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GLOBAL MONODROMY MODULO 5 OF THE QUINTIC-MIRROR FAMILY

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Abstract

The quintic-mirror family is a well-known one-parameter family of Calabi–Yau threefolds. A complete description of the global monodromy group of this family is not yet known. In this paper, we give a presentation of the global monodromy group in the general linear group of degree 4 over the ring of integers modulo 5.

1. Introduction

The quintic-mirror family $(W_\lambda)_{\lambda\in\mathbb{P}^1} \to \mathbb{P}^1$ is a family, whose restriction $f\colon (W_\lambda)_{\lambda\in U} \to U$ on $U := \mathbb{P}^1 - \{0, 1, \infty\}$ is a smooth projective family of Calabi–Yau manifolds. Fix $b \in U$ and let $\langle\ ,\ \rangle$ be the anti-symmetric bilinear form on $H^3(W_b, \mathbb{Z})$ defined by the cup product. The global monodromy group Γ is the image of the representation $\pi_1(U, b) \to \mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$ corresponding to the local system $R^3 f_*\mathbb{Z}$ with the fiber $H^3(W_b, \mathbb{Z})$ over b. When we take a symplectic basis, we can identify $\mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$ with $\mathrm{Sp}(4, \mathbb{Z})$.

In this paper, we are concerned with a description of Γ. Matrix presentations of the generators of Γ are well studied and it is also known that Γ is Zariski dense in $\mathrm{Sp}(4,\mathbb{Z})$ (e.g. [1], [3]). However, it is not known whether the index of Γ in $\mathrm{Sp}(4,\mathbb{Z})$ is finite or not (e.g. [2]). A direct approach for this problem is to describe Γ explicitly. In the main theorem of this paper, we give a presentation of Γ in $\mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z})$, which is a small attempt toward a description of Γ.

On the other hand, Chen, Yang and Yui find a congruence subgroup $\Gamma(5, 5)$ of $\mathrm{Sp}(4,\mathbb{Z})$ of finite index, which contains Γ in [2]. Combining their result and our main theorem, we can construct a smaller congruence subgroup $\tilde{\Gamma}(5, 5)$ of $\mathrm{Sp}(4,\mathbb{Z})$ of finite index, which contains Γ. However this result is merely the fact that $\tilde{\Gamma}(5, 5)$ contains Γ. After all, the index of Γ in $\mathrm{Sp}(4,\mathbb{Z})$ is still unknown.

2. The quintic-mirror family

The quintic-mirror family was constructed by Greene and Plesser. We review the construction of the quintic-mirror family after [4].

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Let $\psi \in \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, and let

$$Q_\psi = \{x \in \mathbb{P}^4 \mid x_1^5 + x_2^5 + x_3^5 + x_4^5 + x_5^5 - 5\psi x_1x_2x_3x_4x_5 = 0\}.$$

A finite group G, which is abstractly isomorphic to $(\mathbb{Z}/5\mathbb{Z})^3$, acts on Q_ψ as follows.

$$\begin{aligned}
&\mu_5\colon \text{the multiplicative group of the 5-th root of } 1 \in \mathbb{C},\\
&\tilde{G} = (\mu_5)^5/\{(\alpha_1, \dots, \alpha_5) \in (\mu_5)^5 \mid \alpha_1 = \cdots = \alpha_5\},\\
&G = \{(\alpha_1, \dots, \alpha_5) \in \tilde{G} \mid \alpha_1 \cdots \alpha_5 = 1\},\\
&G \times Q_\psi \to Q_\psi, \quad ((\alpha_1, \dots, \alpha_5), (x_1, \dots, x_5)) \mapsto (\alpha_1x_1, \dots, \alpha_5x_5).
\end{aligned}$$

When we take the quotient of the hypersurface Q_ψ by G, canonical singularities appear. For $\psi \in \mathbb{C} \subset \mathbb{P}^1$, it is known that there is a simultaneous minimal desingularization of these singularities, and we have the one-parameter family $(W_\psi)_{\psi\in\mathbb{P}^1}$ whose fibres are listed as follows:

- When ψ belongs to $\mu_5 \subset \mathbb{C} \subset \mathbb{P}^1$, W_ψ has one ordinary double point.
- W_∞ is a normal crossing divisor in the total space.
- The other fibres of $(W_\psi)_{\psi\in\mathbb{P}^1}$ are smooth with Hodge numbers $h^{p,q} = 1$ for $p + q = 3,\ p, q \geq 0$.

