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ON THE DISTRIBUTION OF αp MODULO ONE FOR PRIMES p OF A SPECIAL FORM

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Abstract

In this paper it is proved that for any irrational α and some $0 < \theta \leq 1.5/100$, there are infinitely many primes p such that $p+2$ has at most two prime factors and $\|\alpha p + \beta\| < p^{-\theta}$ which improves K. Matomäki's result $\theta < 1/1000$.

1. Introduction

Let α be a irrational real number and $\|x\|$ denote the distance from x to nearest integers. Earlier work about the distribution of the fractional parts of the sequence αp was considered by I.M. Vinogradov [16] who showed that for any real number β, there are infinitely many primes p such that if $\theta = 1/5 - \varepsilon$, then

$$\|\alpha p + \beta\| < p^{-\theta}, \tag{1}$$

where and below $\varepsilon > 0$ is arbitrarily small. Later the exponent θ was improved by several authors (Harman [3, 4], Jia [8, 9], Heath-Brown and Jia [5]). So far the best result is given by Matomäki with $\theta = 1/3 - \varepsilon$.

Let P_r denote an almost prime with at most r prime factors, counted according to multiplicity. The famous prime twins conjecture states that there exist infinitely many primes p such that $p+2$ is a prime too. Up to now this conjecture is still open, but many approximation to it established. One of the most interesting of them is due to J.R. Chen [2], who showed in 1973 that there exist infinitely many primes p such that $p+2 = P_2$.

In [14] Todorova and Tolev considered the distribution of αp modulo one with primes of the form specified above, and showed that for $\theta = 1/100$, there are infinitely many solutions in primes p to (1) such that $p+2 = P_4$. Later Matomäki [11] has shown that this actually holds with $p+2 = P_2$ and $\theta = 1/1000$.

In this paper, our purpose is to improve the range θ and we shall prove the following result.

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Theorem 1.1. *Let $\alpha \in \mathcal{R} \setminus \mathcal{Q}$, $\beta \in \mathcal{R}$ and $0 < \theta \le 1.5/100$. Then there are infinitely many primes p satisfying $p + 2 = P_2$ and such that*

$$\|\alpha p + \beta\| < p^{-\theta}. \tag{2}$$

NOTATION. Let α be a real number with a rational approximation a/q satisfying

$$\left|\alpha - \frac{a}{q}\right| < \frac{1}{q^2}, \quad \text{where} \quad (a, q) = 1, \quad \text{and} \quad q \ge 1.$$

Here $K \ge 1$, $k \sim H$ means $H < k \le 2H$ and $0 < \theta \le 1.5/100$. As usual let $\Lambda(n)$ and $\phi(n)$ respectively denote Von Mangoldt's function and Euler's function. For simplicity instead of $m \equiv n \pmod{k}$, $e^{2\pi i x}$ we write $m \equiv n(k)$, $e(x)$ respectively. Letter C is a positive constant, which is not necessarily the same at each occurrence.

2. Some lemmas

In order to prove Theorem 1.1, we need the following lemmas.

Lemma 2.1 ([11, Theorem 1]). *For any well-factorable function λ of level D, we have*

$$\begin{aligned} &\sum_{\substack{d \sim D \\ (d,c)=1}} \lambda_d \sum_{k \sim H} c_k \sum_{\substack{n \sim x \\ n \equiv c(d)}} \Lambda(n) e(\alpha n k) \\ &\ll H (\log x)^C x^{3/4+\varepsilon} \left(\frac{x}{q} + \frac{q}{H} + D^2 + x^{7/9+4\varepsilon} + \min\left\{ D^{4+20\varepsilon}, \frac{x}{D} \right\} \right)^{1/4-\varepsilon}. \end{aligned} \tag{3}$$

