we have $d_X(\gamma(s), \gamma(t)) = |s - t|$, and γ is a *normal geodesic* if for each $t \in I$, there exists an interval $J \subset I$ with $t \in J$ such that $\gamma|_J$ is a normal minimal geodesic. A metric space (X, d_X) is said to be a *geodesic space* if for every pair of two points in X, there exists a normal minimal geodesic connecting them. A metric space is *proper* if all closed bounded subsets of the space are compact. The Hopf-Rinow theorem for length spaces (see e.g., Theorem 2.5.23 in [3]) states that if a length space (X, d_X) is complete and locally compact, and if $d_X < \infty$, then (X, d_X) is a proper geodesic space.

2.2. Riemannian manifolds with boundary. For $n \geq 2$, let M be an n-dimensional, connected Riemannian manifold with (smooth) boundary with Riemannian metric g. For $p \in \operatorname{Int} M$, let T_pM be the tangent space at p on M, and let U_pM be the unit tangent sphere at p on M. We denote by $\|\cdot\|$ the standard norm induced from g. If $v_1, \dots, v_k \in T_pM$ are linearly independent, then we see $\|v_1 \wedge \cdots \wedge v_k\| = \sqrt{\det(g(v_i, v_j))}$. Let d_M be the length metric induced from g. If M is complete with respect to d_M, then the Hopf-Rinow theorem for length spaces tells us that the metric space (M, d_M) is a proper geodesic space.

For $x \in \partial M$, and the tangent space $T_x\partial M$ at x on ∂M, let $T_x^\perp \partial M$ be the orthogonal complement of $T_x\partial M$ in the tangent space at x on M. Take $u \in T_x^\perp \partial M$. For the second fundamental form S of ∂M, let $A_u : T_x\partial M \to T_x\partial M$ be the *shape operator* for u defined as

$$g(A_u v, w) := g(S(v, w), u).$$

Let $u_x \in T_x^\perp \partial M$ denote the unit inner normal vector at x. The *mean curvature* H_x at x is defined by

$$H_x := \frac{1}{n-1} \operatorname{trace} A_{u_x}.$$

For the normal tangent bundle $T^\perp \partial M := \bigcup_{x \in \partial M} T_x^\perp \partial M$ of ∂M, let $0(T^\perp \partial M)$ be the zero-section $\bigcup_{x \in \partial M} \{ 0_x \in T_x^\perp \partial M \}$ of $T^\perp \partial M$. For $r > 0$, we put

$$U_r(0(T^\perp \partial M)) := \bigcup_{x \in \partial M} \{ t\, u_x \in T_x^\perp \partial M \mid t \in [0, r) \}.$$

For $x \in \partial M$, we denote by $\gamma_x : [0, T) \to M$ the normal geodesic with initial conditions $\gamma_x(0) = x$ and $\gamma_x'(0) = u_x$. Note that γ_x is a normal geodesic in the usual sense in Riemannian geometry. On an open neighborhood of $0(T^\perp \partial M)$ in $T^\perp \partial M$, the normal exponential map $\exp^\perp$ of ∂M is defined as follows: For $x \in \partial M$ and $u \in T_x^\perp \partial M$, put $\exp^\perp(x, u) := \gamma_x(\|u\|)$.

Since the boundary ∂M is smooth, there exists an open neighborhood U of ∂M satisfying the following: (1) the map $\exp^{\perp}|_{(\exp^{\perp})^{-1}(U\setminus\partial M)}$ is a diffeomorphism onto $U\setminus\partial M$; (2) for every $p \in U$, there exists a unique point $x \in \partial M$ such that $d_M(p,x) = d_M(p,\partial M)$; in this case, $\gamma_x|_{[0,d_M(p,\partial M)]}$ is a unique normal minimal geodesic in M from x to p. We call such an open set U a *normal neighborhood of* ∂M. If ∂M is compact, then for some $r > 0$, the set $U_r(\partial M)$ is a normal neighborhood of ∂M.

We say that a Jacobi field Y along γ_x is a *∂M-Jacobi field* if Y satisfies the following initial conditions:

$$Y(0) \in T_x\partial M, \quad Y'(0) + A_{u_x}Y(0) \in T_x^{\perp}\partial M.$$

We say that $\gamma_x(t_0)$ is a *conjugate point* of ∂M along γ_x if there exists a non-zero ∂M-Jacobi field Y along γ_x with $Y(t_0) = 0$. Let $\tau_1(x)$ denote the first conjugate value for ∂M along γ_x. It is well-known that for all $x \in \partial M$ and $t > \tau_1(x)$, we have $t > d_M(\gamma_x(t), \partial M)$.

For all $x \in \partial M$ and $t \in [0, \tau_1(x))$, we denote by $\theta(t,x)$ the absolute value of the Jacobian of $\exp^{\perp}$ at $(x, tu_x) \in T^{\perp}\partial M$. For each $x \in \partial M$, we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. For each $i = 1, \dots, n-1$, let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x}e_{x,i}$. Note that for all $x \in \partial M$ and $t \in [0, \tau_1(x))$, we have $\theta(t,x) = \|Y_{x,1}(t) \wedge \cdots \wedge Y_{x,n-1}(t)\|$. This does not depend on the choice of the orthonormal basis.

2.3. Distance rigidity and metric rigidity. For $i = 1, 2$, let M_i be connected Riemannian manifolds with boundary with Riemannian metric g_i. For each i, the boundary ∂M_i carries the induced Riemannian metric h_i.

Definition 2.1. We say that a homeomorphism $\Phi : M_1 \to M_2$ is a *Riemannian isometry with boundary* from M_1 to M_2 if Φ satisfies the following conditions:

(1) $\Phi|_{\mathrm{Int}\, M_1} : \mathrm{Int}\, M_1 \to \mathrm{Int}\, M_2$ is smooth, and $(\Phi|_{\mathrm{Int}\, M_1})^*(g_2) = g_1$;
(2) $\Phi|_{\partial M_1} : \partial M_1 \to \partial M_2$ is smooth, and $(\Phi|_{\partial M_1})^*(h_2) = h_1$.

If there exists a Riemannian isometry $\Phi : M_1 \to M_2$ with boundary, then the inverse Φ^{-1} is also a Riemannian isometry with boundary.

The following is well-known for manifolds without boundary (see e.g., Theorem 11.1 in [19]).

Lemma 2.2. *Let M and N be connected Riemannian manifolds (without boundary) with Riemannian metric g_M and with g_N, respectively. Let d_M and d_N be the Riemannian distances on M and on N, respectively. Suppose that a map $\Psi : M \to N$ is an isometry between the metric spaces (M, d_M) and (N, d_N). Then Ψ is smooth, and $\Psi^* g_N = g_M$. Namely, Ψ is a Riemannian isometry from (M, g_M) to (N, g_N).*

For manifolds with boundary, we show the following:

Lemma 2.3. *For $i = 1, 2$, let M_i be connected Riemannian manifolds with boundary with Riemannian metric g_i. Then there exists a Riemannian isometry with boundary from M_1 to M_2 if and only if the metric space (M_1, d_{M_1}) is isometric to (M_2, d_{M_2}).*

Proof. For $i = 1, 2$, we denote by $\|\cdot\|_{g_i}$ and by $\|\cdot\|_{h_i}$ the standard norms induced from g_i and from h_i, respectively. For a piecewise smooth curve γ in M_i, we denote by $L_{g_i}(\gamma)$ the length of γ induced from g_i.

First, we show that if $\Phi : M_1 \to M_2$ is a Riemannian isometry with boundary, then it is an isometry between the metric spaces (M_1, d_{M_1}) and (M_2, d_{M_2}). It suffices to show that Φ is a 1-Lipschitz map from (M_1, d_{M_1}) to (M_2, d_{M_2}). Pick $p, q \in M_1$. Take $\epsilon > 0$. There exists a piecewise smooth curve $\gamma : [0, l] \to M_1$ such that $L_{g_1}(\gamma) < d_{M_1}(p, q) + \epsilon$. Assume that γ is smooth at $t \in [0, l]$. If $\gamma(t)$ belongs to $\operatorname{Int} M_1$, then $\|(\Phi \circ \gamma)'(t)\|_{g_2}$ is equal to $\|\gamma'(t)\|_{g_1}$. If $\gamma(t)$ belongs to ∂M_1, then $\|(\Phi \circ \gamma)'(t)\|_{h_2}$ is equal to $\|\gamma'(t)\|_{h_1}$, and hence $L_{g_2}(\Phi \circ \gamma)$ is equal to $L_{g_1}(\gamma)$. We have $d_{M_2}(\Phi(p), \Phi(q)) < d_{M_1}(p, q) + \epsilon$. This implies that Φ is 1-Lipschitz.

Next, we show that if $\Psi : M_1 \to M_2$ is an isometry between the metric spaces (M_1, d_{M_1}) and (M_2, d_{M_2}), then it is a Riemannian isometry with boundary. To do this, we first show that $\Psi|_{\operatorname{Int} M_1} : \operatorname{Int} M_1 \to \operatorname{Int} M_2$ is smooth, and $(\Psi|_{\operatorname{Int} M_1})^*(g_2) = g_1$. Take $p \in \operatorname{Int} M_1$. There exists a sufficiently small $r \in (0, \infty)$ such that $U_r(p)$ and $U_r(\Psi(p))$ are strongly convex in $(\operatorname{Int} M_1, g_1)$ and in $(\operatorname{Int} M_2, g_2)$, respectively. Then $\Psi|_{U_r(p)}$ becomes an isometry between the metric subspaces $U_r(p)$ and $U_r(\Psi(p))$. Applying Lemma 2.2 to the open Riemannian submanifolds $U_r(p)$ and $U_r(\Psi(p))$, we see that $\Psi|_{U_r(p)}$ is a smooth Riemannian isometry. This implies that $\Psi|_{\operatorname{Int} M_1} : \operatorname{Int} M_1 \to \operatorname{Int} M_2$ is smooth, and $(\Psi|_{\operatorname{Int} M_1})^*(g_2) = g_1$.

We second show that the map $\Psi|_{\partial M_1} : \partial M_1 \to \partial M_2$ is smooth, and $(\Psi|_{\partial M_1})^*(h_2) = h_1$. To do this, we prove that $\Psi|_{\partial M_1}$ is an isometry between the metric spaces $(\partial M_1, d_{\partial M_1})$ and $(\partial M_2, d_{\partial M_2})$, where $d_{\partial M_1}$ and $d_{\partial M_2}$ are the Riemannian distances on ∂M_1 and on ∂M_2, respectively. It suffices to show that $\Psi|_{\partial M_1}$ is a 1-Lipschitz map from $(\partial M_1, d_{\partial M_1})$ to $(\partial M_2, d_{\partial M_2})$. Take $x, y \in \partial M_1$. For every $\epsilon > 0$, there exists a piecewise smooth curve $\gamma : [0, l] \to \partial M_1$ such that $L_{h_1}(\gamma) < d_{\partial M_1}(x, y) + \epsilon$. Fix $t \in [0, l]$ at which γ is smooth. Since Ψ is an isometry between (M_1, d_{M_1}) and (M_2, d_{M_2}), we have

$$\begin{aligned}\|\gamma'(t)\|_{h_1} = \|\gamma'(t)\|_{g_1} &= \lim_{\delta \to 0} \frac{d_{M_1}(\gamma(t), \gamma(t+\delta))}{\delta} \\ &= \lim_{\delta \to 0} \frac{d_{M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t+\delta))}{\delta}.\end{aligned}$$

Since ∂M_2 is smooth, and since h_2 is induced from g_2, for every $z_0 \in \partial M_2$ we have

$$\lim_{z \to z_0} \frac{d_{\partial M_2}(z_0, z)}{d_{M_2}(z_0, z)} = 1,$$

where the limit is taken with respect to $d_{\partial M_2}$. Hence, we have

$$\lim_{\delta \to 0} \frac{d_{\partial M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t+\delta))}{d_{M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t+\delta))} = 1;$$

in particular,

$$\|\gamma'(t)\|_{h_1} = \lim_{\delta \to 0} \frac{d_{\partial M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t+\delta))}{\delta}.$$

It follows that

$$L_{h_1}(\gamma) = \int_0^l \lim_{\delta \to 0} \frac{d_{\partial M_2}((\Psi \circ \gamma)(t), (\Psi \circ \gamma)(t+\delta))}{\delta} \, dt.$$

The right hand side coincides with the length of $\Psi \circ \gamma$ with respect to $d_{\partial M_2}$ (see e.g., Section 2.7 in [3]), and is greater than or equal to $d_{\partial M_2}(\Psi(x), \Psi(y))$. Therefore, $d_{\partial M_2}(\Psi(x), \Psi(y)) < d_{\partial M_1}(x, y) + \epsilon$. This implies that $\Psi|_{\partial M_1}$ is 1-Lipschitz. Thus, we conclude that $\Psi|_{\partial M_1}$ is an isometry between $(\partial M_1, d_{\partial M_1})$ and $(\partial M_2, d_{\partial M_2})$. Applying Lemma 2.2 to ∂M_1 and ∂M_2, we see that $\Psi|_{\partial M_1}$ is smooth, and $(\Psi|_{\partial M_1})^*(h_2) = h_1$.

This completes the proof of Lemma 2.3. □

2.4. Comparison theorem. For $\kappa \in \mathbb{R}$, let $s_\kappa(t)$ be a unique solution of the so-called Jacobi-equation $f''(t) + \kappa f(t) = 0$ with initial conditions $f(0) = 0$ and $f'(0) = 1$.

The *Laplacian* Δ of a smooth function on a Riemannian manifold is defined by the minus of the trace of its Hessian.

It is well-known that we have the following Laplacian comparison theorem for the distance function from a single point (see e.g., Proposition 3.6 in [34]).