By the action of

$$\alpha \in \mu_5, \quad (x_1, \dots, x_5) \mapsto (x_1, \dots, x_4, \alpha^{-1}x_5),$$

we have the isomorphism from the fibre over ψ to the fibre over $\alpha\psi$. Let $\lambda = \psi^5$ and let

$$\begin{array}{ccc}
(W_\lambda)_{\lambda\in\mathbb{P}^1} & = & ((W_\psi)_{\psi\in\mathbb{P}^1})/\mu_5\\
\downarrow & & \downarrow\\
(\lambda\text{-plane}) & = & (\psi\text{-plane})/\mu_5.
\end{array}$$

This family $(W_\lambda)_{\lambda\in\mathbb{P}^1}$ is the so-called quintic-mirror family. (For more details of the above construction, see e.g. [4], [5].)

3. Monodromy

Let $b \in \mathbb{P}^1 - \{0, 1, \infty\}$ on the λ-plane. In [1], Candelas, de la Ossa, Green and Parks constructed a symplectic basis $\{A^1, A^2, B_1, B_2\}$ of $H_3(W_b, \mathbb{Z})$ and calculated the monodromies around $\lambda = 0, 1, \infty$ on the period integrals of a holomorphic 3-form on this basis. By the relation in [5, Appendix C] between the symplectic basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ of $H^3(W_b, \mathbb{Z})$, which is defined to be the dual basis of $\{B_1, B_2, A^1, A^2\}$, and the period integrals, we have the matrix representations of the local monodromies for the basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$. We recall their results.

Matrix representations A, T, T_∞ of local monodromies around $\lambda = 0, 1, \infty$ for the basis $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ are as follows:

$$A = \begin{pmatrix} 11 & 8 & -5 & 0 \\ 5 & -4 & -3 & 1 \\ 20 & 15 & -9 & 0 \\ 5 & -5 & -3 & 1 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad T_\infty = \begin{pmatrix} -9 & -3 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ -20 & -5 & 11 & 0 \\ -15 & 5 & 8 & 1 \end{pmatrix}.$$

In particular, the above A and T are the inverse matrices of the matrices A and T in the lists of [1], respectively.

Let $\langle\ ,\ \rangle$ be the anti-symmetric bilinear form on $H^3(W_b, \mathbb{Z})$ defined by the cup product. The global monodromy Γ is $\mathrm{Im}(\pi_1(\mathbb{P}^1 - \{0, 1, \infty\}) \to \mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$. When we take $\{\beta^1, \beta^2, \alpha_1, \alpha_2\}$ as the basis of $H^3(W_b, \mathbb{Z})$, $\mathrm{Aut}(H^3(W_b, \mathbb{Z}), \langle\ ,\ \rangle)$ is identified with $\mathrm{Sp}(4, \mathbb{Z})$, and Γ is the subgroup of $\mathrm{Sp}(4, \mathbb{Z})$ which is generated by A and T.

We can partially normalize A and T simultaneously as follows.

Lemma. *There exists* $P \in \mathrm{GL}(4, \mathbb{Q})$ *such that*

$$P^{-1}A^{-1}P = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -1 \\ 5 & 5 & 5 & -4 \end{pmatrix}, \quad P^{-1}T^{-1}P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof. We take $P = \begin{pmatrix} 5 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 10 & -5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. The assertion follows. □

4. Main result

Let $\Gamma' = \{P^{-1}XP \in \mathrm{GL}(4, \mathbb{Z}) \mid X \in \Gamma\}$, and let $\rho\colon \mathrm{GL}(4, \mathbb{Z}) \to \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z})$ be the natural projection. Define $\tilde{\Gamma} = \rho(\Gamma')$. We will study $\tilde{\Gamma}$.