Lemma 2.2 ([10, 13]). *Let $x > 1$, $z = x^{1/u}$. Then for $u \ge 1$, we have*

$$\sum_{\substack{n \le x \\ (n, P(z))=1}} 1 = w(u) \frac{x}{\log z} + O\left(\frac{x}{\log^2 z} \right),$$

where $w(u)$ is determined by the following differential-difference equation

$$\begin{cases} w(u) = \dfrac{1}{u}, & \text{if} \quad 1 < u \le 2, \\ (u w(u))' = w(u-1), & \text{if} \quad u \ge 2. \end{cases}$$

Lemma 2.3 ([13]). *For any given constant $A > 10$, there exists a constant $B = B(A) > 0$ such that*

$$\sum_{d \le D} \max_{(l,d)=1} \max_{y \le x} \left| \sum_{\substack{k \le E(x) \\ (k,d)=1}} g(x, k) H(y; k, d, l) \right| \ll \frac{x}{\log^A x},$$

where

$$H(y; k, d, l) = \sum_{\substack{kp\le y\\ kp\equiv l(d)}} 1 - \frac{1}{\phi(d)} \sum_{kp\le y} 1,$$

$$\frac{1}{2} \le E(x) \ll x^{1-\vartheta}, \quad 0 < \vartheta \le 1,$$

$$g(x, k) \ll d_r(k), \quad D = x^{1/2} \log^{-B} x.$$

Lemma 2.4 ([13]). *Let the condition of* Lemma 2.3 *be given and* $r_1(y)$ *be a positive function depending on* x *and satisfying* $r_1(y) \ll x^{\vartheta}$ *for* $y \le x$. *Then we have*

$$\sum_{d\le D} \max_{(l,d)=1} \max_{y\le x} \left| \sum_{\substack{k\le E(x)\\ (k,d)=1}} g(x, k) H(kr_1(y); k, d, l) \right| \ll \frac{x}{\log^A x}.$$

Lemma 2.5 ([13]). *Let the condition of* Lemma 2.3 *be given and* $r_2(y)$ *be a positive function depending on* x, y *and satisfying* $kr_2(y) \ll x$ *for* $k \le E(x)$, $y \le x$. *Then we have*

$$\sum_{d\le D} \max_{(l,d)=1} \max_{y\le x} \left| \sum_{\substack{k\le E(x)\\ (k,d)=1}} g(x, k) H(kr_2(y); k, d, l) \right| \ll \frac{x}{\log^A x}.$$

3. Proof of Theorem 1.1

As in [14] we begin with a periodic function $\chi(t)$ with period 1 such that

$$\chi(t) \begin{cases} \in (0, 1) & \text{if} \quad -\Delta < t < \Delta, \\ = 0 & \text{if} \quad \Delta \le t \le 1 - \Delta, \end{cases}$$

and which has a Fourier series

$$\chi(t) = \Delta + \sum_{|k|>0} g(k) e(kt) \tag{4}$$

with coefficients satisfying

$$\begin{aligned} &g(0) = \Delta, \\ &g(k) \ll \Delta, \quad \text{for all} \quad k, \\ &\sum_{|k|>H} |g(k)| \ll N^{-1}, \end{aligned} \tag{5}$$

where

$$\Delta = \Delta(N) = N^{-\theta} \quad \text{and} \quad H = \Delta^{-1} \log^2 N. \tag{6}$$

Next we will use sieve methods. As usual, for any sequence $\mathcal{E}$ of integers weighted by the numbers f_n, $n \in \mathcal{E}$, we set

$$S(\mathcal{E}, z) = \sum_{\substack{n \in \mathcal{E} \\ (n, P(z))=1}} f_n,$$

and denote by $\mathcal{E}_d$ be the subsequence of elements $n \in \mathcal{E}$ with $n \equiv 0 \pmod d$. We write

$$P(z) = \prod_{p<z} p$$

and

$$V(z) = \prod_{p \mid P(z)} \left(1 - \frac{\omega(p)}{p}\right).$$

Let further

$$C_0 = \prod_{p>2} \left(1 - \frac{1}{(p-1)^2}\right),$$

and we will use the following form of the linear sieve due to Iwaniec [6].