Lemma 2.4. *Let M be an n-dimensional, connected complete Riemmanian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$. Take $p \in \mathrm{Int}\, M$ and $u \in U_pM$. Let $\rho_p : M \to \mathbb{R}$ be the function defined as $\rho_p(q) := d_M(p, q)$, and let $\gamma_u : [0, t_0) \to M$ be the normal minimal geodesic with initial conditions $\gamma_u(0) = p$ and $\gamma_u'(0) = u$ such that γ_u lies in $\mathrm{Int}\, M$. Then for all $t \in (0, t_0)$, we have*

$$\Delta\rho_p(\gamma_u(t)) \geq -(n-1)\frac{s_\kappa'(t)}{s_\kappa(t)}.$$

3. Cut locus for the boundary

Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g.

3.1. Foot points. For a point $p \in M$, we call $x \in \partial M$ a *foot point* on ∂M of p if $d_M(p, x) = d_M(p, \partial M)$. Since (M, d_M) is proper, every point in M has at least one foot point on ∂M.

Lemma 3.1. *For $p \in \mathrm{Int}\, M$, let $x \in \partial M$ be a foot point on ∂M of p. Then there exists a unique normal minimal geodesic $\gamma : [0, l] \to M$ from x to p such that $\gamma = \gamma_x|_{[0,l]}$, where $l = \rho_{\partial M}(p)$. In particular, $\gamma'(0) = u_x$ and $\gamma|_{(0,l]}$ lies in $\mathrm{Int}\, M$.*

Proof. Since (M, d_M) is a geodesic space, there exists a normal minimal geodesic $\gamma : [0, l] \to M$ from x to p. Since x is a foot point on ∂M of p, we see that $\gamma|_{(0,l]}$ lies in $\mathrm{Int}\, M$. We take a normal neighborhood U of ∂M. If $p \in U \setminus \partial M$, then x is a unique foot point on ∂M of p, and $\gamma = \gamma_x|_{[0,l]}$; in particular, we have $\gamma'(0) = u_x$. Even if $p \notin U \setminus \partial M$, then for every sufficiently small $t > 0$, we see that x is the foot point on ∂M of $\gamma(t)$. Hence, $\gamma'(0) = u_x$. This implies $\gamma = \gamma_x|_{[0,l]}$. □

3.2. Cut locus. Let $\tau : \partial M \to \mathbb{R} \cup \{\infty\}$ be the function defined as

$$\tau(x) := \sup\{\, t > 0 \mid \rho_{\partial M}(\gamma_x(t)) = t \,\}.$$

Recall that for all $x \in \partial M$ and $t > \tau_1(x)$, we have $t > \rho_{\partial M}(\gamma_x(t))$. Therefore, for all $x \in \partial M$, we have $0 < \tau(x) \leq \tau_1(x)$.

To study the cut locus, we show the following:

Lemma 3.2. *The function τ is continuous on ∂M.*

Proof. Assume $x_i \to x$ in ∂M. First, we show the upper semi-continuity of τ. We assume $\limsup_{i\to\infty} \tau(x_i) < \infty$. Take a subsequence $\{\tau(x_j)\}$ of $\{\tau(x_i)\}$ with $\tau(x_j) \to \limsup_{i\to\infty} \tau(x_i)$ as $j \to \infty$. Put $p_j := \gamma_{x_j}(\tau(x_j))$ and $p := \gamma_x(\limsup_{i\to\infty} \tau(x_i))$. Since geodesics in $(\mathrm{Int}\, M, g)$ depend continuously on the initial direction and the parameter, we see $p_j \to p$ in M as $j \to \infty$. By the definition of τ, for all j we have $\rho_{\partial M}(p_j) = \tau(x_j)$. By letting $j \to \infty$, we obtain $\rho_{\partial M}(p) = \limsup_{i\to\infty} \tau(x_i)$. Hence, $\limsup_{i\to\infty} \tau(x_i) \leq \tau(x)$. In a similar way, we see that if $\limsup_{i\to\infty} \tau(x_i) = \infty$, then $\tau(x) = \infty$. Therefore, we have shown the upper semi-continuity.

Next, we show the lower semi-continuity of τ. We may assume $\liminf_{i\to\infty} \tau(x_i) < \infty$. The proof is done by contradiction. We suppose $\liminf_{i\to\infty} \tau(x_i) < \tau(x)$. Choose $\delta > 0$ such that $\liminf_{i\to\infty} \tau(x_i) + \delta < \tau(x)$. Take a subsequence $\{\tau(x_j)\}$ of $\{\tau(x_i)\}$ with $\tau(x_j) \to \liminf_{i\to\infty} \tau(x_i)$ as $j \to \infty$. By the definition of τ, we have $\tau(x_j)+\delta > d_M(\gamma_{x_j}(\tau(x_j)+\delta), \partial M)$. Since $\gamma_{x_j}(\tau(x_j) + \delta) \to \gamma_x(\liminf_{i\to\infty} \tau(x_i) + \delta)$ in M, we have

$$\liminf_{i\to\infty} \tau(x_i) + \delta > \rho_{\partial M}(\gamma_x(\liminf_{i\to\infty} \tau(x_i) + \delta)).$$

On the other hand, $\liminf_{i\to\infty} \tau(x_i) + \delta < \tau(x)$. This contradicts the definition of τ. Hence, we have shown the lower semi-continuity. □

By Lemma 3.1, we have the following:

Lemma 3.3. *For all $r > 0$, we have*

$$B_r(\partial M) = \exp^{\perp}\left(\bigcup_{x\in\partial M} \{tu_x \mid t \in [0, \min\{r, \tau(x)\}]\}\right).$$

Proof. Take $p \in B_r(\partial M)$, and let x be a foot point on ∂M of p. By Lemma 3.1, there exists a unique normal minimal geodesic $\gamma : [0, l] \to M$ from x to p such that $\gamma = \gamma_x|_{[0,l]}$, where $l = \rho_{\partial M}(p)$. Since x is a foot point on ∂M of p, we have $l \leq r$, and $l \leq \tau(x)$. Hence,

$$B_r(\partial M) \subset \exp^{\perp}\left(\bigcup_{x\in\partial M} \{tu_x \mid t \in [0, \min\{r, \tau(x)\}]\}\right).$$

On the other hand, take $x \in \partial M$ and $t \in [0, \min\{r, \tau(x)\}]$. By the definition of τ, the point x is a foot point on ∂M of $\gamma_x(t)$. Therefore, $\rho_{\partial M}(\gamma_x(t)) = t \leq r$. This implies the opposite inclusion. □

For the inscribed radius $D(M, \partial M)$ of M, from the definition of τ, it follows that $\sup_{x\in\partial M} \tau(x) \leq D(M, \partial M)$. Lemma 3.1 implies the opposite. Hence, we have $D(M, \partial M) = \sup_{x\in\partial M} \tau(x)$.

We put

$$TD_{\partial M} := \bigcup_{x\in\partial M} \{\, t\, u_x \in T_x^{\perp}\partial M \mid t \in [0, \tau(x))\,\},$$

$$T\mathrm{Cut}\, \partial M := \bigcup_{x\in\partial M} \{\, \tau(x)\, u_x \in T_x^{\perp}\partial M \mid \tau(x) < \infty\,\},$$

and define $D_{\partial M} := \exp^{\perp}(TD_{\partial M})$ and $\mathrm{Cut}\,\partial M := \exp^{\perp}(T\mathrm{Cut}\,\partial M)$. We call $\mathrm{Cut}\,\partial M$ the *cut locus for the boundary* ∂M. By Lemma 3.1, we have $\mathrm{Int}\,M = (D_{\partial M} \setminus \partial M) \cup \mathrm{Cut}\,\partial M$ and $M = D_{\partial M} \cup \mathrm{Cut}\,\partial M$.

The continuity of τ tells us the following:

Lemma 3.4. *Suppose ∂M is compact. Then $D(M, \partial M) < \infty$ if and only if M is compact.*

Proof. If $D(M, \partial M) < \infty$, then $\sup_{x \in \partial M} \tau(x) < \infty$. By the continuity of τ, the set $TD_{\partial M} \cup T\mathrm{Cut}\,\partial M$ is closed in $T^{\perp}\partial M$. Since ∂M is compact, the set is compact in $T^{\perp}\partial M$. The set $D_{\partial M} \cup \mathrm{Cut}\,\partial M$ coincides with M. The continuity of $\exp^{\perp}|_{TD_{\partial M} \cup T\mathrm{Cut}\,\partial M}$ implies that M is compact. On the other hand, if M is compact, then the function $\rho_{\partial M}$ is finite on M; in particular, $D(M, \partial M) < \infty$. □

Furthermore, we have:

Proposition 3.5. $\mathrm{vol}_g \mathrm{Cut}\,\partial M = 0$.

Proof. By Lemma 3.2, and by the Fubini theorem, the graph

$$\{ (x, \tau(x)) \mid x \in \partial M,\ \tau(x) < \infty \}$$

of τ is a null set of $\partial M \times [0, \infty)$. A map $\Psi : \partial M \times [0, \infty) \to T^{\perp}\partial M$ defined by $\Psi(x, t) := (x, tu_x)$ is smooth. In particular, the set $T\mathrm{Cut}\,\partial M$ is also a null set of $T^{\perp}\partial M$. By the definition of τ, the set $\mathrm{Cut}\,\partial M$ is contained in $\mathrm{Int}\,M$. Hence, $\exp^{\perp}$ is smooth on an open neighborhood of $T\mathrm{Cut}\,\partial M$ in $T^{\perp}\partial M$. Therefore, we see $\mathrm{vol}_g \mathrm{Cut}\,\partial M = 0$. □

We next show the following characterization of τ:

Lemma 3.6. *Let $T > 0$. Take $x \in \partial M$ with $\tau(x) < \infty$. Then $T = \tau(x)$ if and only if $T = \rho_{\partial M}(\gamma_x(T))$, and at least one of the following holds:*

(1) *$\gamma_x(T)$ is the first conjugate point of ∂M along γ_x;*
(2) *there exists a foot point $y \in \partial M \setminus \{x\}$ on ∂M of $\gamma_x(T)$.*

Proof. First, we assume $T = \rho_{\partial M}(\gamma_x(T))$. By the definition of τ, we have $T \leq \tau(x)$. If (1) holds, then T is equal to $\tau_1(x)$; in particular, $T = \tau(x)$. Suppose that (2) holds. We assume $T < \tau(x)$, and take $\delta > 0$ such that $T + \delta < \tau(x)$. If $\gamma_x'(T) = -\gamma_y'(T)$ at $\gamma_x(T)$, then $\gamma_x(T + \delta) = \gamma_y(T - \delta)$. Since $T \leq \tau(y)$, we have

$$\rho_{\partial M}(\gamma_x(T + \delta)) = \rho_{\partial M}(\gamma_y(T - \delta)) = T - \delta.$$

This is in contradiction with $T + \delta < \tau(x)$. If $\gamma_x'(T) \neq -\gamma_y'(T)$ at $\gamma_x(T)$, then for all $t \in (T, T + \delta]$, we have

$$\rho_{\partial M}(\gamma_x(t)) < d_M(\gamma_x(t), \gamma_x(T)) + d_M(\gamma_x(T), y) \leq t.$$

This contradicts $t \leq T + \delta < \tau(x)$. Hence, we see $T = \tau(x)$.

Next, we assume $T = \tau(x)$. Then we have $T = \rho_{\partial M}(\gamma_x(T))$. Put $p := \gamma_x(T)$. Assuming that p is not the first conjugate point of ∂M along γ_x, we will prove (2). Take an open neighborhood $\bar{U}$ of (x, Tu_x) in $T^{\perp}\partial M$ such that $\exp^{\perp}|_{\bar{U}} : \bar{U} \to \exp^{\perp}(\bar{U})$ is a diffeomorphism. Put $U := \exp^{\perp}(\bar{U})$. For every sufficiently large $i \in \mathbb{N}$, we put $p_i := \gamma_x(T + 1/i)$, and take a foot point x_i on ∂M of p_i. By Lemma 3.1, there exists a unique normal minimal geodesic $\gamma_i : [0, l_i] \to M$ from x_i to p_i such that $\gamma_i = \gamma_{x_i}|_{[0,l_i]}$, where $l_i = \rho_{\partial M}(p_i)$. Since (M, d_M) is proper, by taking a subsequence if necessary, we may assume that for some $y \in \partial M$, we have $x_i \to y$ in ∂M. Since x_i is a foot point on ∂M of p_i and $p_i \to p$ in M, we see that y is a foot point on ∂M of p. If $x = y$, then for every sufficiently large $i \in \mathbb{N}$, we have $(x_i, l_i\, u_{x_i}) \in \bar{U}$ and $\exp^{\perp}(x, (T + 1/i)\, u_x) = \exp^{\perp}(x_i, l_i\, u_{x_i})$. By the injectivity of $\exp^{\perp}|_{\bar{U}}$, we have $T + 1/i = l_i$. This is in contradiction with $T + 1/i > l_i$. Hence, we see $x \neq y$. This completes the proof. □

From Lemma 3.6, we derive the following:

Lemma 3.7. *We have* $\mathrm{Cut}\,\partial M \cap D_{\partial M} = \emptyset$. *In particular,*

$$\mathrm{Int}\, M = (D_{\partial M} \setminus \partial M) \sqcup \mathrm{Cut}\,\partial M, \quad M = D_{\partial M} \sqcup \mathrm{Cut}\,\partial M.$$

Proof. Suppose that there exists $p \in \mathrm{Cut}\,\partial M \cap D_{\partial M}$. Then there exist x, $y \in \partial M$ and $l \in (0, \tau(y))$ such that $p = \gamma_x(\tau(x)) = \gamma_y(l)$. By the definition of τ, we have $l = \tau(x)$; in particular, $x \neq y$. Furthermore, by the definition of τ, we see that x and y are foot points on ∂M of p. By Lemma 3.6, we have $l = \tau(y)$. This is a contradiction. Therefore, we have $\mathrm{Cut}\,\partial M \cap D_{\partial M} = \emptyset$. Since $\mathrm{Int}\, M = (D_{\partial M} \setminus \partial M) \cup \mathrm{Cut}\,\partial M$ and $M = D_{\partial M} \cup \mathrm{Cut}\,\partial M$, we prove the lemma. □

For the connectedness of the boundary, we show:

Lemma 3.8. *If* $\mathrm{Cut}\,\partial M = \emptyset$*, then* ∂M *is connected.*

Proof. Suppose that ∂M is not connected. Let $\{\partial M_i\}_{i \geq 2}$ be the connected components of ∂M. By Lemma 3.6, for every $p \in D_{\partial M} \setminus \partial M$, there exists a unique foot point on ∂M of p. For each i, we denote by $D_{\partial M_i}$ the set of all points in $D_{\partial M} \setminus \partial M$ whose foot points are contained in ∂M_i. By the continuity of τ, the sets $D_{\partial M_i} \setminus \partial M$, $i \geq 2$, are mutually disjoint domains in $\mathrm{Int}\, M$. Lemma 3.7 implies that $\mathrm{Int}\, M$ coincides with $(\bigsqcup_{i \geq 2} D_{\partial M_i}) \sqcup \mathrm{Cut}\,\partial M$. Since $\mathrm{Cut}\,\partial M = \emptyset$, the set $\mathrm{Int}\, M$ is not connected. This is a contradiction. □

By the continuity of τ, the set $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$ is a domain in $T^{\perp}\partial M$. Using Lemma 3.6, we see the following:

Lemma 3.9. $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$ *is a maximal domain in* $T^{\perp}\partial M$ *on which* $\exp^{\perp}$ *is regular and injective.*

We show the smoothness of $\rho_{\partial M}$ on the set $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$.