Let $\tilde{A} = \rho(P^{-1}A^{-1}P)$, $\tilde{T} = \rho(P^{-1}T^{-1}P) \in \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z})$. By a simple calculation, we obtain

$$\tilde{A}^n = \begin{pmatrix} 1 & n & 3n(n+4) & n(n+1)(4n+1) \\ 0 & 1 & n & 2n(n+1) \\ 0 & 0 & 1 & 4n \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z}).$$

Let $\hat{\Gamma}$ be

$$\left\{ \begin{pmatrix} 1 & n & 3n^2+2n & a \\ 0 & 1 & n & b \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z}) \;\middle|\; n, a, b, c \in \mathbb{Z}/5\mathbb{Z} \right\}.$$

$\hat{\Gamma}$ is a subgroup of $\mathrm{GL}(4, \mathbb{Z}/5\mathbb{Z})$ which contains $\tilde{A}$ and $\tilde{T}$. The following Theorem and Corollary are the main results of this paper.

Theorem. $\tilde{\Gamma} = \hat{\Gamma}$.

Proof. $\tilde{\Gamma} \subset \hat{\Gamma}$ follows from what we just mentioned. So we shall prove the converse inclusion.

From the presentations of elements of $\hat{\Gamma}$, we see that $\hat{\Gamma}$ is generated by

$$\tilde{A},\, \tilde{T},\, E_1 = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Therefore, it is enough to show E_1 and E_2 belong to $\tilde{\Gamma}$. In fact, we have

$$E_2 = \tilde{A}\tilde{T}\tilde{A}^4\tilde{T}^4, \quad E_1 = (E_2^2\tilde{A}^2\tilde{T}^4\tilde{A}^3\tilde{T})^4.$$

Hence $E_1, E_2 \in \tilde{\Gamma}$. □

Corollary. *Let $X \in \Gamma$. Then the characteristic polynomial of X is*

$$x^4 + (5m+1)x^3 + (5n+1)x^2 + (5m+1)x + 1,$$

where m, n are some integers. In particular, if X is not the unit matrix and the order of X is finite, then the order of X is 5 *and the eigenvalues of X are* $\exp(2\pi i/5)$, $\exp(4\pi i/5)$, $\exp(6\pi i/5)$, $\exp(8\pi i/5)$.

Proof. We shall prove the first part. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be the eigenvalues of X. Then the the characteristic polynomial $p(X)$ of X is

$$x^4 - \left(\sum_{1\le i\le j\le k\le 4} \lambda_i\lambda_j\lambda_k\right)x^3 + \left(\sum_{1\le i\le j\le 4} \lambda_i\lambda_j\right)x^2 - \left(\sum_{1\le i\le 4} \lambda_i\right)x + 1.$$

On the other hand, the the characteristic polynomial $p(X^{-1})$ of X^{-1} is

$$x^4 - \left(\sum_{1\le i\le j\le k\le 4} \frac{1}{\lambda_i}\frac{1}{\lambda_j}\frac{1}{\lambda_k}\right)x^3 + \left(\sum_{1\le i\le j\le 4} \frac{1}{\lambda_i}\frac{1}{\lambda_j}\right)x^2 - \left(\sum_{1\le i\le 4} \frac{1}{\lambda_i}\right)x + 1$$
$$= x^4 - \left(\sum_{1\le i\le 4} \lambda_i\right)x^3 + \left(\sum_{1\le i\le j\le 4} \lambda_i\lambda_j\right)x^2 - \left(\sum_{1\le i\le j\le k\le 4} \lambda_i\lambda_j\lambda_k\right)x + 1.$$

Since $X \in \mathrm{Sp}(4, \mathbb{Z})$, $p(X) = p(X^{-1})$. So $p(X)$ is the form $x^4 + ax^3 + bx^2 + ax + 1$, where $a, b \in \mathbb{Z}$. It follows from the theorem that $a \equiv -4$, $b \equiv 6 \bmod 5$. Hence the claim of the first part follows.