Lemma 3.1. *Let $2 \le z \le D^{1/2}$ and let $s = \log D/\log z$. If*
(A$_1$) $|\mathcal{E}_d| = (\omega(d)/d)X + r(\mathcal{E}, d)$, $\mu(d) \neq 0$;
(A$_2$) $\sum_{z_1 \le p < z_2} \omega(p)/p = \log(\log z_2/\log z_1) + O(1/\log z_1)$, $z_2 > z_1 \ge 2$,
where $\omega(d)$ is a multiplicative function, $0 \le \omega(p) < p$, $X > 1$ is independent of d. Then

$$S(\mathcal{E}, z) \le XV(z)(F(s) + o(1)) + \sum_{l<L} \sum_{d \mid P(z)} \lambda_l^+(d) r(\mathcal{E}, d),$$
$$S(\mathcal{E}, z) \ge XV(z)(f(s) - o(1)) - \sum_{l<L} \sum_{d \mid P(z)} \lambda_l^+(d) r(\mathcal{E}, d),$$

where $L = O(1)$, $\lambda^\pm$ are well-factorable bounded functions of level D, $f(s)$, $F(s)$ are determined by the following differential-difference equation

$$\begin{cases} F(s) = \dfrac{2e^\gamma}{s}, \quad f(s) = 0, & \textit{if} \quad 0 < s \le 2, \\ (sF(s))' = f(s-1), \quad (sf(s))' = F(s-1), & \textit{if} \quad s \ge 2, \end{cases}$$

where γ denote the Euler's constant.

So, if we define $\mathcal{A}$ to be the sequence of integers $n \le N$ weighted by

$$a_n = \begin{cases} \chi(\alpha(n-2)+\beta) & \text{if} \quad n-2 \in \mathbb{P}, \\ 0 & \text{else.} \end{cases}$$

Then to prove Theorem 1.1, it suffice to show that

$$S(\mathcal{A}, N^{1/3}) = \sum_{\substack{p+2 \le N \\ (p+2, P(N^{1/3}))=1}} \chi(\alpha p + \beta) > 0. \tag{7}$$

However, we cannot quite do that, but need to use a more sophisticated weighted sieve method. Indeed following Cai (see [1], Lemma 5), let $k = 1/12$, $l = 1/3.1$, and we consider

$$S \ge \sum_{\substack{n \in \mathcal{A} \\ (n, N^{1/12})=1}} a_n \left(1 - \frac{1}{2} \sum_{\substack{N^{1/12} \le p < N^{1/3.1} \\ p \mid n}} 1 - \frac{1}{2} \sum_{\substack{n = p_1 p_2 p_3 \\ N^{1/12} \le p_1 < N^{1/3.1} \\ N^{1/3.1} \le p_2 < (N/p_1)^{1/2}}} 1 - \sum_{\substack{n = p_1 p_2 p_3 \\ N^{1/3.1} \le p_1 < p_2 < (N/p_1)^{1/2}}} 1 \right)$$
$$+ O(N^{11/12}).$$

Here we notice that the weight of n is a_n if and only if n has no prime factors $< N^{1/3.1}$ in which case clearly $n = P_2$. If the weight of n is $a_n/2$, then a_n has one prime factor in the interval $[N^{1/12}, N^{1/3.1})$ and the third, fourth sum is 0. But this again implies that $n = P_2$. Thus the weight of n is positive only if

$$n = P_2, \quad n-2 \in \mathbb{P} \quad \text{and} \quad \|\alpha(n-2)+\beta\| < N^{-\theta},$$

and so it is enough to show that $S > 0$.