Proposition 3.10. *The function* $\rho_{\partial M}$ *is smooth on* $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$. *Moreover, for each* $p \in \mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$*, the gradient vector* $\nabla\rho_{\partial M}(p)$ *of* $\rho_{\partial M}$ *at* p *is given by* $\nabla\rho_{\partial M}(p) = \gamma'(l)$*, where* $\gamma : [0, l] \to M$ *is the normal minimal geodesic from the foot point on* ∂M *of* p *to* p.

Proof. By Lemma 3.9, the map $\exp^{\perp}|_{TD_{\partial M} \setminus 0(T^{\perp}\partial M)}$ is a diffeomorphism onto $D_{\partial M} \setminus \partial M$. Lemma 3.7 implies $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M = D_{\partial M} \setminus \partial M$. For all $q \in \mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$, we have $\rho_{\partial M}(q) = \|(\exp^{\perp})^{-1}(q)\|$. Hence, $\rho_{\partial M}$ is smooth on $\mathrm{Int}\, M \setminus \mathrm{Cut}\,\partial M$.

For any vector $v \in T_pM$, we take a smooth curve $c : (-\epsilon, \epsilon) \to \operatorname{Int} M$ tangent to v at $p = c(0)$. We may assume $c(s) \in \operatorname{Int} M \setminus \operatorname{Cut} \partial M$ when $|s|$ is sufficiently small. By Lemma 3.6, there exists a unique foot point $\bar{c}(s)$ on ∂M of $c(s)$. By Lemma 3.1, we obtain a smooth variation of γ by taking normal minimal geodesics in M from $\bar{c}(s)$ to $c(s)$. The first variation formula for the variation implies $(\rho_{\partial M} \circ c)'(0) = g(v, \gamma'(l))$. Therefore, we have $\nabla \rho_{\partial M}(p) = \gamma'(l)$. □

4. Comparison theorems

In this section, we prove Theorem 1.1.

4.1. Basic comparison. We refer to the following absolute comparison inequality that has been shown by Heintze and Karcher in Subsection 3.4 in [18].

Lemma 4.1 ([18])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. Take a point $x \in \partial M$. Suppose that for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$, we have $\operatorname{Ric}_g(\gamma_x'(t)) \geq (n-1)\kappa$, and suppose $H_x \geq \lambda$. Then for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$, we have*

$$\frac{\theta'(t,x)}{\theta(t,x)} \leq (n-1)\frac{s_{\kappa,\lambda}'(t)}{s_{\kappa,\lambda}(t)}.$$

Remark 4.2. In the case in Lemma 4.1, we choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$, and let $\{Y_{x,i}\}_{i=1}^{n-1}$ be the ∂M-Jacobi fields along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y_{x,i}'(0) = -A_{u_x} e_{x,i}$. Then there exists $t_0 \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ such that

$$\frac{\theta'(t_0,x)}{\theta(t_0,x)} = (n-1)\frac{s_{\kappa,\lambda}'(t_0)}{s_{\kappa,\lambda}(t_0)}.$$

if and only if for all $i = 1, \dots, n-1$ and $t \in [0, t_0]$, we have $Y_{x,i}(t) = s_{\kappa,\lambda}(t)\, E_{x,i}(t)$, where $E_{x,i}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$ (see [18]).

The following Laplacian comparison theorem has been stated by Kasue in Corollary 2.42 in [22].

Theorem 4.3 ([22])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. Take $x \in \partial M$. Suppose that for all $t \in (0, \tau(x))$, we have $\operatorname{Ric}_g(\gamma_x'(t)) \geq (n-1)\kappa$, and suppose $H_x \geq \lambda$. Then for all $t \in (0, \tau(x))$, we have*

$$\Delta \rho_{\partial M}(\gamma_x(t)) \geq -(n-1)\frac{s_{\kappa,\lambda}'(t)}{s_{\kappa,\lambda}(t)}.$$

Remark 4.4. In the case in Theorem 4.3, for all $t \in (0, \tau(x))$, we have $\Delta \rho_{\partial M}(\gamma_x(t)) = -\theta'(t,x)/\theta(t,x)$. Therefore, the equality case in Theorem 4.3 results into that in Lemma 4.1 (see Remark 4.2).

By Lemma 4.1, we have the following relative comparison inequality.

Lemma 4.5. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g. Take a point $x \in \partial M$. Suppose that for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$, we have $\mathrm{Ric}_g(\gamma_x'(t)) \geq (n-1)\kappa$, and suppose $H_x \geq \lambda$. Then for all $s, t \in [0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ with $s \leq t$,*

$$\frac{\theta(t,x)}{\theta(s,x)} \leq \frac{s_{\kappa,\lambda}^{n-1}(t)}{s_{\kappa,\lambda}^{n-1}(s)};$$

in particular, $\theta(t,x) \leq s_{\kappa,\lambda}^{n-1}(t)$. Moreover, if κ and λ satisfy the ball-condition, then $\tau_1(x) \leq C_{\kappa,\lambda}$.

Proof. Take $\tilde{x} \in \partial M^n_{\kappa,\lambda}$. By Lemma 4.1, for all $t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$,

$$\frac{d}{dt} \log \frac{\theta(t,\tilde{x})}{\theta(t,x)} = \frac{\theta'(t,\tilde{x})}{\theta(t,\tilde{x})} - \frac{\theta'(t,x)}{\theta(t,x)} \geq 0.$$

Hence, for all $s, t \in (0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ with $s \leq t$, we have

$$\frac{\theta(t,x)}{\theta(s,x)} \leq \frac{\theta(t,\tilde{x})}{\theta(s,\tilde{x})}.$$

In the inequality, by letting $s \to 0$, we have $\theta(t,x) \leq \theta(t,\tilde{x})$. Hence, for all $s, t \in [0, \min\{\tau_1(x), \bar{C}_{\kappa,\lambda}\})$ with $s \leq t$, we have the desired inequality.

Let κ and λ satisfy the ball-condition. We suppose $C_{\kappa,\lambda} < \tau_1(x)$. For all $t \in [0, C_{\kappa,\lambda})$, we have $\theta(t,x) \leq s_{\kappa,\lambda}^{n-1}(t)$. By letting $t \to C_{\kappa,\lambda}$, we have $\theta(C_{\kappa,\lambda}, x) = 0$. Since $C_{\kappa,\lambda} < \tau_1(x)$, the point $\gamma_x(C_{\kappa,\lambda})$ is not a conjugate point of ∂M along γ_x. Hence, there exists a nonzero ∂M-Jacobi field Y along γ_x such that $Y(C_{\kappa,\lambda}) = 0$; in particular, $\gamma_x(C_{\kappa,\lambda})$ is a conjugate point of ∂M along γ_x. This is a contradiction. Therefore, we have $\tau_1(x) \leq C_{\kappa,\lambda}$. □

4.2. Inscribed radius comparison.

Using Lemma 4.5, we will give a proof of the following lemma that has been already proved by Kasue in Theorem A in [23].

Lemma 4.6 ([23])**.** *Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. Let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Then for all $x \in \partial M$, we have $\tau(x) \leq C_{\kappa,\lambda}$; in particular, $D(M, \partial M) \leq C_{\kappa,\lambda}$.*

Proof. Take $x \in \partial M$. By the definition of τ, the geodesic $\gamma_x|_{(0,\tau(x)]}$ lies in $\mathrm{Int}\, M$. If $C_{\kappa,\lambda} < \tau(x)$, then by Lemma 4.5, we see that $\gamma_x(C_{\kappa,\lambda})$ is a conjugate point of ∂M along γ_x. We obtain $\tau_1(x) < \tau(x)$. This contradicts the relation between τ and τ_1. Hence, $\tau(x) \leq C_{\kappa,\lambda}$. □

The following rigidity theorem has been proved in Theorem A in [23].

Theorem 4.7 ([23])**.** *Let $\kappa \in \mathbb{R}$ and $\lambda \in \mathbb{R}$ satisfy the ball-condition. Let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. If there exists a point $p \in M$ such that $\rho_{\partial M}(p) = C_{\kappa,\lambda}$, then the metric space (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$.*

4.3. Volume comparison. By the coarea formula (see e.g., Theorem 3.2.3 in [12]), we have the following:

Lemma 4.8. *Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g. Suppose ∂M is compact. Let r be a positive number such that $U_r(\partial M)$ is a normal neighborhood of ∂M. Then we have*

$$\mathrm{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^r \theta(t, x)\, dt\, d\,\mathrm{vol}_h .$$

From Lemma 4.8, we derive the following:

Lemma 4.9. *Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g. Suppose ∂M is compact. Then for all $r > 0$, we have*

$$\mathrm{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^{\min\{r,\tau(x)\}} \theta(t, x)\, dt\, d\,\mathrm{vol}_h .$$

Proof. Take $r > 0$. By Lemma 3.3, we have

$$B_r(\partial M) = \exp^{\perp}\left(\bigcup_{x\in\partial M} \{tu_x \mid t \in [0, \min\{r, \tau(x)\}]\} \right).$$

From Lemma 3.9, it follows that the map $\exp^{\perp}$ is diffeomorphic on $\bigcup_{x\in\partial M}\{tu_x \mid t \in (0, \min\{r, \tau(x)\})\}$. Therefore, by Proposition 3.5 and Lemma 4.8, we have the desired equality. □

We prove Theorem 1.1. Proof of Theorem 1.1. We define a function $\bar{\theta} : [0, \infty)\times\partial M \to \mathbb{R}$ by

$$\bar{\theta}(t, x) := \begin{cases} \theta(t, x) & \text{if } t \leq \tau(x), \\ 0 & \text{if } t > \tau(x). \end{cases}$$

By Lemma 4.9, we have

$$\mathrm{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^r \bar{\theta}(t, x)\, dt\, d\,\mathrm{vol}_h .$$

Lemma 4.6 implies that for each $x \in \partial M$, we have $\tau(x) \leq \bar{C}_{\kappa,\lambda}$. Therefore, from Lemma 4.5, it follows that for all $s, t \in [0, \infty)$ with $s \leq t$,

$$\bar{\theta}(t, x)\ \bar{s}^{n-1}_{\kappa,\lambda}(s) \leq \bar{\theta}(s, x)\ \bar{s}^{n-1}_{\kappa,\lambda}(t).$$

Integrating the both sides of the above inequality over $[0, r]$ with respect to s, and then doing that over $[r, R]$ with respect to t, we see

$$\frac{\int_r^R \bar{\theta}(t, x)\, dt}{\int_0^r \bar{\theta}(s, x)\, ds} \leq \frac{\int_r^R \bar{s}^{n-1}_{\kappa,\lambda}(t)\, dt}{\int_0^r \bar{s}^{n-1}_{\kappa,\lambda}(s)\, ds}.$$

Hence, we have

$$\begin{aligned}\frac{\operatorname{vol}_g B_R(\partial M)}{\operatorname{vol}_g B_r(\partial M)} &= 1 + \frac{\int_{\partial M} \int_r^R \bar{\theta}(t,x)\,dt\,d\operatorname{vol}_h}{\int_{\partial M} \int_0^r \bar{\theta}(s,x)\,ds\,d\operatorname{vol}_h} \\ &\leq 1 + \frac{\int_r^R \bar{s}_{\kappa,\lambda}^{n-1}(t)\,dt}{\int_0^r \bar{s}_{\kappa,\lambda}^{n-1}(s)\,ds} = \frac{\operatorname{vol} B_R(\partial M^n_{\kappa,\lambda})}{\operatorname{vol} B_r(\partial M^n_{\kappa,\lambda})}.\end{aligned}$$

This completes the proof of Theorem 1.1. □

Remark 4.10. In the case in Theorem 1.1, we suppose that there exists $R > 0$ such that for all $r \in (0, R]$, we have

$$\frac{\operatorname{vol}_g B_R(\partial M)}{\operatorname{vol}_g B_r(\partial M)} = \frac{\operatorname{vol} B_R(\partial M^n_{\kappa,\lambda})}{\operatorname{vol} B_r(\partial M^n_{\kappa,\lambda})}.$$

In this case, for all $t \in (0, R]$ and $x \in \partial M$, we have $\bar{\theta}(t,x) = \bar{s}_{\kappa,\lambda}^{n-1}(t)$. We choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all $i = 1, \dots, n-1$, and for all $t \in [0, \min\{R, \bar{C}_{\kappa,\lambda}\}]$ and $x \in \partial M$, we have $Y_{x,i}(t) = s_{\kappa,\lambda}(t)\, E_{x,i}(t)$, where $E_{x,i}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$.