Next we shall prove the latter part. Let λ be an eigenvalue of X. It follows from $p(X) = p(\bar{X})$ and $p(X) = p(X^{-1})$ that $\bar{\lambda}$, $1/\lambda$, $1/\bar{\lambda}$ are also eigenvalues of X. Since the determinant of X is 1, if 1 or -1 is an eigenvalue of X, its multiplicity is even. Since the order of X is finite, we can express eigenvalues of X by $\exp(i\theta_1)$, $\exp(-i\theta_1)$, $\exp(i\theta_2)$, $\exp(-i\theta_2)$ $(0 \le \theta_1,\ \theta_2 \le \pi)$. Then the characteristic polynomial of X is

$$x^4 - 2(\cos\theta_1 + \cos\theta_2)x^3 + 2(\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2) + 1)x^2 - 2(\cos\theta_1 + \cos\theta_2)x + 1.$$

By the claim of the first part of this Corollary, we have

$$-2(\cos\theta_1 + \cos\theta_2) = 5m+1, \quad 2(\cos(\theta_1+\theta_2) + \cos(\theta_1-\theta_2) + 1) = 5n+1, \quad m, n \in \mathbb{Z}.$$

By the addition theorem for cosines, we have

$$2(\cos\theta_1 + \cos\theta_2) = -5m - 1, \quad 4\cos\theta_1\cos\theta_2 = 5n - 1.$$

It follows from $-4 \le 2(\cos\theta_1 + \cos\theta_2) \le 4$ that $m = 0$ or -1. If $m = -1$, then $\cos\theta_1$, $\cos\theta_2 = 1$ and all eigenvalues of X are 1. Since the order of X is finite, X is the unit matrix. This contradicts the assumption that X is not the unit matrix. Hence $m = 0$ and

$$\cos\theta_1 + \cos\theta_2 = -\frac{1}{2}.$$

It follows from $-4 \le 4\cos\theta_1\cos\theta_2 \le 4$ that $n = 0$ or 1. If $n = 1$, then $\cos\theta_1 = \pm 1$, $\cos\theta_2 = \pm 1$. This contradicts the fact that $\cos\theta_1 + \cos\theta_2 = -1/2$. Hence $n = 0$ and

$$\cos\theta_1 \cos\theta_2 = -\frac{1}{4}.$$

Combining these two equations, we have

$$\cos^2\theta_1 + \frac{1}{2}\cos\theta_1 - \frac{1}{4} = 0.$$

When we solve this equation for $\cos\theta_1$,

$$\cos\theta_1 = \frac{-1\pm\sqrt{5}}{4},\quad \sin\theta_1 = \frac{\sqrt{10\pm 2\sqrt{5}}}{4},$$
$$\cos\theta_2 = \frac{-1\mp\sqrt{5}}{4},\quad \sin\theta_2 = \frac{\sqrt{10\mp 2\sqrt{5}}}{4}.$$

Then we can verify easily that $(\exp(i\theta_1))^5$ and $(\exp(i\theta_2))^5 = 1$. Hence $(\theta_1, \theta_2) = (2\pi/5, 4\pi/5)$ or $(4\pi/5, 2\pi/5)$. $\square$

5. A relation to the other result

In this section, we shall compare the main result of this paper with the result of Chen, Yang and Yui. In [2], they find the congruence subgroup $\Gamma(5,5)$ which contains the global monodromy Γ. Combining their result and our theorem, we can find a smaller group which contains Γ.

The congruence subgroup $\Gamma(5,5)$ is defined by

$$\Gamma(5,5) = \left\{ X \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & * & 1 \end{pmatrix} \pmod 5 \right\}.$$

$\Gamma(5,5)$ contains the principal congruence group $\Gamma(5) = \mathrm{Ker}(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/5\mathbb{Z}))$ as a normal subgroup of finite index.