Using the sieve notation, we can write

$$\begin{aligned} S &\ge S(\mathcal{A}, N^{1/12}) - \frac{1}{2} \sum_{N^{1/12} \le p < N^{1/3.1}} S(\mathcal{A}_p, N^{1/12}) - \frac{1}{2} \sum_{\substack{N^{1/12} \le p_1 < N^{1/3.1} \\ N^{1/3.1} \le p_2 < (N/p_1)^{1/2}}} S(\mathcal{A}_{p_1 p_2}, p_2) \\ &\quad - \sum_{N^{1/3.1} \le p_1 < p_2 < (N/p_1)^{1/2}} S(\mathcal{A}_{p_1 p_2}, p_2) + O(N^{11/12}) \\ &=: S_1 - \frac{1}{2} S_2 - \frac{1}{2} S_3 - S_4 + O(N^{11/12}). \end{aligned} \tag{8}$$

Consider a square-free number d. If $2 \mid d$, then we write $|\mathcal{A}_d| = |r(\mathcal{A}, d)| \le 1$. Otherwise we have by the Fourier expansion of $\chi(n)$

$$\begin{aligned} |\mathcal{A}_d| &= \sum_{\substack{md \le N \\ md-2 \in \mathbb{P}}} \chi(\alpha(md-2)+\beta) \\ &= \sum_{\substack{p \le N-2 \\ p \equiv -2(d)}} \chi(\alpha p + \beta) \\ &= \sum_{\substack{p \le N \\ p \equiv -2(d)}} \left(\Delta + \Delta \sum_{0<|k|<H} c_k e(\alpha k p) + O(N^{-1}) \right) \\ &= \Delta \left(\frac{\mathrm{Li}\, N}{\phi(d)} + R_1(d) + R_2(d) + O\left(\frac{N}{d(\log N)^C} \right) \right), \end{aligned}$$

where $c_k \ll 1$, and

$$\begin{aligned} R_1(d) &= \sum_{\substack{p \le N \\ p \equiv -2(d)}} 1 - \frac{\mathrm{Li}\, N}{\phi(d)}, \\ R_2(d) &= \sum_{\substack{p \le N \\ p \equiv -2(d)}} \sum_{0<|k|<H} c_k e(\alpha k p). \end{aligned}$$

Applying Bombieri–Vinogradov theorem (see [7], Theorem 17.1) implies that

$$\sum_{d \le N^{1/2}/\log^C N} |R_1(d)| \ll \frac{N}{\log^A N}.$$

On the other hand, Lemma 2.1 implies that for a well-factorable function λ of level $D < N^{1/2}/(H^2 \log^C N)$, we get

$$\sum_{d \le D} \lambda_d R_2(d) \ll \frac{N}{\log^A N},$$

when $N = q^2$, where a/q is a convergent to α with a large enough denominator.

Therefore we apply Lemma 3.1 with

$$\omega(d) = \begin{cases} 0 & \text{if } \ 2 \mid d, \\ \dfrac{d}{\phi(d)} & \text{otherwise}, \end{cases} \quad X = \Delta\, \mathrm{Li}\, N, \quad \text{and} \quad D < \frac{N^{1/2}}{H^2 \log^C N},$$

to S_1 and obtain

$$\begin{aligned} S_1 &\geq \Delta \operatorname{Li} N V(N^{1/12}) f(6-24\theta)(1+o(1)) \\ &= \frac{8}{1-4\theta}\left(\log(5-24\theta) + \int_3^{5-24\theta} \frac{1}{t}\,dt \int_2^{t-1} \frac{\log(s-1)}{s}\,ds\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \qquad (9) \\ &\geq 13.471 \frac{C_0 \Delta N}{\log^2 N}. \end{aligned}$$

Since $(\mathcal{A}_p)_d = \mathcal{A}_{pd}$, we can use Lemma 3.1 also to S_2 by using the same method. In this case one faces the sum

$$\sum_{N^{1/12} \leq p < N^{1/3.1}} \sum_{d \leq D} \lambda_d R_2(pd),$$

by Remark 10 in [11], the above sum is at most

$$\ll \frac{N}{\log^A N},$$

then

$$\omega(d) = \begin{cases} 0 & \text{if} \quad 2 \mid d, \\ \dfrac{d}{\phi(d)} & \text{otherwise,} \end{cases} \qquad X = \frac{\Delta \operatorname{Li} N}{\phi(p)}, \quad \text{and} \quad D < \frac{N^{1/2}}{pH^2 \log^C N}.$$