5. Volume growth rigidity

5.1. Volume growth. By Theorem 1.1, we have the following:

Proposition 5.1. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\operatorname{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Let h denote the induced Riemannian metric on ∂M. Then*

$$\limsup_{r \to \infty} \frac{\operatorname{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \leq \operatorname{vol}_h \partial M.$$

Proof. Take $r > 0$. By Lemma 4.9, we have

$$\operatorname{vol}_g B_r(\partial M) = \int_{\partial M} \int_0^{\min\{r,\tau(x)\}} \theta(t,x)\,dt\,d\operatorname{vol}_h.$$

By Lemma 4.5, for all $x \in \partial M$ and $t \in (0, \min\{r, \tau(x)\})$, we have $\theta(t,x) \leq s_{\kappa,\lambda}^{n-1}(t)$. Integrating the both sides of the inequality over $(0, \min\{r, \tau(x)\})$ with respect to t, and then doing that over ∂M with respect to x, we see $\operatorname{vol}_g B_r(\partial M)/f_{n,\kappa,\lambda}(r) \leq \operatorname{vol}_h \partial M$. Letting $r \to \infty$, we obtain the desired inequality. □

5.2. Volume growth rigidity. In the equality case in Theorem 1.1, τ satisfies the following property:

Lemma 5.2. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Assume that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for all $r \in (0, R]$, we have*

$$\frac{\mathrm{vol}_g B_R(\partial M)}{\mathrm{vol}_g B_r(\partial M)} = \frac{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_R(\partial M^n_{\kappa,\lambda})}{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_r(\partial M^n_{\kappa,\lambda})}.$$

Then for all $x \in \partial M$, we have $\tau(x) \geq R$.

Proof. Suppose that for some $x_0 \in \partial M$, we have $\tau(x_0) < R$. Put $t_0 := \tau(x_0)$. Take $\epsilon > 0$ with $t_0 + \epsilon < R$. By the continuity of τ, there exists a closed geodesic ball B in ∂M centered at x_0 such that for all $x \in B$, we have $\tau(x) \leq t_0 + \epsilon$. By Lemmas 4.5 and 4.9, we see that $\mathrm{vol}_g B_R(\partial M)$ is not larger than

$$\int_{\partial M \setminus B} \int_0^{\min\{R, \tau(x)\}} s^{n-1}_{\kappa,\lambda}(t)\, dt\, d\,\mathrm{vol}_h + \int_B \int_0^{t_0+\epsilon} s^{n-1}_{\kappa,\lambda}(t)\, dt\, d\,\mathrm{vol}_h\,.$$

This is smaller than $(\mathrm{vol}_h \partial M)\, f_{n,\kappa,\lambda}(R)$. On the other hand, by the assumption, we see that $f_{n,\kappa,\lambda}(R)$ is equal to $\mathrm{vol}_g B_R(\partial M)/\,\mathrm{vol}_h \partial M$. This is a contradiction. □

In the case in Lemma 5.2, for every $r \in (0, R)$, the level set $\rho^{-1}_{\partial M}(r)$ is an $(n-1)$-dimensional submanifold of M. In particular, $(B_r(\partial M), g)$ is an n-dimensional (not necessarily, connected) complete Riemannian manifold with boundary. We denote by $d_{B_r(\partial M)}$ and by $d_{\kappa,\lambda,r}$ the Riemannian distances on $(B_r(\partial M), g)$ and on $[0, r] \times_{\kappa,\lambda} \partial M$, respectively.

Proposition 5.3. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Assume that there exists $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ such that for all $r \in (0, R]$, we have*

$$\frac{\mathrm{vol}_g B_R(\partial M)}{\mathrm{vol}_g B_r(\partial M)} = \frac{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_R(\partial M^n_{\kappa,\lambda})}{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_r(\partial M^n_{\kappa,\lambda})}.$$

Then for every $r \in (0, R)$, the metric space $(B_r(\partial M), d_{B_r(\partial M)})$ is isometric to $([0, r] \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda,r})$.

Proof. Take $r \in (0, R)$. By Lemma 5.2, for all $x \in \partial M$, we have $\tau(x) > r$; in particular, $B_r(\partial M) \cap \mathrm{Cut}\,\partial M = \emptyset$. Each connected component of ∂M one-to-one corresponds to the connected component of $B_r(\partial M)$. Therefore, we may assume that $B_r(\partial M)$ is connected.

By Lemma 4.5, for all $t \in (0, R]$ and $x \in \partial M$, we have $\theta(t, x) = s^{n-1}_{\kappa,\lambda}(t)$. Choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. For each $i = 1, \dots, n-1$, let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. For all $t \in [0, \min\{R, \bar{C}_{\kappa,\lambda}\}]$ and $x \in \partial M$, we have $Y_{x,i}(t) = s_{\kappa,\lambda}(t)\, E_{x,i}(t)$, where $E_{x,i}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$ (see Remark 4.10). Define a map $\Phi : [0, r] \times \partial M \to B_r(\partial M)$ by $\Phi(t, x) := \gamma_x(t)$. For every $p \in (0, r) \times \partial M$, the map $D(\Phi|_{(0,r)\times\partial M})_p$ sends an orthonormal basis of $T_p([0, r] \times \partial M)$ to that of $T_{\Phi(p)} B_r(\partial M)$, and for every $x \in \{0, r\} \times \partial M$, the map $D(\Phi|_{\{0,r\}\times\partial M})_x$ sends an orthonormal basis of $T_x(\{0, r\} \times \partial M)$

to that of $T_{\Phi(x)}\partial(B_r(\partial M))$. Hence, Φ is a Riemannian isometry with boundary from $[0,r]\times_{\kappa,\lambda}\partial M$ to $B_r(\partial M)$. $\square$

5.3. Proof of Theorem 1.6. Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. We assume

$$\liminf_{r\to\infty} \frac{\mathrm{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \geq \mathrm{vol}_h \partial M.$$

By Theorem 1.1 and Proposition 5.1, for all $r, R \in (0,\infty)$ with $r \leq R$,

$$\frac{\mathrm{vol}_g B_R(\partial M)}{f_{n,\kappa,\lambda}(R)} = \frac{\mathrm{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} = \mathrm{vol}_h \partial M.$$

If κ and λ satisfy the ball-condition, then for all $r \in (0, C_{\kappa,\lambda}]$ we have

$$\frac{\mathrm{vol}_g B_{C_{\kappa,\lambda}}(\partial M)}{\mathrm{vol}_g B_r(\partial M)} = \frac{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_{C_{\kappa,\lambda}}(\partial M^n_{\kappa,\lambda})}{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_r(\partial M^n_{\kappa,\lambda})};$$

in particular, Lemmas 4.6 and 5.2 imply that τ is equal to $C_{\kappa,\lambda}$ on ∂M. If κ and λ do not satisfy the ball-condition, then for all $R \in (0,\infty)$ and $r \in (0,R]$ we have

$$\frac{\mathrm{vol}_g B_R(\partial M)}{\mathrm{vol}_g B_r(\partial M)} = \frac{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_R(\partial M^n_{\kappa,\lambda})}{\mathrm{vol}_{g^n_{\kappa,\lambda}} B_r(\partial M^n_{\kappa,\lambda})};$$

in particular, Lemma 5.2 implies that for all $x \in \partial M$, we have $\tau(x) = \infty$. It follows that τ coincides with $\bar{C}_{\kappa,\lambda}$ on ∂M.

If κ and λ satisfy the ball-condition, then Lemmas 3.4 and 4.6 imply that M is compact; in particular, there exists a point $p \in M$ such that $\rho_{\partial M}(p) = D(M, \partial M) = C_{\kappa,\lambda}$. Hence, from Theorem 4.7, it follows that (M, d_M) is isometric to $(B^n_{\kappa,\lambda}, d_{B^n_{\kappa,\lambda}})$.

If κ and λ do not satisfy the ball-condition, then $\mathrm{Cut}\,\partial M = \emptyset$. From Lemma 3.8, it follows that ∂M is connected. Take a sequence $\{r_i\}$ with $r_i \to \infty$. By Proposition 5.3, for each r_i, we obtain a Riemannian isometry $\Phi_i : [0, r_i] \times_{\kappa,\lambda} \partial M \to B_{r_i}(\partial M)$ with boundary from $[0, r_i] \times_{\kappa,\lambda} \partial M$ to $B_{r_i}(\partial M)$ defined by $\Phi_i(t,x) := \gamma_x(t)$. Since for all $x \in \partial M$ it holds that $\tau(x) = \infty$, there exists a Riemannian isometry $\Phi : [0,\infty) \times_{\kappa,\lambda} \partial M \to M$ with boundary from $[0,\infty) \times_{\kappa,\lambda} \partial M$ to M defined by $\Phi(t,x) := \gamma_x(t)$ satisfying $\Phi|_{[0,r_i]\times_{\kappa,\lambda}\partial M} = \Phi_i$. Hence, (M, d_M) is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. We complete the proof. $\square$

5.4. Curvature of the boundary. It seems that the following is well-known, especially in a submanifold setting (see e.g., Proposition 9.36 in [1]). For the sake of the readers, we give a proof in our setting.

Lemma 5.4. *Let M be an n-dimensional Riemannian manifold with boundary with Riemannian metric g. Let h denote the induced Riemannian metric on ∂M. Take a point $x \in \partial M$, and choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x\partial M$. Put $u := e_{x,1}$. Then*

$$\mathrm{Ric}_h(u) = \mathrm{Ric}_g(u) - K_g(u_x, u) + \mathrm{trace}\, A_{S(u,u)} - \sum_{i=1}^{n-1} \|S(u, e_{x,i})\|^2,$$

where $K_g(u_x, u)$ is the sectional curvature at x in (M, g) determined by u_x and u.

Proof. Note that $\mathrm{Ric}_h(u) = \sum_{i=2}^{n-1} K_h(u, e_{x,i})$. By the Gauss formula,

$$\mathrm{Ric}_h(u) = \sum_{i=2}^{n-1} \Big(K_g(u, e_{x,i}) + g(S(u, u), S(e_{x,i}, e_{x,i})) - \|S(u, e_{x,i})\|^2 \Big).$$

Since $u, e_{x,2}, \ldots, e_{x,n-1}, u_x$ are orthogonal to each other, we have

$$\mathrm{Ric}_g(u) = \sum_{i=2}^{n-1} K_g(u, e_{x,i}) + K_g(u, u_x).$$

On the other hand, we see

$$\sum_{i=1}^{n-1} g(S(u, u), S(e_{x,i}, e_{x,i})) = \sum_{i=1}^{n-1} g(A_{S(u,u)} e_{x,i}, e_{x,i}) = \operatorname{trace} A_{S(u,u)}.$$

Combining these equalities, we have the formula. □

To study our rigidity cases, we need the following:

Lemma 5.5. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \geq (n-1)\kappa$. *If (M, d_M) is isometric to $([0, \infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$, then we have* $\mathrm{Ric}_{\partial M} \geq (n-2)(\kappa + \lambda^2)$.

Proof. There exists a Riemannian isometry with boundary from M to $[0, \infty) \times_{\kappa,\lambda} \partial M$. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. For each $i = 1, \ldots, n-1$, let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. We have $Y_{x,i}(t) = s_{\kappa,\lambda}(t) E_{x,i}(t)$, where $E_{x,i}$ are the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$. Then $A_{u_x} e_{x,i} = -Y'_{x,i}(0) = \lambda e_{x,i}$ and $Y''_{x,1}(0) = \kappa e_{x,1}$. Hence, $\operatorname{trace} A_{u_x} = (n-1)\lambda$ and $K_g(u_x, e_{x,1}) = \kappa$. For all i we have $S(e_{x,i}, e_{x,i}) = \lambda u_x$, and for all $i \neq j$ we have $S(e_{x,i}, e_{x,j}) = 0_x$. By Lemma 5.4 and $\mathrm{Ric}_M \geq (n-1)\kappa$, we have $\mathrm{Ric}_{\partial M} \geq (n-2)(\kappa + \lambda^2)$. □

5.5. Complement rigidity. For $\kappa > 0$, let M be an n-dimensional, connected complete Riemmanian manifold (without boundary) with Riemannian metric g such that $\mathrm{Ric}_M \geq (n-1)\kappa$. By the Bishop volume comparison theorem ([2]), $\mathrm{vol}_g M \leq \mathrm{vol}\, M^n_\kappa$; the equality holds if and only if M is isometric to M^n_κ.

The following is concerned with the complements of metric balls.

Corollary 5.6. *Let $\kappa \in \mathbb{R}$ and $-\lambda \in \mathbb{R}$ satisfy the ball-condition. Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Let h denote the induced Riemannian metric on ∂M. If*

$$\liminf_{r \to \infty} \frac{\mathrm{vol}_g B_r(\partial M)}{f_{n,\kappa,\lambda}(r)} \geq \mathrm{vol}_h \partial M, \quad \mathrm{vol}_h \partial M \geq \mathrm{vol}_{h^{n-1}_{\kappa,-\lambda}} \partial B^n_{\kappa,-\lambda},$$

then (M, d_M) is isometric to $(M^n_\kappa \setminus \operatorname{Int} B^n_{\kappa,-\lambda}, d_{M^n_\kappa \setminus \operatorname{Int} B^n_{\kappa,-\lambda}})$.

