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A subdirect representation of a group

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A normal subgroup P of a group G is called prime if $[A, B]$[1] $\subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for two normal subgroups A and B of G. It is then easily verified that the following three conditions are equivalent to one another. (1) P is prime, (2) $[A, B] \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for two finitely generated normal subgroups A and B of G, and (3) $[N(x), N(y)] \subseteq P$ implies $N(x) \subseteq P$ or $N(y) \subseteq P$ for two normal subgroups $N(x)$ and $N(y)$ generated by x and y respectively. A group G is called prime if its unit subgroup is prime. Then G is prime if and only if its unit subgroup is a meet-irreducible radical group[2], and also if and only if G has a unique minimal normal non-abelian subgroup[3].

The aim of the present short note is to prove the following

THEOREM. *A group G is isomorphic to a subdirect product of a finite or infinite number of prime groups if and only if the normal subgroup $N(x)$ generated by x is not solvable for an arbitrary element x $(\neq e)$*[4] *of G.*

1. LEMMAS.

We shall now establish a few definitions and several preliminary lemmas for use of the proof of the "if part" of Theorem.

DEFINITION 1. A family $\mathfrak{F}$ of finitely generated normal subgroups of a group G is called a C-family of G, if it satisfies the following two conditions:

1) $A, B \in \mathfrak{F}$ implies $[A, B] \in \mathfrak{F}$, and
2) $[A, B] \in \mathfrak{F}$ implies $A, B \in \mathfrak{F}$.

LEMMA 1. *Let P be a prime normal subgroup of a group G. Then the set $\mathfrak{F}[P]$ of all finitely generated normal subgroups none of which is contained in P is a C-family of G.*

Proof. This is immediate.

LEMMA 2. *Let $\mathfrak{F}$ be a C-family of a group G, and let N be a normal subgroup of G such that no member of $\mathfrak{F}$ is contained in N. Then there exists a maximal normal subgroup P of G such that P contains N and no member in $\mathfrak{F}$ is contained in P. P is necessarily prime in G.*

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1) $[A, B]$ will denote the commutator subgroup of normal subgroups A and B of G.
2), 3) Cf. [4; p. 377].
4) The unit element of G will be denoted by e.

Proof. The existence of P follows at once from Zorn's lemma. We prove that P is prime. Suppose that A and B are two finitely generated normal subgroups such that both A and B are not contained in P. Then the maximality of P implies that AP contains a subgroup A_0 in $\mathfrak{F}$, and likewise BP contains a subgroup B_0 in $\mathfrak{F}$. Hence we have $[A_0, B_0]\subseteq[AP, BP]=[A, B][A, P][P, B][P, P]\subseteq[A, B]P$. If we suppose that $[A, B]$ is contained in P, then $[A_0, B_0]$ is contained in P. This is a contradiction, since $[A_0, B_0]$ is a member of $\mathfrak{F}$. This shows that $[A, B]$ is not contained in P, that is, P is prime in G.

LEMMA 3. *Let A be a normal subgroup of G, and $\mathfrak{F}$ a C-family such that no member in $\mathfrak{F}$ is contained in A. Then $\mathfrak{F}$ is contained in a maximal C-family, none of whose members is contained in A.*

Proof. This is immediate by Zorn's lemma.

DEFINITION 2. A prime normal subgroup P of a group G is called a minimal prime of a normal subgroup A if and only if $A\subseteq P$ and there exists no prime P_0 such as $A\subseteq P_0<P$[5].