And applying partial summation, prime number theory, we have

$$\begin{aligned} S_2 &\leq \sum_{N^{1/12} \leq p < N^{1/3.1}} \frac{\Delta \operatorname{Li} N}{\phi(p)} V(N^{1/12}) F\left(6 - 24\theta - 12\frac{\log p}{\log N}\right)(1+o(1)) \\ &= 6\int_{1/12}^{1/3.1} \frac{F(6-24\theta-12t)}{t}\,dt\, \frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\ &= 8\left(\int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1-4\theta-2t)}\left(1 + \int_2^{5-24\theta-12t} \frac{\log(s-1)}{s}\,ds\right)\right. \\ &\qquad \left. + \int_{(1-8\theta)/4}^{1/3.1} \frac{1}{t(1-4\theta-2t)}\,dt\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \qquad (10) \\ &= 8\left(\int_{1/12}^{(1-8\theta)/4} \frac{dt}{t(1-4\theta-2t)} \int_2^{5-24\theta-12t} \frac{\log(s-1)}{s}\,ds\right. \\ &\qquad \left. + \int_{1/12}^{1/3.1} \frac{1}{t(1-4\theta-2t)}\,dt\right)\frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\ &\leq 21.3643 \frac{C_0 \Delta N}{\log^2 N}. \end{aligned}$$

For the sum S_3, we write

$$\begin{aligned} S_3 &= \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} \sum_{\substack{np_1p_2\le N\\ np_1p_2-2\in\mathbb{P},\ (n,P(p_2))=1}} \chi(\alpha(np_1p_2-2)+\beta) \\ &= \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} \sum_{\substack{p=np_1p_2-2\\ 1\le n\le N/(p_1p_2),\ (n,P(p_2))=1}} \chi(\alpha p+\beta) \\ &\le \sum_{N^{1/3.1}\le p_2<N^{11/24}} \sum_{\substack{1\le n\le N^{11/12}/p_2\\ (n,P(p_2))=1}} \sum_{\substack{p=np_1p_2-2\\ N^{1/12}\le p_1<\min(N^{1/3.1},N/(np_2))}} 1. \end{aligned}$$

Let's consider the set

$$\mathcal{E}=\left\{e\ \middle|\ e=np_2,\ N^{1/3.1}\le p_2<N^{11/24},\ 1\le n\le \frac{N^{11/12}}{p_2},\ (n,\ P(p_2))=1\right\}.$$

By the definition of the set $\mathcal{E}$, it is easy to see that for every $e\in\mathcal{E}$, p_2 is determined by e uniquely. Let $p_2=r(e)$, then we have

$$N^{1/3.1}\le r(e)<N^{11/24}\quad\text{and}\quad er(e)<N.$$

Let

$$\mathcal{L}=\left\{l\ \middle|\ l=ep_1-2,\ e\in\mathcal{E},\ N^{1/12}\le p_1<\min\left(N^{1/3.1},\frac{N}{np_2}\right)\right\}.$$

Then

$$N^{1/3.1}<e<N^{11/12}\quad\text{for}\quad e\in\mathcal{E}$$

and

$$|\mathcal{E}|\le N^{11/12},\qquad \sum_{l\in\mathcal{L},\ l\le N^{1/3.1}} 1\ll N^{11/12},$$

and also we have

$$S_3\le S(\mathcal{L},z)+O(N^{11/12})\quad\text{for}\quad z\le N^{1/3}. \tag{11}$$