Proof. By Theorem 1.6, (M, d_M) is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Lemma 5.5 implies $\mathrm{Ric}_{\partial M} \geq (n-2)(\kappa+\lambda^2)$. Since κ and $-\lambda$ satisfy the ball-condition, $(\partial M, h)$ is a connected complete Riemannian manifold of positive Ricci curvature. By the assumption $\mathrm{vol}_h \partial M \geq \mathrm{vol}_{h^{n-1}_{\kappa,-\lambda}} \partial B^n_{\kappa,-\lambda}$, and by the Bishop volume comparison theorem, $(\partial M, h)$ is isometric to $(\partial B^n_{\kappa,-\lambda}, h^{n-1}_{\kappa,-\lambda})$. It turns out that M and $M^n_\kappa \setminus \mathrm{Int}\, B^n_{\kappa,-\lambda}$ are isometric to each other as metric spaces. □

6. Splitting theorems

Let M be a connected complete Riemannian manifold with boundary. A normal geodesic $\gamma : [0,\infty) \to M$ is said to be a *ray* if for all $s, t \in [0,\infty)$, we have $d_M(\gamma(s), \gamma(t)) = |s-t|$. For a ray $\gamma : [0,\infty) \to M$, the function $b_\gamma : M \to \mathbb{R}$ defined as

$$b_\gamma(p) := \lim_{t\to\infty}(t - d_M(p, \gamma(t)))$$

is called the *busemann function* of γ.

Lemma 6.1. *Let M be a connected complete Riemannian manifold with boundary. Suppose that for some $x_0 \in \partial M$, we have $\tau(x_0) = \infty$. Take a point $p \in \mathrm{Int}\, M$. If $b_{\gamma_{x_0}}(p) = \rho_{\partial M}(p)$, then $p \notin \mathrm{Cut}\, \partial M$. Moreover, for the unique foot point x on ∂M of p, we have $\tau(x) = \infty$.*

Proof. Since $\tau(x_0) = \infty$, the normal geodesic $\gamma_{x_0} : [0,\infty) \to M$ is a ray. Since $\rho_{\partial M}$ is 1-Lipschitz, for all $q \in M$, we have $b_{\gamma_{x_0}}(q) \leq \rho_{\partial M}(q)$.

Take a foot point x on ∂M of p. Suppose $p \in \mathrm{Cut}\, \partial M$. We have $\tau(x) < \infty$ and $p = \gamma_x(\tau(x))$. Take $\epsilon > 0$ with $B_\epsilon(p) \subset \mathrm{Int}\, M$, and a sequence $\{t_i\}$ with $t_i \to \infty$. For each i, we take a normal minimal geodesic $\gamma_i : [0, l_i] \to M$ from p to $\gamma_{x_0}(t_i)$. Then $\gamma_i|_{[0,\epsilon)}$ lies in $\mathrm{Int}\, M$. Put $u_i := \gamma_i'(0) \in U_pM$. By taking a subsequence, for some $u \in U_pM$, we have $u_i \to u$ in U_pM. We denote by $\gamma_u : [0, T) \to M$ the normal geodesic with initial conditions $\gamma_u(0) = p$ and $\gamma_u'(0) = u$. We have

$$t_i - d_M(p, \gamma_{x_0}(t_i)) = -\epsilon + (t_i - d_M(\gamma_i(\epsilon), \gamma_{x_0}(t_i))).$$

By letting $i \to \infty$, we have $b_{\gamma_{x_0}}(p) = -\epsilon + b_{\gamma_{x_0}}(\gamma_u(\epsilon))$. From the assumption $b_{\gamma_{x_0}}(p) = \rho_{\partial M}(p)$, it follows that $\rho_{\partial M}(p) \leq -\epsilon + \rho_{\partial M}(\gamma_u(\epsilon))$. On the other hand, since $\rho_{\partial M}$ is 1-Lipschitz, we have the opposite. Therefore, $d_M(x, \gamma_u(\epsilon))$ is equal to $d_M(x, p) + d_M(p, \gamma_u(\epsilon))$; in particular, we see $u = \gamma_x'(\tau(x))$. Furthermore, $\rho_{\partial M}(\gamma_x(\tau(x)+\epsilon)) = \tau(x) + \epsilon$. This contradicts the definition of τ. Hence, $p \notin \mathrm{Cut}\, \partial M$, and x is the unique foot point on ∂M of p.

Put $l := \rho_{\partial M}(p)$. We see that for every sufficiently small $\epsilon > 0$, we have $b_{\gamma_{x_0}}(\gamma_x(l+\epsilon)) = \rho_{\partial M}(\gamma_x(l+\epsilon))$. In particular, for all $t \in [l, \infty)$, we have $b_{\gamma_{x_0}}(\gamma_x(t)) = \rho_{\partial M}(\gamma_x(t))$. It follows that $\tau(x) = \infty$. □

Let M be a connected complete Riemannian manifold with boundary, and let $\gamma : [0,\infty) \to M$ be a ray. Take $p \in \mathrm{Int}\, M$, and a sequence $\{t_i\}$ with $t_i \to \infty$. For each i, let $\gamma_i : [0, l_i] \to M$ be a normal minimal geodesic from p to $\gamma(t_i)$. Since γ is a ray, we have $l_i \to \infty$. Take a sequence $\{T_j\}$ with $T_j \to \infty$. Since M is proper, there exists a subsequence $\{\gamma_{1,i}\}$ of $\{\gamma_i\}$, and a normal minimal geodesic $\gamma_{p,1} : [0, T_1] \to M$ from p to $\gamma_{p,1}(T_1)$ such that $\gamma_{1,i}|_{[0,T_1]}$ uniformly converges to $\gamma_{p,1}$. Furthermore, there exists a subsequence $\{\gamma_{2,i}\}$ of $\{\gamma_{1,i}\}$, and a normal minimal geodesic $\gamma_{p,2} : [0, T_2] \to M$ from p to $\gamma_{p,2}(T_2)$ such that $\gamma_{2,i}|_{[0,T_2]}$ uniformly converges to $\gamma_{p,2}$, where $\gamma_{p,2}|_{[0,T_1]} = \gamma_{p,1}$. By a diagonal argument, we obtain a subsequence $\{\gamma_k\}$ of $\{\gamma_i\}$, and a ray $\gamma_p : [0, \infty) \to M$ such that for every $t \in (0, \infty)$, we have $\gamma_k(t) \to \gamma_p(t)$ as $k \to \infty$. We call such a ray γ_p an *asymptote for γ from p*.

Lemma 6.2. *Let M be a connected complete Riemannian manifold with boundary. Suppose that for some $x_0 \in \partial M$, we have $\tau(x_0) = \infty$. Take $l > 0$, and put $p := \gamma_{x_0}(l)$. Then there exists $\epsilon > 0$ such that for all $q \in B_\epsilon(p)$, all asymptotes for the ray γ_{x_0} from q lie in* Int M.

Proof. The proof is by contradiction. Suppose that there exists a sequence $\{q_i\}$ in Int M with $q_i \to p$ such that for each i, there exists an asymptote γ_i for γ_{x_0} from q_i such that γ_i does not lie in Int M. Now, M is proper. Therefore, by taking a subsequence of $\{\gamma_i\}$, we may assume that there exists a ray $\gamma_p : [0, \infty) \to M$ such that for every $t \in [0, \infty)$, we have $\gamma_i(t) \to \gamma_p(t)$ as $i \to \infty$.

Fix i. Since γ_i is an asymptote for γ_{x_0} from q_i, there exists a sequence $\{t_{i_k}\}$ with $t_{i_k} \to \infty$ as $k \to \infty$, and for every k there exists a normal minimal geodesic γ_{i_k} in M from q_i to $\gamma_{x_0}(t_{i_k})$ such that for every $t \in (0, \infty)$ we have $\gamma_{i_k}(t) \to \gamma_i(t)$ as $k \to \infty$. For a fixed $t \in (0, \infty)$, and for every k, we have

$$t_{i_k} - d_M(q_i, \gamma_{x_0}(t_{i_k})) = -t + \left(t_{i_k} - d_M(\gamma_{i_k}(t), \gamma_{x_0}(t_{i_k}))\right).$$

Letting $k \to \infty$, we have $b_{\gamma_{x_0}}(q_i) = -t + b_{\gamma_{x_0}}(\gamma_i(t))$. By letting $i \to \infty$, we obtain $b_{\gamma_{x_0}}(p) = -t + b_{\gamma_{x_0}}(\gamma_p(t))$.

Since $\rho_{\partial M}$ is 1-Lipschitz, and since $\tau(x_0) = \infty$, we have $b_{\gamma_{x_0}} \leq \rho_{\partial M}$ on M, and the equality holds at p. Furthermore, for every $t \in (0, \infty)$ we have $b_{\gamma_{x_0}}(p) = -t + b_{\gamma_{x_0}}(\gamma_p(t))$. Therefore, for every $t \in (0, \infty)$,

$$\begin{aligned} d_M(\gamma_p(t), x_0) \geq \rho_{\partial M}(\gamma_p(t)) \geq b_{\gamma_{x_0}}(\gamma_p(t)) &= t + \rho_{\partial M}(p) \\ &= d_M(\gamma_p(t), p) + d_M(p, x_0). \end{aligned}$$

From the triangle inequality, it follows that $d_M(\gamma_p(t), x_0)$ is equal to $d_M(\gamma_p(t), p) + d_M(p, x_0)$. In particular, $\gamma_p|_{[0,\infty)}$ coincides with $\gamma_{x_0}|_{[l,\infty)}$. Since $q_i \in \operatorname{Int} M$ for each i, we have $u_i := \gamma_i'(0) \in U_{q_i}M$. We have $q_i \to p$ in M. Therefore, by taking a subsequence of $\{u_i\}$, for some $u \in U_pM$ we have $u_i \to u$ in the unit tangent bundle on Int M. Since $\gamma_p|_{[0,\infty)}$ coincides with $\gamma_{x_0}|_{[l,\infty)}$, we have $u = \gamma_{x_0}'(l)$. Put

$$t_i := \sup\{t > 0 \mid \gamma_i([0, t)) \subset \operatorname{Int} M\}$$

and $x_i := \gamma_i(t_i) \in \partial M$. Since all γ_i are asymptotes for γ_{x_0}, and since $\rho_{\partial M}(x_i) = 0$ for all i, we have

$$b_{\gamma_{x_0}}(q_i) = -t_i + b_{\gamma_{x_0}}(x_i) \leq -t_i.$$

We see $b_{\gamma_{x_0}}(q_i) \to l$ as $i \to \infty$. Therefore, the sequence $\{t_i\}$ does not diverge. We may assume that for some $x \in \partial M$, the sequence $\{x_i\}$ converges to x in ∂M. Since $u = \gamma_{x_0}'(l)$, the ray γ_{x_0} passes through x. This contradicts that $\gamma_{x_0}|_{(0,\infty)}$ lies in Int M. □

Let M be a connected complete Riemannian manifold with boundary. Take a point $p \in \operatorname{Int} M$, and a continuous function $f : M \to \mathbb{R}$. We say that a function $\bar{f} : M \to \mathbb{R}$ is a *support function of f at p* if we have $\bar{f}(p) = f(p)$, and for all $q \in M$, we have $\bar{f}(q) \leq f(q)$.

Take a domain U in $\operatorname{Int} M$. We say that f is *subharmonic in a barrier sense on U* if for each $\epsilon > 0$, and for each $p \in U$, there exists a support function $f_{p,\epsilon} : M \to \mathbb{R}$ of f at p such that $f_{p,\epsilon}$ is smooth on an open neighborhood of p, and $\Delta f_{p,\epsilon}(p) \leq \epsilon$. The Calabi maximal principle in [4] tells us that if a function that is subharmonic in a barrier sense on U takes the maximal value at a point in U, then the function must be constant.

We prove Theorem 1.8 by using the Calabi maximal principle in [4]. Proof of Theorem 1.8. For $\kappa \leq 0$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \sqrt{|\kappa|}$. Assume that for $x \in \partial M$, we have $\tau(x) = \infty$. Let ∂M_0 be the connected component of ∂M containing x. Put

$$\Omega := \{y \in \partial M_0 \mid \tau(y) = \infty\}.$$

The assumption implies $\Omega \neq \emptyset$. By the continuity of the function τ, we see that Ω is closed in ∂M_0.

We show the openness of Ω in ∂M_0. Let $x_0 \in \Omega$. Take $l > 0$, and put $p_0 := \gamma_{x_0}(l)$. By Lemma 6.2, there exists a sufficiently small open neighborhood U of p_0 in $\operatorname{Int} M$ with $U \subset D_{\partial M}$ such that for each $q \in U$, the unique foot point on ∂M of q belongs to ∂M_0, and all asymptotes for γ_{x_0} from q lie in $\operatorname{Int} M$.

We prove that the function $b_{\gamma_{x_0}} - \rho_{\partial M}$ is subharmonic in a barrier sense on U. By Proposition 3.10, $\rho_{\partial M}$ is smooth on U. Fix a point $q_0 \in U$, and take an asymptote $\gamma_{q_0} : [0,\infty) \to M$ for γ_{x_0} from q_0. For $t > 0$, define a function $b_{\gamma_{x_0},t} : M \to \mathbb{R}$ by

$$b_{\gamma_{x_0},t}(p) := b_{\gamma_{x_0}}(q_0) + t - d_M(p, \gamma_{q_0}(t)).$$

We see that $b_{\gamma_{x_0},t} - \rho_{\partial M}$ is a support function of $b_{\gamma_{x_0}} - \rho_{\partial M}$ at q_0. Since γ_{q_0} is a ray contained in $\operatorname{Int} M$, for every $t \in (0,\infty)$, the function $b_{\gamma_{x_0},t}$ is smooth on a neighborhood of q_0 in $\operatorname{Int} M$. By Lemma 2.4, we have $\Delta b_{\gamma_{x_0},t}(q_0) \leq (n-1)(s'_\kappa(t)/s_\kappa(t))$. Note that $s'_\kappa(t)/s_\kappa(t) \to \sqrt{|\kappa|}$ as $t \to \infty$. On the other hand, by Theorem 4.3, for all $q \in U$, we have $\Delta \rho_{\partial M}(q) \geq (n-1)\sqrt{|\kappa|}$. Hence, $b_{\gamma_{x_0}} - \rho_{\partial M}$ is subharmonic in a barrier sense on U. The function $b_{\gamma_{x_0}} - \rho_{\partial M}$ takes the maximal value 0 at p_0. The Calabi maximal principle in [4] implies that $b_{\gamma_{x_0}}$ coincides with $\rho_{\partial M}$ on U. From Lemma 6.1, it follows that Ω is open in ∂M_0.