LEMMA 4. *In order that P is a minimal prime of A, it is necessary and sufficient that $\mathfrak{F}[P]$ is a maximal C-family, none of whose members is contained in A.*

Proof. It is convenient to prove first the sufficient part. Suppose that $\mathfrak{F}[P]$ is maximal in the class $\boldsymbol{F}$ of C-families such that no subgroup in each family in $\boldsymbol{F}$ is contained in A. Take a maximal normal subgroup P^* with the properties such that P^* contains A and no subgroup in $\mathfrak{F}[P]$ is contained in P^*. Then by Lemma 2 P^* is prime. Now no subgroup in $\mathfrak{F}[P^*]$ is contained in A, since no subgroup in $\mathfrak{F}[P^*]$ is contained in P^*. Hence $\mathfrak{F}[P^*]$ is a member of $\boldsymbol{F}$. Clearly, $\mathfrak{F}[P]$ is contained in $\mathfrak{F}[P^*]$. Hence the maximal property of $\mathfrak{F}[P]$ implies that $\mathfrak{F}[P]=\mathfrak{F}[P^*]$, and hence $P=P^*$[6]. Now there can exist no prime normal subgroup P_0 such that $A\subseteq P_0<P$, since this would imply that $\mathfrak{F}[P_0]$ is a C-family which contains $\mathfrak{F}[P]$ strictly and no subgroup in $\mathfrak{F}[P_0]$ is contained in A. This is impossible because of the maximality of $\mathfrak{F}[P]$. Hence P is a minimal prime of A.

Next we prove the necessary part: If P is a minimal prime of A, then $\mathfrak{F}[P]$ is a member of $\boldsymbol{F}$, and $\mathfrak{F}[P]$ is contained in a maximal C-family $\mathfrak{F}^*$ in $\boldsymbol{F}$ (Lemma 3). Take a maximal normal subgroup P^* with the properties that P^* contains P and no subgroup in $\mathfrak{F}^*$ is contained in P^*. Then clearly P^* is a maximal normal subgroup with the properties that P^* contains A and no subgroup in $\mathfrak{F}^*$ is contained in P^*. Now since $\mathfrak{F}[P^*]$ contains $\mathfrak{F}^*$ and no subgroup in $\mathfrak{F}[P^*]$ is contained in A, $\mathfrak{F}[P^*]$ would coincide with $\mathfrak{F}^*$. Hence, by the sufficient part

5) By $P_0<P$ we mean that P_0 is strictly contained in P.
6) It is easily verified that $\mathfrak{F}[A]=\mathfrak{F}[B]$ implies $A=B$.

of this lemma, P^* must be a minimal prime of A. Since $P\subseteq P^*$, it follows that $P=P^*$. This completes the proof.

In [4] Schenkman has already defined a radical of a normal subgroup A of a group G. Now we shall define a P-radical of A, which is an analogue of one of an ideal of a ring[7]. This concept is useful in our argument.

DEFINITION 3. The intersection of all prime normal subgroups containing a normal subgroup A of a group G is called a P-radical of A. In symbol: $R(A)$. In particular the P-radical of the unit subgroup of G is called a P-radical of the group G. In symbol: $R(G)$.

REMARK. It is easily proved that $R(A)$ is contained in the radical of A, and equal to the intersection of all the minimal primes of A.

LEMMA 5. *$R(A)$ is equal to the subgroup N^* generated by the set-union of all normal subgroups N such that every C-family containing a subgroup of N contains a subgroup of A.*

Proof. Let $\mathfrak{F}$ be any C-family which contains a subgroup of $R(A)$[8]. Then $\mathfrak{F}$ contains a subgroup of A. Because, if not so, there exists by Lemma 2 a prime normal subgroup P such that $P\supseteq A$ and no subgroup in $\mathfrak{F}$ is contained in P. Since $R(A)$ is contained in P, no subgroup in $\mathfrak{F}$ is contained in $R(A)$, which is a contradiction. Hence $R(A)$ is contained in N^*.

Conversely, if P is any prime which contains A, then P contains N^*. Because, if P does not contain N^*, we can take an element x of N^* such that x is not contained in P. Let $N(x)$ be the normal subgroup generated by x. Since $N(x)$ is contained in $\mathfrak{F}[P]$ and also contained in N^*, $\mathfrak{F}[P]$ contains a subgroup of A. Clearly, this is impossible, since A is contained in P. Hence P contains N^*. We have therefore $N^*\subseteq \bigcap_{P\geqq A} P=R(A)$. This completes the proof.