We write

$$z^2=D=N^{1/2}\log^{-B}N,$$

then

$$S(\mathcal{L},z)\le 8\frac{C_0|\mathcal{L}|}{\log N}+R_3+R_4, \tag{12}$$

where

$$R_3 = \sum_{\substack{d\le D\\(d,N)=1}} \left| \sum_{\substack{e\in\mathcal{E}\\(e,d)=1}} \left(\sum_{\substack{N^{1/12}\le p_1<\min(N^{1/3.1},N/e)\\ ep_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/e)} 1 \right) \right|,$$

$$R_4 = \sum_{d\le D,\,(d,N)=1} \frac{1}{\phi(d)} \sum_{\substack{e\in\mathcal{E}\\(e,d)>1}} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},\,N/e)} 1.$$

Let

$$Q(k) = \sum_{e=k,\,e\in\mathcal{E}} 1,$$

then

$$R_3 = \sum_{d\le D} \left| \sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)=1}} Q(k) \left(\sum_{\substack{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)\\ kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)} 1 \right) \right|,$$

$$R_4 = \sum_{d\le D} \frac{1}{\phi(d)} \sum_{\substack{N^{1/3}<k<N^{11/12}\\(k,d)>N^{1/3.1}}} \sum_{N^{1/12}\le p_1<\min(N^{1/3.1},N/k)} 1.$$

It is easy to show

$$Q(k) \le 1.$$

Then we have

$$\begin{aligned} R_4 &\ll \sum_{d\le D} \frac{1}{\phi(d)} \sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)>N^{1/3.1}}} \frac{N}{k} \\ &\ll N \sum_{d\le D} \frac{1}{\phi(d)} \sum_{h\mid d,\,h\ge N^{1/3.1}} \sum_{\substack{k<N^{11/12}\\(k,d)=h}} \frac{1}{k} \\ &\ll N\log N \sum_{d\le D} \frac{1}{\phi(d)} \sum_{h\mid d,\,h\ge N^{1/3.1}} \frac{1}{h} \\ &\ll N\log N \sum_{N^{1/3.1}\le h\le D} \frac{1}{h\phi(h)} \sum_{d\le D/h} \frac{1}{\phi(d)} \\ &\ll N^{2.1/3.1}\log^2 N, \end{aligned} \tag{13}$$

and

$$R_3 \le R_5 + R_6 + R_7, \tag{14}$$

where

$$R_5 = \sum_{d\le D,\,(d,N)=1}\left|\sum_{\substack{N^{1/3.1}<k<N^{2.1/3.1}\\(k,d)=1}} Q(k)\left(\sum_{\substack{p_1<N^{1/3.1}\\kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)}\sum_{p_1<N^{1/3.1}} 1\right)\right|,$$

$$R_6 = \sum_{d\le D,\,(d,N)=1}\left|\sum_{\substack{N^{2.1/3.1}<k<N^{11/12}\\(k,d)=1}} Q(k)\left(\sum_{\substack{kp_1<N\\kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)}\sum_{kp_1<N} 1\right)\right|,$$

$$R_7 = \sum_{d\le D,\,(d,N)=1}\left|\sum_{\substack{N^{1/3.1}<k<N^{11/12}\\(k,d)=1}} Q(k)\left(\sum_{\substack{p_1<N^{1/12}\\kp_1\equiv 2(d)}} 1 - \frac{1}{\phi(d)}\sum_{p_1<N^{1/12}} 1\right)\right|.$$

Due to Lemma 2.3–2.5,

$$R_j \ll \frac{N}{\log^4 N}, \quad j = 5, 6, 7. \tag{15}$$

By Lemma 2.2 and prime number theorem, we have

$$\begin{aligned}
|\mathcal{L}| &= \sum_{e\in\mathcal{E}} \sum_{N^{1/12}\le p_1<N^{1/3.1}} 1 \\
&= \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} \sum_{\substack{1\le n\le N/(p_1p_2)\\(n,P(p_2))=1}} 1 + O(N^{11/12}) \\
&< (1+o(1)) \sum_{N^{1/12}\le p_1<N^{1/3.1}\le p_2<(N/p_1)^{1/2}} w\left(\frac{\log(N/(p_1p_2))}{\log p_2}\right)\frac{N}{p_1p_2\log p_2} \\
&\quad + O(N^{11/12}) \\
&\le \left(\int_{1/12}^{1/3.1}\frac{dt}{t}\int_{1/3.1}^{(1-t)/2}\frac{ds}{s(1-t-s)}\right)\frac{N}{\log N}.
\end{aligned} \tag{16}$$