For all $x \in \partial M_0$, we have $\tau(x) = \infty$. We put

$$TD_{\partial M_0} := \bigcup_{x \in \partial M_0} \{t\, u_x \mid t \in (0,\infty)\}.$$

By Lemma 3.9, $\exp^\perp|_{TD_{\partial M_0}} : TD_{\partial M_0} \to \exp^\perp(TD_{\partial M_0})$ is a diffeomorphism. The set $TD_{\partial M_0}$ is open and closed in $TD_{\partial M} \setminus 0(T^\perp \partial M)$. Therefore, $\exp^\perp(TD_{\partial M_0})$ is also open and closed in $\operatorname{Int} M$. Since $\operatorname{Int} M$ is connected, $\exp^\perp(TD_{\partial M_0})$ coincides with $\operatorname{Int} M$; in particular, ∂M is connected and $\operatorname{Cut} \partial M = \emptyset$. Note that $\rho_{\partial M}$ is smooth on $\operatorname{Int} M$.

Take $p \in \operatorname{Int} M$ and the unique foot point x_p on ∂M of p. Since $\tau(x_p) = \infty$, the maximal principle argument implies that $b_{\gamma_{x_p}}$ coincides with $\rho_{\partial M}$ on a neighborhood V of p in $\operatorname{Int} M$; in particular, $b_{\gamma_{x_p}}$ is smooth on V, and $\Delta \rho_{\partial M}(p) = (n-1)\sqrt{|\kappa|}$. It follows that the equality in Theorem 4.3 holds on $\operatorname{Int} M$. For each $x \in \partial M$, choose an orthonormal basis $\{e_{x,i}\}_{i=1}^{n-1}$ of $T_x \partial M$. For each $i = 1, \dots, n-1$, let $Y_{x,i}$ be the ∂M-Jacobi field along γ_x with initial conditions $Y_{x,i}(0) = e_{x,i}$ and $Y'_{x,i}(0) = -A_{u_x} e_{x,i}$. Then we have $Y_{x,i}(t) = s_{\kappa,\sqrt{|\kappa|}}(t) E_{x,i}(t)$, where $E_{x,i}$ is the parallel vector fields along γ_x with initial condition $E_{x,i}(0) = e_{x,i}$ (see Remark 4.4).

Define a map $\Phi : [0,\infty)\times\partial M \to M$ by $\Phi(t,x) := \gamma_x(t)$. For every $p \in (0,\infty)\times\partial M$, the map $D(\Phi|_{(0,\infty)\times\partial M})_p$ sends an orthonormal basis of $T_p((0,\infty)\times\partial M)$ to that of $T_{\Phi(p)}M$, and for every $x \in \{0\}\times\partial M$, the map $D(\Phi|_{\{0\}\times\partial M})_x$ sends an orthonormal basis of $T_x(\{0\}\times\partial M)$ to that of $T_{\Phi(x)}\partial M$. Therefore, Φ is a Riemannian isometry with boundary from $[0,\infty)\times_{\kappa,\sqrt{|\kappa|}}\partial M$ to M. We complete the proof of Theorem 1.8. □

The Cheeger-Gromoll splitting theorem ([8]) states that if M is an n-dimensional, connected complete Riemmanian manifold of non-negative Ricci curvature, and if M contains a line, then there exists an $(n-1)$-dimensional Riemannian manifold N of non-negative Ricci curvature such that M is isometric to the standard product $\mathbb{R}\times N$.

Corollary 6.3. *For $\kappa \le 0$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that* $\mathrm{Ric}_M \ge (n-1)\kappa$ *and $H_{\partial M} \ge \sqrt{|\kappa|}$. Suppose that for some $x \in \partial M$, we have $\tau(x) = \infty$. Then there exist $k \in \{0,\dots,n-1\}$, and an $(n-1-k)$-dimensional, connected complete Riemannian manifold N of non-negative Ricci curvature containing no line such that $(\partial M, d_{\partial M})$ is isometric to the standard product metric space* $(\mathbb{R}^k\times N, d_{\mathbb{R}^k\times N})$. *In particular, (M, d_M) is isometric to* $([0,\infty)\times_{\kappa,\sqrt{|\kappa|}}(\mathbb{R}^k\times N), d_{\kappa,\sqrt{|\kappa|}})$.

Proof. From Theorem 1.8, it follows that the metric space (M, d_M) is isometric to $([0,\infty)\times_{\kappa,\sqrt{|\kappa|}}\partial M, d_{\kappa,\sqrt{|\kappa|}})$. Lemma 5.5 implies $\mathrm{Ric}_{\partial M} \ge 0$. Applying the Cheeger-Gromoll splitting theorem to ∂M inductively, we see that $(\partial M, d_{\partial M})$ is isometric to $(\mathbb{R}^k\times N, d_{\mathbb{R}^k\times N})$ for some k. □

7. The first eigenvalues

7.1. Lower bounds. Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g. For a relatively compact domain Ω in M such that $\partial\Omega$ is a smooth hypersurface in M, we denote by $\mathrm{vol}_{\partial\Omega}$ the Riemannian volume measure on $\partial\Omega$ induced from the induced Riemannian metric on $\partial\Omega$. For $\alpha \in (0,\infty)$, the *Dirichlet α-isoperimetric constant $ID_\alpha(M)$* of M is defined as

$$ID_\alpha(M) := \inf_{\Omega} \frac{\mathrm{vol}_{\partial\Omega}\,\partial\Omega}{\left(\mathrm{vol}_g\,\Omega\right)^{1/\alpha}},$$

where the infimum is taken over all relatively compact domains Ω in M such that $\partial\Omega$ is a smooth hypersurface in M and $\partial\Omega\cap\partial M = \emptyset$. The *Dirichlet α-Sobolev constant $SD_\alpha(M)$* of M is defined as

$$SD_\alpha(M) := \inf_{f\in W_0^{1,1}(M)} \frac{\int_M \|\nabla f\|\, d\,\mathrm{vol}_g}{\left(\int_M |f|^\alpha\, d\,\mathrm{vol}_g\right)^{1/\alpha}}.$$

For all $\alpha \in (0,\infty)$, we have $ID_\alpha(M) = SD_\alpha(M)$. This relationship between the isoperimetric constant and the Sobolev constant has been formally established by Federer and Fleming in [13] (see e.g., Theorem 4 in Chapter 4 in [5], Theorem 9.5 in [29]), and later used by Cheeger in [6] for the estimate of the first Dirichlet eigenvalue of the Laplacian.

The following volume estimate has been proved by Kasue in Proposition 4.1 in [25].

Proposition 7.1 ([25])**.** *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \ge (n-1)\kappa$ *and $H_{\partial M} \ge \lambda$.*

Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M. Then

$$\mathrm{vol}_g\,\Omega \le \mathrm{vol}_{\partial\Omega}\,\partial\Omega \sup_{t\in(\delta_1(\Omega),\delta_2(\Omega))} \frac{\int_t^{\delta_2(\Omega)} s_{\kappa,\lambda}^{n-1}(s)\,ds}{s_{\kappa,\lambda}^{n-1}(t)},$$

where $\delta_1(\Omega) := \inf_{p\in\Omega} \rho_{\partial M}(p)$ and $\delta_2(\Omega) := \sup_{p\in\Omega} \rho_{\partial M}(p)$.

The equality case in Proposition 7.1 has been also studied in [25].

We prove Theorem 1.10. Proof of Theorem 1.10. Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \ge (n-1)\kappa$, $H_{\partial M} \ge \lambda$ and $D(M, \partial M) \le D$. Suppose ∂M is compact. Recall that the positive constant $C(n,\kappa,\lambda,D)$ is defined as

$$C(n,\kappa,\lambda,D) := \sup_{t\in[0,D)} \frac{\int_t^{D} s_{\kappa,\lambda}^{n-1}(s)\,ds}{s_{\kappa,\lambda}^{n-1}(t)}.$$

Let Ω be a relatively compact domain in M such that $\partial\Omega$ is a smooth hypersurface in M and $\partial\Omega \cap \partial M = \emptyset$. By Proposition 7.1,

$$\mathrm{vol}_g\,\Omega \le \mathrm{vol}_{\partial\Omega}\,\partial\Omega \sup_{t\in(0,D)} \frac{\int_t^{D} s_{\kappa,\lambda}^{n-1}(s)\,ds}{s_{\kappa,\lambda}^{n-1}(t)} = C(n,\kappa,\lambda,D)\ \mathrm{vol}_{\partial\Omega}\,\partial\Omega.$$

From the relationship $ID_1(M) = SD_1(M)$, it follows that $SD_1(M) \ge C(n,\kappa,\lambda,D)^{-1}$. Therefore, for all $\phi \in W_0^{1,1}(M)$, we have the following Poincaré inequality:

$$\int_M |\phi|\,d\,\mathrm{vol}_g \le C(n,\kappa,\lambda,D) \int_M \|\nabla\phi\|\,d\,\mathrm{vol}_g\,.$$

For a fixed $p \in (1,\infty)$, let ψ be a non-zero function in $W_0^{1,p}(M)$. Put $q := p\,(1-p)^{-1}$. In the Poincaré inequality, by replacing ϕ with $|\psi|^p$, and by the Hölder inequality, we see

$$\begin{aligned}\int_M |\psi|^p\,d\,\mathrm{vol}_g &\le p\,C(n,\kappa,\lambda,D) \int_M |\psi|^{p-1}\,\|\nabla\psi\|\,d\,\mathrm{vol}_g\\ &\le p\,C(n,\kappa,\lambda,D)\left(\int_M |\psi|^p\,d\,\mathrm{vol}_g\right)^{1/q}\left(\int_M \|\nabla\psi\|^p\,d\,\mathrm{vol}_g\right)^{1/p}.\end{aligned}$$

Considering the Rayleigh quotient $R_p(\psi)$, we obtain the inequality $\mu_{1,p}(M) \ge (p\,C(n,\kappa,\lambda,D))^{-p}$. This proves Theorem 1.10. □

We next prove Theorem 1.13. Proof of Theorem 1.13. Let $\kappa < 0$ and $\lambda := \sqrt{|\kappa|}$. Let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \ge (n-1)\kappa$ and $H_{\partial M} \ge \lambda$. Suppose ∂M is compact. We put $D := D(M,\partial M) \in (0,\infty]$. We have

$$C(n,\kappa,\lambda,D) = ((n-1)\lambda)^{-1}\left(1 - e^{-(n-1)\lambda D}\right).$$

The right hand side is monotone increasing as $D \to \infty$. By Theorem 1.10, for all $p \in (1,\infty)$ we have $\mu_{1,p}(M) \ge ((n-1)\lambda/p)^p$.

We assume $\mu_{1,p}(M) = ((n-1)\lambda/p)^p$. By Theorem 1.10, we have $D = \infty$. Therefore, the compactness of ∂M and Lemma 3.4 imply that M is noncompact. It has been proved in Theorem C in [23] as a splitting theorem (see Subsection 1.2) that if M is noncompact and ∂M is compact, then (M, d_M) is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Therefore, (M, d_M) is isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$.