Let $X \in \Gamma(5,5)$ and express X by

$$\begin{pmatrix} 5x_{11}+1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22}+1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33}+1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44}+1 \end{pmatrix},\quad x_{ij} \in \mathbb{Z} \quad (1 \le i, j \le 4).$$

Then we have

$$\mathrm{GL}(4,\mathbb{Z}) \ni P^{-1}XP \equiv \begin{pmatrix} 1 & -9x_{31} & -x_{12}+3x_{32} & -x_{14}+3x_{34} \\ 0 & 1 & -2x_{12} & -2x_{14} \\ 0 & 0 & 1 & x_{24} \\ 0 & 0 & 0 & 1 \end{pmatrix} \pmod 5.$$

By the main theorem, if $X \in \Gamma$, then $\rho(P^{-1}XP) \in \tilde{\Gamma}$ and

$$-9x_{31} \equiv n,\quad -2x_{12} \equiv n,\quad -x_{12}+3x_{32} \equiv 3n^2+2n \pmod 5.$$

where n is some integer. From a simple calculation, the above equation is equivalent to

$$x_{31} \equiv 3x_{12}, \quad x_{32} \equiv 4x_{12}^2 + 4x_{12} \quad (\mathrm{mod}\ 5).$$

So we define

$$\tilde{\Gamma}(5,5) = \left\{ \begin{pmatrix} 5x_{11}+1 & x_{12} & x_{13} & x_{14} \\ 5x_{21} & 5x_{22}+1 & x_{23} & x_{24} \\ 5x_{31} & 5x_{32} & 5x_{33}+1 & 5x_{34} \\ 5x_{41} & 5x_{42} & x_{43} & 5x_{44}+1 \end{pmatrix} \in \mathrm{Sp}(4,\mathbb{Z}) \;\middle|\; \begin{array}{l} x_{31} \equiv 3x_{12}, \\ x_{32} \equiv 4x_{12}^2+4x_{12} \\ (\mathrm{mod}\ 5) \end{array} \right\}.$$

Then we have the following Corollary.

Corollary. (i) $\tilde{\Gamma}(5,5)$ *is a subgroup of* $\Gamma(5,5)$.
(ii) $\Gamma \subset \tilde{\Gamma}(5,5) \subsetneq \Gamma(5,5)$.
(iii) $\tilde{\Gamma}(5,5)$ *is a congruence subgroup of* $\mathrm{Sp}(4,\mathbb{Z})$ *of finite index.*

Proof. Let $\rho' \colon \Gamma(5,5) \to \mathrm{GL}(4,\mathbb{Z})$, $X \mapsto P^{-1}XP$ and let $\pi = \rho \circ \rho' \colon \Gamma(5,5) \to \mathrm{GL}(4,\mathbb{Z}/5\mathbb{Z})$. $\tilde{\Gamma}(5,5) = \pi^{-1}(\tilde{\Gamma})$ follows from what we just mentioned. Since π is a group homomorphism, $\pi^{-1}(\tilde{\Gamma})$ is a subgroup of $\Gamma(5,5)$. Hence the claim of (i) follows.

We can verify easily that A and T belong to $\tilde{\Gamma}(5,5)$. Therefore $\tilde{\Gamma}(5,5)$ contains Γ. We shall show that $\tilde{\Gamma}(5,5)$ is a proper subgroup of $\Gamma(5,5)$. We take $X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Then $X \in \Gamma(5) \subset \Gamma(5,5)$ and $X \notin \tilde{\Gamma}(5,5)$. Hence the claim of (ii) follows.

Finally, we shall show the claim of (iii). $\tilde{\Gamma}(5,5)$ contains the principal congruence subgroup $\Gamma(25) = \mathrm{Ker}(\mathrm{Sp}(4,\mathbb{Z}) \to \mathrm{Sp}(4,\mathbb{Z}/25\mathbb{Z}))$ as a normal subgroup. Hence we obtain $|\tilde{\Gamma}(5,5) : \mathrm{Sp}(4,\mathbb{Z})| < |\Gamma(25) : \mathrm{Sp}(4,\mathbb{Z})| = |\mathrm{Sp}(4,\mathbb{Z}/25\mathbb{Z}))| < \infty$. $\square$

QUESTION. There are other 13 mirror families of Calabi–Yau threefolds with $h^{2,1} = 1$ as discussed in [2]. Is it possible to find smaller subgroups in those 13 cases as well?

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