By (11)–(16), we obtain

$$\begin{aligned}
S_3 &\le 8\left(\int_{1/12}^{1/3.1}\frac{dt}{t}\int_{1/3.1}^{(1-t)/2}\frac{ds}{s(1-t-s)}\right)\frac{C_0\Delta N}{\log^2 N}(1+o(1)) \\
&\le 5.52946\frac{C_0\Delta N}{\log^2 N}.
\end{aligned}$$

We also use the same idea to S_4,

$$\begin{aligned} S_4 &\le 8\left(\int_{1/3.1}^{1/3} \frac{dt}{t} \int_t^{(1-t)/2} \frac{ds}{s(1-t-s)}\right) \frac{C_0 \Delta N}{\log^2 N}(1+o(1)) \\ &\le 0.018745 \frac{C_0 \Delta N}{\log^2 N}. \end{aligned} \tag{18}$$

Combining (7)–(10), (17) and (18), then we obtain

$$S > S_1 - \frac{1}{2}S_2 - \frac{1}{2}S_3 - S_4 \gg \frac{\Delta N}{\log^2 N},$$

which concludes the proof of Theorem 1.1.

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References

[1] Y. Cai: *On Chen's theorem*, II, J. Number Theory **128** (2008), 1336–1357.

[2] J.R. Chen: *On the representation of a larger even integer as the sum of a prime and the product of at most two primes*, Sci. Sinica **16** (1973), 157–176.

[3] G. Harman: *On the distribution of αp modulo one*, J. London Math. Soc. (2) **27** (1983), 9–18.

[4] G. Harman: *On the distribution of αp modulo one*, II, Proc. London Math. Soc. (3) **72** (1996), 241–260.

[5] D.R. Heath-Brown and C. Jia: *The distribution of αp modulo one*, Proc. London Math. Soc. (3) **84** (2002), 79–104.

[6] H. Iwaniec: *A new form of the error term in the linear sieve*, Acta Arith. **37** (1980), 307–320.

[7] H. Iwaniec and E. Kowalski: Analytic Number Theory, American Mathematical Society Colloquium Publications **53**, Amer. Math. Soc., Providence, RI, 2004.

[8] C.H. Jia: *On the distribution of αp modulo one*, J. Number Theory **45** (1993), 241–253.

[9] C.H. Jia: *On the distribution of αp modulo one*, II, Sci. China Ser. A **43** (2000), 703–721.

[10] C.H. Jia: *Almost all short intervals containing prime numbers*, Acta Arith. **76** (1996), 21–84.

[11] K. Matomäki: *A Bombieri-Vinogradov type exponential sum result with applications*, J. Number Theory **129** (2009), 2214–2225.

[12] K. Matomäki: *The distribution of αp modulo one*, Math. Proc. Cambridge Philos. Soc. **147** (2009), 267–283.

[13] C.D. Pan and C.B. Pan: Goldbach Conjecture, Science Press, Beijing, China, 1981.

[14] T.L. Todorova and D.I. Tolev: *On the distribution of αp modulo one for primes p of a special form*, Math. Slovaca **60** (2010), 771–786.

[15] R.C. Vaughan: *On the distribution of αp modulo* 1, Mathematika **24** (1977), 135–141.

[16] I.M. Vinogradov: The Method of Trigonometrical Sums in the Theory of Numbers, translated from the Russian, revised and annotated by K.F. Roth and A. Davenport, reprint of the 1954 translation, Dover, Mineola, NY, 2004.

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