Let $p = 2$, and let (M, d_M) be isometric to $([0,\infty) \times_{\kappa,\lambda} \partial M, d_{\kappa,\lambda})$. Let $\phi_{n,\kappa,\lambda} : [0,\infty) \to [0,\infty)$ be a smooth function defined by

$$\phi_{n,\kappa,\lambda}(t) := t\, e^{\frac{(n-1)\lambda t}{2}}.$$

Then the smooth function $\phi_{n,\kappa,\lambda} \circ \rho_{\partial M}$ on M satisfies

$$\Delta_2(\phi_{n,\kappa,\lambda} \circ \rho_{\partial M}) = \left(\frac{(n-1)\lambda}{2}\right)^2 (\phi_{n,\kappa,\lambda} \circ \rho_{\partial M})$$

on M; in particular,

$$\mu_{1,2}(M) \le R_2(\phi_{n,\kappa,\lambda} \circ \rho_{\partial M}) = \left(\frac{(n-1)\lambda}{2}\right)^2.$$

Therefore, $\mu_{1,2}(M) = ((n-1)\lambda/2)^2$. This proves Theorem 1.13. □

7.2. Segment inequality. For $n \ge 2$, $\kappa, \lambda \in \mathbb{R}$, and $D \in (0, \bar{C}_{\kappa,\lambda}]$, let $C_1(n,\kappa,\lambda,D)$ be the positive constant defined as

$$C_1(n,\kappa,\lambda,D) := \sup_{l \in (0,D)} \sup_{t \in (0,l)} \frac{s_{\kappa,\lambda}^{n-1}(l)}{s_{\kappa,\lambda}^{n-1}(t)}.$$

We prove the following segment inequality:

Proposition 7.2. *For $D \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \ge (n-1)\kappa$, $H_{\partial M} \ge \lambda$ and $D(M, \partial M) \le D$. Let $f : M \to \mathbb{R}$ be a non-negative integrable function on M, and define a function $E_f : M \to \mathbb{R}$ by*

$$E_f(p) := \inf_{x \in \partial M} \int_0^{\rho_{\partial M}(p)} f(\gamma_x(t))\, dt,$$

where the infimum is taken over all foot points x on ∂M of p. Then

$$\int_M E_f \, d\,\mathrm{vol}_g \le C_1(n,\kappa,\lambda,D) D \int_M f \, d\,\mathrm{vol}_g.$$

Proof. Put $C_1 := C_1(n,\kappa,\lambda,D)$. Fix $x \in \partial M$ and $l \in (0, \tau(x))$. Observe that x is the unique foot point on ∂M of $\gamma_x(l)$, and $\gamma_x|_{(0,l]}$ lies in $\mathrm{Int}\, M$. By Lemma 4.5, for all $t \in [0, l]$ we have

$$E_f(\gamma_x(l))\theta(l,x) \le C_1 \int_0^l f(\gamma_x(t))\theta(t,x)\, dt.$$

Integrating the both sides, we see

$$\int_0^{\tau(x)} E_f(\gamma_x(l))\theta(l,x)\, dl \le C_1 D \int_0^{\tau(x)} f(\gamma_x(t))\theta(t,x)\, dt.$$

Lemma 3.7 implies $M = \exp^{\perp}(\bigcup_{x\in\partial M}\{tu_x \mid t \in [0,\tau(x)]\})$. From Lemma 3.9, it follows that $\exp^{\perp}|_{TD_{\partial M}\setminus 0(T^{\perp}\partial M)}$ is a diffeomorphism onto $D_{\partial M} \setminus \partial M$. By Proposition 3.5, we have $\operatorname{vol}_g \operatorname{Cut}\partial M = 0$. Integrating the both sides of the above inequality over ∂M with respect to x, we obtain the desired segment inequality. □

From Proposition 7.2, we derive the following Poincaré inequality:

Lemma 7.3. *For $D \in (0,\bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\operatorname{Ric}_M \geq (n-1)\kappa$, $H_{\partial M} \geq \lambda$ and $D(M,\partial M) \leq D$. Let $\psi : M \to \mathbb{R}$ be a smooth integrable function on M with $\psi|_{\partial M} = 0$. Assume $\int_M \|\nabla\psi\|\, d\operatorname{vol}_g < \infty$. Then*

$$\int_M |\psi|\, d\operatorname{vol}_g \leq C_1(n,\kappa,\lambda,D)D \int_M \|\nabla\psi\|\, d\operatorname{vol}_g .$$

Proof. Put $f := \|\nabla\psi\|$, and let E_f be the function defined in Proposition 7.2. For each $p \in D_{\partial M}$, let x be the foot point on ∂M of p. By the Cauchy-Schwarz inequality, we have

$$|\psi(p) - \psi(x)| \leq \int_0^{\rho_{\partial M}(p)} \left|g(\nabla\psi, \gamma_x'(t))\right| dt \leq E_f(p).$$

Since $\psi|_{\partial M} = 0$, we have $|\psi(p)| \leq E_f(p)$. Integrate the both sides of the inequality over $D_{\partial M}$ with respect to p. By Proposition 7.2 and $\operatorname{vol}_g \operatorname{Cut}\partial M = 0$, we arrived at the desired inequality. □

As one of the applications of our segment inequality in Proposition 7.2, we show the following:

Proposition 7.4. *For $D \in (0,\bar{C}_{\kappa,\lambda}]$, let M be an n-dimensional, connected complete Riemannian manifold with boundary such that $\operatorname{Ric}_M \geq (n-1)\kappa$, $H_{\partial M} \geq \lambda$ and $D(M,\partial M) \leq D$. Let M be compact. Then for all $p \in (1,\infty)$, we have*

$$\mu_{1,p}(M) \geq (\, p\, C_1(n,\kappa,\lambda,D)\, D\,)^{-p}.$$

Proof. For a fixed $p \in (1,\infty)$, let ψ be a non-zero function in $W_0^{1,p}(M)$. We may assume that ψ is smooth on M. In Lemma 7.3, by replacing ψ with $|\psi|^p$, we have

$$\int_M |\psi|^p\, d\operatorname{vol}_g \leq p\, C_1(n,\kappa,\lambda,D)\, D \int_M |\psi|^{p-1}\, \|\nabla\psi\|\, d\operatorname{vol}_g .$$

From the Hölder inequality, we derive $R_p(\psi) \geq (\, p\, C_1(n,\kappa,\lambda,D)\, D\,)^{-p}$. This proves Proposition 7.4. □

Remark 7.5. Proposition 7.4 is weaker than Theorem 1.10. We can prove that the lower bound $(p\, C_1(n,\kappa,\lambda,D)\, D)^{-p}$ for $\mu_{1,p}$ in Proposition 7.4 is at most the lower bound $(p\, C(n,\kappa,\lambda,D))^{-p}$ in Theorem 1.10.

8. Measure contraction property

Let M be a connected complete Riemannian manifold with boundary with Riemannian metric g.

8.1. Measure contraction inequalities. Let $t \in (0,1)$. For a point $p \in M$, we say that $q \in M$ is a *t-extension point from ∂M of p* if q satisfies the following: (1) $\rho_{\partial M}(p)/\rho_{\partial M}(q) = t$; (2) there exists a foot point x on ∂M of p with $q = \gamma_x(\rho_{\partial M}(q))$. We denote by W_t the set of all points $p \in M$ for which there exists a t-extension point from ∂M of p.

We first show the following:

Lemma 8.1. *For every $t \in (0,1)$, and for every $p \in W_t$, there exists a unique foot point on ∂M of p. In particular, every $p \in W_t$ has a unique t-extension point from ∂M.*

Proof. Take $p \in W_t$. Let q be a t-extension point from ∂M of p. There exists a foot point x on ∂M of p such that $q = \gamma_x(\rho_{\partial M}(q))$. The definition of τ implies $\rho_{\partial M}(q) \leq \tau(x)$. Since $\rho_{\partial M}(p) = t\rho_{\partial M}(q)$, it follows that $\rho_{\partial M}(p) < \tau(x)$. From Lemma 3.1, we derive $p = \gamma_x(\rho_{\partial M}(p))$. Lemma 3.6 tells us that x is a unique foot point on ∂M of p.

Suppose that there exist distinct t-extension points $q_1, q_2 \in M$ from ∂M of p. By the definition, it holds that $\rho_{\partial M}(q_1) = \rho_{\partial M}(q_2)$. Furthermore, for each $i = 1, 2$, there exists a foot point x_i on ∂M of p with $q_i = \gamma_{x_i}(\rho_{\partial M}(q_i))$. Since $q_1 \neq q_2$, we have $x_1 \neq x_2$. This contradicts the property that p has a unique foot point on ∂M. □

By Lemma 8.1, for every $t \in (0,1)$, we can define a map $\Phi_t : W_t \to M$ by $\Phi_t(p) := q$, where q is a unique t-extension point from ∂M of p. We call Φ_t the *t-extension map from ∂M*. Notice that for every $t \in (0,1)$, the t-extension map Φ_t from ∂M is surjective and continuous.

Let Ω be a subset of M. We say that $x \in \partial M$ is a *foot point on ∂M of* Ω if there exists a point $p \in \Omega$ such that x is a foot point on ∂M of p. We denote by $\Pi(\Omega)$ the set of all foot points on ∂M of Ω.

We have the following property of the t-extension map Φ_t from ∂M:

Lemma 8.2. *For $t \in (0,1)$, let Φ_t be the t-extension map from ∂M. Let Ω be a subset of M. Then $\Pi(\Phi_t^{-1}(\Omega)) = \Pi(\Omega)$.*

Proof. First, we show $\Pi(\Omega) \subset \Pi(\Phi_t^{-1}(\Omega))$. Take $x \in \Pi(\Omega)$. There exists $p \in \Omega$ such that x is a foot point on ∂M of p. Put $p_t := \gamma_x(t\rho_{\partial M}(p))$. It suffices to show that x is a foot point on ∂M of p_t, and p_t belongs to $\Phi_t^{-1}(\Omega)$. Lemma 3.1 implies $p = \gamma_x(\rho_{\partial M}(p))$. By the definition of τ, we see $\rho_{\partial M}(p) \leq \tau(x)$; in particular, $t\rho_{\partial M}(p)$ is smaller than $\tau(x)$. From Lemma 3.6, it follows that x is a unique foot point on ∂M of p_t. Furthermore, we have $\rho_{\partial M}(p_t) = t\rho_{\partial M}(p)$. Hence, p is a t-extension point from ∂M of p_t. By Lemma 8.1, p is a unique t-extension point from ∂M. Since $p = \Phi_t(p_t)$ and $p \in \Omega$, we see $p_t \in \Phi_t^{-1}(\Omega)$. This implies $x \in \Pi(\Phi_t^{-1}(\Omega))$.

Next, we show the opposite. Take $x \in \Pi(\Phi_t^{-1}(\Omega))$. There exists $p \in \Phi_t^{-1}(\Omega)$ such that x is a foot point on ∂M of p. By Lemma 8.1, x is a unique foot point on ∂M of p. By the definition of the t-extension point from ∂M, we see $\Phi_t(p) = \gamma_x(\rho_{\partial M}(\Phi_t(p)))$. Thus, we have $\rho_{\partial M}(\Phi_t(p)) \leq \tau(x)$. Hence, x is a foot point on ∂M of $\Phi_t(p)$. Since $\Phi_t(p) \in \Omega$, we have

$x \in \Pi(\Omega)$. This proves the lemma. $\square$

For $t \in (0,1)$, let Φ_t be the t-extension map from ∂M. Let Ω be a subset of M. For $x \in \Pi(\Omega)$, we put

$$I_{\Omega,t,x} := \{\, s \in (0, t\tau(x)) \mid \gamma_x(s) \in \Phi_t^{-1}(\Omega) \,\}.$$

We prove the following:

Lemma 8.3. *For $t \in (0,1)$, let Φ_t be the t-extension map from ∂M. Suppose that a subset Ω of M is measurable, and satisfies $\mathrm{vol}_g \Phi_t^{-1}(\Omega) < \infty$. Then we have*

$$\mathrm{vol}_g \Phi_t^{-1}(\Omega) = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,x}} \theta(s,x)\, ds\, d\,\mathrm{vol}_h .$$

Proof. We put

$$A := \{\, \gamma_x(t\tau(x)) \in \Phi_t^{-1}(\Omega) \mid x \in \Pi(\Omega), \tau(x) < \infty\},$$
$$B := \{\, \gamma_x(s) \mid x \in \Pi(\Omega), s \in I_{\Omega,t,x}\}.$$

Note that A and B are disjoint.

We show $\Phi_t^{-1}(\Omega) \setminus \partial M = A \sqcup B$. The definition of $I_{\Omega,t,x}$ implies $A \sqcup B \subset \Phi_t^{-1}(\Omega) \setminus \partial M$. To show the opposite, take $p \in \Phi_t^{-1}(\Omega) \setminus \partial M$, and take a foot point x on ∂M of p. By Lemma 3.1, we see $p = \gamma_x(\rho_{\partial M}(p))$. From Lemma 8.2, we derive $x \in \Pi(\Omega)$. Now, p belongs to W_t. Hence, by Lemma 8.1, x is a unique foot point on ∂M of p, and there exists a unique t-extension point $q \in M$ from ∂M of p. The t-extension point q from ∂M of p satisfies $t\rho_{\partial M}(q) = \rho_{\partial M}(p)$ and $q = \gamma_x(\rho_{\partial M}(q))$. The definition of τ implies $\rho_{\partial M}(q) \le \tau(x)$. It holds that $\rho_{\partial M}(p) \le t\tau(x)$. Since $x \in \Pi(\Omega)$ and $\rho_{\partial M}(p) \in (0, t\tau(x)]$, it follows that $\Phi_t^{-1}(\Omega) \setminus \partial M \subset A \sqcup B$.

We next show that A is a null set of M. We put

$$\bar{A} := \bigcup_{x \in \Pi(\Omega)} \{t\tau(x)u_x \mid \tau(x) < \infty\}.$$

Note that $A = \exp^{\perp}(\bar{A})$. By Lemma 3.2, and by the Fubini theorem, the graph $\{(x, t\tau(x)) \mid x \in \partial M,\ \tau(x) < \infty\}$ of $t\tau$ is a null set of $\partial M \times [0,\infty)$. Since a map $\Psi : \partial M \times [0,\infty) \to T^{\perp}\partial M$ defined by $\Psi(x,s) := su_x$ is smooth, the set $\bar{A}$ is also a null set of $T^{\perp}\partial M$. By the definition of τ, the set A is contained in $\mathrm{Int}\, M$. From the smoothness of $\exp^{\perp}$, it follows that A is a null set of M.