LEMMA 6. *Let $\{f(X_1, \cdots, X_\rho)\}$ be the set of all commutator-forms[9] f of the weight $\rho=1, 2, \cdots$, and let N be a finitely generated normal subgroup of a group G. Then $\{f(N, \cdots, N)\}$ forms a C-family of G, and N is solvable if (and only if) $\{f(N, \cdots, N)\}$ contains the unit subgroup of G.*

Proof. The first part is easy to see. If $f(N, \cdots, N)$ is equal to the unit subgroup for some commutator-form f, then by using the properties[10] (1) $[N^{(\rho)}, N^{(\sigma)}]\supseteq[N^{(\tau)}, N^{(\tau)}]$, $\tau=\mathrm{Max}\{\rho, \sigma\}$ and (2) $[N, N]^{(\rho)}=[N^{(\rho)}, N^{(\rho)}]$ we can prove that $N^{(\kappa)}$ is equal to the unit subgroup for a sufficiently large whole number κ, that is, the derived chain of N terminates in the unit subgroup after a finite number of steps.

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7) Cf. [3; p. 333].
8) The existence of such a C-family is assured by the existence of the C-family of all the finitely generated normal subgroups of G.
9) Cf. [5; p. 59].
10) Cf. [2; p. 31].

LEMMA 7. *In order that G has no solvable normal subgroup $N(x)$, $x \neq e$, it is necessary and sufficient that the P-radical $R(G)$ is equal to the unit subgroup of G.*

Proof. Suppose that $R(G)$ is not the unit subgroup. Take a non-unit element x in $R(G)$. Then every C-family, which contains a subgroup contained in the normal subgroup $N(x)$ generated by x, contains the unit subgroup of G. In particular $\{f(N(x), \cdots, N(x))\}$ contains the unit subgroup. Hence by Lemma 6 $N(x)$ is solvable. Conversely, if $N(x)$ is solvable it is contained in every prime normal subgroup of G. This implies that $N(x)$ is contained in $R(G)$.

2. PROOF OF THEOREM.

Let G be any group which has no solvable normal subgroup $N(x)$ except the unit subgroup, and let $\{P_\lambda; \lambda \in \Lambda\}$ be the set of all prime normal subgroups of G. Then the mapping

$$a \to (\cdots, aP_\lambda, \cdots)$$

gives an isomorphism from G into the direct product $\Pi^{\otimes}_{\lambda \in \Lambda} G/P_\lambda$. Because, if $(\cdots, aP_\lambda, \cdots) = (\cdots, bP_\lambda, \cdots)$, then $a^{-1}b \in P_\lambda$ for every $\lambda \in \Lambda$. Hence $a^{-1}b \in \bigcap_{\lambda \in \Lambda} P_\lambda = R(G)$. By Lemma 7 $R(G)$ is equal to the unit subgroup of G. We have therefore $a=b$. Since G/P_λ is a prime group, we complete the proof of the "if part" of Theorem.

The "only if part" is easy to see. Let G be isomorphic to a subgroup of a direct product $\Pi^{\otimes}_{\lambda \in \Lambda} G_\lambda$ of prime groups G_λ. If there exists a solvable normal subgroup N of G, the λ-component $\varphi_\lambda(N) = \{a_\lambda; a \in N\}$ of N is a solvable normal subgroup of G_λ for every λ. Hence there exists a whole number ρ_λ such that $(\varphi_\lambda(N))^{(\rho_\lambda)}$ is equal to the unit subgroup of G_λ. Since G_λ is prime, $\varphi_\lambda(N)$ is equal to the unit subgroup of G_λ for every λ. This implies that N is equal to the unit subgroup of G. In particular $N(x)$ is not solvable for an arbitrary element $x\,(\neq e)$ of G.

Since it is clear that G has no solvable normal subgroup except the unit subgroup if (and only if) G has no solvable normal subgroup $N(x)$ except the unit subgroup $N(e)$, we have

COROLLARY. *A group G is isomorphic to a subdirect product of a finite or infinite number of prime groups if and only if G has no solvable normal subgroup except the unit subgroup.*

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