Since $\Phi_t^{-1}(\Omega) \setminus \partial M = A \sqcup B$, and since A is a null set of M, it suffices to show that

$$\mathrm{vol}_g B = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,x}} \theta(s,x)\, ds\, d\,\mathrm{vol}_h .$$

We put

$$\bar{B} := \bigcup_{x \in \Pi(\Omega)} \{su_x \mid s \in I_{\Omega,t,x}\}.$$

Note that $B = \exp^{\perp}(\bar{B})$. The set $\bar{B}$ is contained in $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$. By Lemma 3.9, the map $\exp^{\perp}|_{TD_{\partial M}\setminus 0(T^{\perp}\partial M)}$ is a diffeomorphism. Hence, by the coarea formula and the Fubini theorem,

$$\operatorname{vol}_g \exp^{\perp}(\bar{B}) = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,x}} \theta(s,x)\, ds\, d\operatorname{vol}_h .$$

Since $B = \exp^{\perp}(\bar{B})$, we arrive at the desired equation. □

Now, we prove the following measure contraction inequality:

Proposition 8.4. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\operatorname{Ric}_M \geq (n-1)\kappa$ *and $H_{\partial M} \geq \lambda$. For $t \in$* $(0,1)$*, let Φ_t be the t-extension map from ∂M. Suppose that a subset Ω of M is measurable. Then we have*

$$\operatorname{vol}_g \Phi_t^{-1}(\Omega) \geq t \int_{\Omega} \frac{s_{\kappa,\lambda}^{n-1} \circ t\rho_{\partial M}}{s_{\kappa,\lambda}^{n-1} \circ \rho_{\partial M}} d\operatorname{vol}_g .$$

Proof. We may assume $\operatorname{vol}_g \Phi_t^{-1}(\Omega) < \infty$. By Lemma 8.3,

$$\operatorname{vol}_g \Phi_t^{-1}(\Omega) = \int_{\Pi(\Omega)} \int_{I_{\Omega,t,x}} \theta(s,x)\, ds\, d\operatorname{vol}_h .$$

From Lemma 4.5, for all $x \in \Pi(\Omega)$ and $s \in I_{\Omega,t,x}$, we derive

$$\frac{\theta(t^{-1}s,x)}{\theta(s,x)} \leq \frac{s_{\kappa,\lambda}^{n-1}(t^{-1}s)}{s_{\kappa,\lambda}^{n-1}(s)}.$$

It follows that

$$\operatorname{vol}_g \Phi_t^{-1}(\Omega) \geq \int_{\Pi(\Omega)} \int_{I_{\Omega,t,x}} \frac{s_{\kappa,\lambda}^{n-1}(s)}{s_{\kappa,\lambda}^{n-1}(t^{-1}s)} \theta(t^{-1}s,x)\, ds\, d\operatorname{vol}_h .$$

For $x \in \Pi(\Omega)$, we put

$$I_{\Omega,x} := \{\, s \in (0,\tau(x)) \mid \gamma_x(s) \in \Omega \,\}.$$

Note that for each $x \in \Pi(\Omega)$, the set $\{l \in (0,\tau(x)) \mid tl \in I_{\Omega,t,x}\}$ coincides with $I_{\Omega,x}$. By putting $l := t^{-1}s$ in the above inequality, we have

$$\operatorname{vol}_g \Phi_t^{-1}(\Omega) \geq t \int_{\Pi(\Omega)} \int_{I_{\Omega,x}} \frac{s_{\kappa,\lambda}^{n-1}(tl)}{s_{\kappa,\lambda}^{n-1}(l)} \theta(l,x)\, dl\, d\operatorname{vol}_h .$$

Now, we put

$$\bar{U} := \bigcup_{x \in \Pi(\Omega)} \{su_x \mid s \in I_{\Omega,x}\}.$$

We show $\exp^{\perp}(\bar{U}) = \Omega \setminus (\operatorname{Cut}\partial M \cup \partial M)$. By the definition of $I_{\Omega,x}$, we have $\exp^{\perp}(\bar{U}) \subset \Omega \setminus$ $(\operatorname{Cut}\partial M \cup \partial M)$. To show the opposite, take $p \in \Omega \setminus (\operatorname{Cut}\partial M \cup \partial M)$, and take a foot point x on ∂M of p. From Lemma 3.1, it follows that $p = \exp^{\perp}(\rho_{\partial M}(p)u_x)$. We see $x \in \Pi(\Omega)$. Since p does not belongs to $\operatorname{Cut}\partial M \cup \partial M$, we have $\rho_{\partial M}(p) \in (0,\tau(x))$. This implies $\rho_{\partial M}(p) \in I_{\Omega,x}$. Hence, the set $\Omega \setminus (\operatorname{Cut}\partial M \cup \partial M)$ is contained in $\exp^{\perp}(\bar{U})$.

The set $\bar{U}$ is contained in $TD_{\partial M} \setminus 0(T^{\perp}\partial M)$. Lemma 3.9 implies that the map $\exp^{\perp}|_{TD_{\partial M}\setminus 0(T^{\perp}\partial M)}$ is a diffeomorphism. By the coarea formula and the Fubini theorem, and by Lemma 3.5, we have

$$\begin{aligned} t\int_{\Pi(\Omega)}\int_{I_{\Omega,x}} \frac{s^{n-1}_{\kappa,\lambda}(tl)}{s^{n-1}_{\kappa,\lambda}(l)}\theta(l,x)\,dl\,d\,\mathrm{vol}_h &= t\int_{\exp^{\perp}(\bar{U})} \frac{s^{n-1}_{\kappa,\lambda}\circ t\rho_{\partial M}}{s^{n-1}_{\kappa,\lambda}\circ \rho_{\partial M}}d\,\mathrm{vol}_g \\ &= t\int_{\Omega} \frac{s^{n-1}_{\kappa,\lambda}\circ t\rho_{\partial M}}{s^{n-1}_{\kappa,\lambda}\circ \rho_{\partial M}}d\,\mathrm{vol}_g\,. \end{aligned}$$

Thus, we arrive at the desired inequality. □

8.2. Another proof of Theorem 1.1. For $r, R \in (0,\infty)$ with $r < R$, we put $A_{r,R}(\partial M) := B_R(\partial M) \setminus B_r(\partial M)$.

By using Proposition 8.4, we have the following:

Lemma 8.5. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and $H_{\partial M} \geq \lambda$. Let $t \in (0,1)$. Suppose ∂M is compact. Then for all $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ and $r \in (0,R)$, we have*

$$\frac{\mathrm{vol}_g A_{r,R}(\partial M)}{\mathrm{vol}_g A_{tr,tR}(\partial M)} \leq \left(t \inf_{s\in(r,R)} \frac{s^{n-1}_{\kappa,\lambda}(ts)}{s^{n-1}_{\kappa,\lambda}(s)} \right)^{-1}.$$

Proof. Take $R \in (0, \bar{C}_{\kappa,\lambda}]\setminus\{\infty\}$ and $r \in (0,R)$. Put $\Omega := A_{r,R}(\partial M)$. Let Φ_t be the t-extension map from ∂M. For all $p \in \Phi_t^{-1}(\Omega)$, we have

$$\rho_{\partial M}(p) = t\,\rho_{\partial M}(\Phi_t(p)) \in (tr, tR].$$

Hence, $\Phi_t^{-1}(\Omega)$ is contained in $A_{tr,tR}(\partial M)$. Applying Proposition 8.4 to Ω, we obtain

$$\mathrm{vol}_g A_{tr,tR}(\partial M) \geq \mathrm{vol}_g \Phi_t^{-1}(\Omega) \geq t \inf_{s\in(r,R)} \frac{s^{n-1}_{\kappa,\lambda}(ts)}{s^{n-1}_{\kappa,\lambda}(s)}\ \mathrm{vol}_g \Omega.$$

This proves the lemma. □

From Lemma 8.5, we derive the following:

Lemma 8.6. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Let $r_2 \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, and let $r_1 \in (0, r_2)$. Put $t := r_1/r_2$. For $k \in \mathbb{N}$, put $r := t^k r_2$. Then we have*

$$\frac{\mathrm{vol}_g A_{r_1,r_2}(\partial M)}{\mathrm{vol}_g B_r(\partial M)} \leq \left(\sum_{i=k}^{\infty} t^i \inf_{s\in(r_1,r_2)} \frac{s^{n-1}_{\kappa,\lambda}(t^i s)}{s^{n-1}_{\kappa,\lambda}(s)} \right)^{-1}.$$

Proof. We see $B_r(\partial M) \setminus \partial M = \bigcup_{i=k}^{\infty} A_{t^i r_1, t^i r_2}(\partial M)$. Lemma 8.5 implies

$$\mathrm{vol}_g B_r(\partial M) = \sum_{i=k}^{\infty} \mathrm{vol}_g A_{t^i r_1, t^i r_2}(\partial M)$$

$$\geq \mathrm{vol}_g A_{r_1,r_2}(\partial M)\left(\sum_{i=k}^{\infty} t^i \inf_{s\in(r_1,r_2)} \frac{s^{n-1}_{\kappa,\lambda}(t^i s)}{s^{n-1}_{\kappa,\lambda}(s)}\right).$$

This completes the proof. □

By Lemma 8.6, we have the following volume estimate:

Lemma 8.7. *Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that* $\mathrm{Ric}_M \geq (n-1)\kappa$ *and* $H_{\partial M} \geq \lambda$. *Suppose ∂M is compact. Let $t \in (0,1)$. Take $l, m \in \mathbb{N}$ with $l < m$. Then for all $r \in (0,\infty)$ with $t^{l-1} r \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$, we have*

$$\frac{\mathrm{vol}_g B_{t^{l-1}r}(\partial M)}{\mathrm{vol}_g B_{t^{m-1}r}(\partial M)} \leq \frac{\sum_{j=l}^{\infty} \sup_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(s)(t^{j-1}r - t^j r)}{\sum_{i=m}^{\infty} \inf_{s\in(t^i r, t^{i-1} r)} s^{n-1}_{\kappa,\lambda}(s)(t^{i-1}r - t^i r)}.$$

Proof. Fix $j \in \{l, \dots, m-1\}$. By Lemma 8.6, we have

$$\begin{aligned}\frac{\mathrm{vol}_g A_{t^j r, t^{j-1} r}(\partial M)}{\mathrm{vol}_g B_{t^{m-1}r}(\partial M)} &\leq \left(\sum_{i=m-j}^{\infty} t^i \inf_{s\in(t^j r, t^{j-1} r)} \frac{s^{n-1}_{\kappa,\lambda}(t^i s)}{s^{n-1}_{\kappa,\lambda}(s)}\right)^{-1}\\ &\leq \left(\sum_{i=m-j}^{\infty} t^i \frac{\inf_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(t^i s)}{\sup_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(s)}\right)^{-1}.\end{aligned}$$

Note that we have

$$\left(\sum_{i=m-j}^{\infty} t^i \frac{\inf_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(t^i s)}{\sup_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(s)}\right)^{-1} = \frac{t^j \sup_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(s)}{\sum_{i=m}^{\infty} t^i \inf_{s\in(t^i r, t^{i-1} r)} s^{n-1}_{\kappa,\lambda}(s)}.$$

It follows that

$$\begin{aligned}\frac{\mathrm{vol}_g B_{t^{l-1}r}(\partial M)}{\mathrm{vol}_g B_{t^{m-1}r}(\partial M)} &= 1 + \sum_{j=l}^{m-1} \frac{\mathrm{vol}_g A_{t^j r, t^{j-1} r}(\partial M)}{\mathrm{vol}_g B_{t^{m-1}r}(\partial M)}\\ &\leq 1 + \sum_{j=l}^{m-1} \frac{t^j \sup_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(s)}{\sum_{i=m}^{\infty} t^i \inf_{s\in(t^i r, t^{i-1} r)} s^{n-1}_{\kappa,\lambda}(s)}\\ &\leq \frac{\sum_{j=l}^{\infty} t^j \sup_{s\in(t^j r, t^{j-1} r)} s^{n-1}_{\kappa,\lambda}(s)}{\sum_{i=m}^{\infty} t^i \inf_{s\in(t^i r, t^{i-1} r)} s^{n-1}_{\kappa,\lambda}(s)}.\end{aligned}$$

This implies the lemma. □

Now, we give another proof of Theorem 1.1.

Proof of Theorem 1.1. Let M be an n-dimensional, connected complete Riemannian manifold with boundary with Riemannian metric g such that $\mathrm{Ric}_M \geq (n-1)\kappa$ and $H_{\partial M} \geq \lambda$. Suppose ∂M is compact. Take $r, R \in (0,\infty)$ with $r \leq R$. By Lemma 4.6, we may assume $R \in (0, \bar{C}_{\kappa,\lambda}] \setminus \{\infty\}$ and $r < R$. Put $r_0 := Rr$. Take a sufficiently large $N \in \mathbb{N}$ such that $N^{-1} \log r \in (0,1)$. We put $t := 1 - (\log r/N)$, and

$$l := N + 1, \quad m := \min\{i \in \mathbb{N} \mid i \geq N(\log R/\log r) + 1\}.$$

We have $l < m$ and $t^{m-1}r_0 \leq r$. Note that if $N \to \infty$, then $t^{l-1}r_0 \to R$ and $t^{m-1}r_0 \to r$. From Lemma 8.7, it follows that

$$\begin{aligned}\frac{\mathrm{vol}_g B_{t^{l-1}r_0}(\partial M)}{\mathrm{vol}_g B_r(\partial M)} &\leq \frac{\mathrm{vol}_g B_{t^{l-1}r_0}(\partial M)}{\mathrm{vol}_g B_{t^{m-1}r_0}(\partial M)} \\ &\leq \frac{\sum_{j=l}^{\infty} \sup_{s\in(t^j r_0, t^{j-1}r_0)} s_{\kappa,\lambda}^{n-1}(s)(t^{j-1}r_0 - t^j r_0)}{\sum_{i=m}^{\infty} \inf_{s\in(t^i r_0, t^{i-1}r_0)} s_{\kappa,\lambda}^{n-1}(s)(t^{i-1}r_0 - t^i r_0)}.\end{aligned}$$

Letting $N \to \infty$, we have

$$\frac{\mathrm{vol}_g B_R(\partial M)}{\mathrm{vol}_g B_r(\partial M)} \leq \frac{\int_0^R s_{\kappa,\lambda}^{n-1}(s)\,ds}{\int_0^r s_{\kappa,\lambda}^{n-1}(s)\,ds}.$$

Thus, we obtain Theorem 1.1. □

Addendum: After completing the first draft of this paper, the author has been informed by Sormani of the paper [33] written by Perales. Let M be a connected complete Riemannian manifold with boundary such that $\mathrm{Ric}_M \geq 0$ and $H_{\partial M} \geq \lambda$. The paper [33] contains a Laplacian comparison theorem for $\rho_{\partial M}$ everywhere in a barrier sense, a theorem of volume estimates of the metric neighborhoods of ∂M, and applications to studies of convergences of such manifolds with boundary.

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