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# *Semi cubical theory on higher obstruction*

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Let $Y$ be a simply connected topological space which has vanishing homotopy groups $\pi_i(Y)$ for $0 \leqq i < n$, $n < i < q$, and $q < i < r < 2q-1$, and let $K$ be a geometric complex with subcomplex $L$ and $f: K^n \cup L \to Y$ be a mapping extensible to a map $K^{q+1} \cup L \to Y$. We discussed the third obstruction to the extension of $f$ in [3].

It is the purpose of this paper to establish the higher obstruction theorems in the general cases by the aid of results of our preceding paper along the line of Eilenberg-MacLane [2]. This paper makes full use of the results and terminologies of the preceding paper of the author [4].

## 1. Preliminary

Let $K$ and $L$ are S.Q. complexes, we shall define the standard maps $f: K \times L \to K \otimes L$ and $g: K \otimes L \to K \times L$ between the cartesian and the tensor product. First map $f$ is defined by

$$f(\sigma \times \tau) = \Sigma_\beta \beta_1^* \sigma \otimes \beta_2^* \tau \qquad \textit{if } \dim \sigma = \dim \tau = r$$

where $\beta$ is going round the family of pairs $(\beta_1, \beta_2)$ such that

$$\begin{aligned} &\beta_i : I^{m_i} \to I^r, \quad 0 \leqq m_i \leqq r, \ m_1 + m_2 = r, \\ &\beta_1(t_1, \cdots, t_{m_1}) = (t_1, \cdots, t_{m_1}, 0, \cdots, 0), \\ &\beta_2(t_1, \cdots, t_{m_2}) = (1, \cdots, 1, t_1, \cdots, t_{m_2}), \end{aligned}$$

namely $\beta_1^* = F^{0 \cdot m_2} = F^0_{m_1+1} \cdots F^0_r$ and $\beta_2^* = {}^{m_1}F^1 = F^1_0 \cdots F^1_{m_1}$. Second map $g$ is defined by

$$g(\sigma \otimes \tau) = \Sigma_\alpha \wp(\alpha) \alpha_1^* \sigma \times \alpha_2^* \tau \qquad \textit{if } \dim \sigma = m_1, \ \dim \tau = m_2$$

where $\alpha$ is going round the family of pairs $(\alpha_1, \alpha_2)$ such that

$$\begin{aligned} &\alpha_i : I^r \to I^{m_i}, \quad r = m_1 + m_2, \\ &\alpha_1(t_1, \cdots, t_r) = (t_{i_1}, \cdots, t_{i_{m_1}}) \qquad i_1 < \cdots < i_{m_1}, \\ &\alpha_2(t_1, \cdots, t_r) = (t_{j_1}, \cdots, t_{j_{m_2}}) \qquad j_1 < \cdots < j_{m_2}, \end{aligned}$$

and

$$\wp(\alpha) = \mathrm{Sgn.}\begin{pmatrix} 1, & \cdots\cdots\cdots\cdots\cdots\cdots, & r \\ i_1, \cdots, i_{m_1}, & j_1, \cdots, & j_{m_2} \end{pmatrix}.$$

LEMMA 1.1. *If $K$ and $L$ are S.Q. complexes, then each of the composites $fg$ and $gf$ is chain homotopic to the appropriate identity map.*

The proof of this lemma is similar to that of Eilenberg-Zilber theorem [1] in the S.S. complexes, and therefore we omit it.

Let $Y$ be a topological space and the homotopy groups $\pi_i(Y)$ of $Y$ vanish without $i=n_1, n_2, \cdots, n_m$ $(1<n_1<n_2<\cdots<n_m)$. We shall denote $\pi_{n_j}(Y)$ with $\Pi^j$ in the following discussion.

It is well known that any minimal subcomplex $M=M(Y)$ of the total singular cubical complex $Q(Y)$ is isomorphic to a Postnikov complex

$$N^m = \mathcal{P}(K(\Pi^1, n_1), \mathbf{k}^1, N^2, \boldsymbol{k}^2, N^3, \cdots, \boldsymbol{k}^{m-1}, N^m),$$

and there are natural injections

$$M_{n_m} \subset M_{n_{m-1}} \subset \cdots \subset M_{n_2} \subset M_{n_1} = M$$

where $M_{n_j}=M\cap Q_{n_j}(Y)$ consists of cubes whose faces in dimensions less than $n_j$ reduce to the base point $y_0$ of $Y$. We shall denote the Postnikov construction of $M_{n_j}$ $(1\leqq j\leqq m)$ as follows;

$$N_j^j = K(\Pi^j, n_j),$$
$$N_j^{j+1} = K(\Pi^j, n_j, \Pi^{j+1}, n_{j+1}, \boldsymbol{k}_j^j)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$M_{n_j} \cong N_j^m = \mathcal{P}(N_j^j, \boldsymbol{k}_j^j, N_j^{j+1}, \boldsymbol{k}_j^{j+1}, \cdots, \mathbf{k}_j^{m-1}, N_j^m)$$

where $\boldsymbol{k}_j^q=i_j^*\boldsymbol{k}^q$ is the image of injection homomorphism

$$i_j^*: H^{n_{q+1}+1}(N^q; \Pi^{q+1}) \to H^{n_{q+1}+1}(N_j^q; \Pi^{q+1}) \qquad j\leqq q\leqq m-1.$$

If the Postnikov invariants $\boldsymbol{k}^1, \boldsymbol{k}^2, \cdots, \boldsymbol{k}^{m-1}$ are additive, we shall define the internal products of S.Q. complexes $N_j^{p+1}=\mathcal{P}(N_j^p, \Pi^{p+1}, n_{p+1}, \boldsymbol{k}_j^p)$ $(1\leqq j\leqq p<m)$ inductively as follows;

$$(\phi, \psi)\circ(\phi', \psi') = (\phi\circ\phi', \psi\circ\psi')$$

for $r$-cubes $(\phi, \psi)$, $(\phi', \psi')$ of $N_j^{p+1}$, where $\phi\circ\phi'$ is the internal product of $r$-cubes of $N_j^p$ defined inductively, and $\psi\circ\psi'$ is the internal product of $r$-cubes of $F(\Pi^{p+1}, n_{p+1})$ defined in (8.1.1) [4], then

$$k_j^p[(\phi\circ\phi')\gamma] = k_j^p(\phi\gamma)+k_j^p(\phi'\gamma) = \Delta\psi(\gamma)+\Delta\psi'(\gamma) = \Delta(\psi\circ\psi')(\gamma)$$
$$\textit{for any } (n_{p+1}+1)\textit{-cube } \gamma \textit{ of } Q(I^r)$$

since $\boldsymbol{k}_j^p$ is additive. Hence $(\phi\circ\phi', \psi\circ\psi')$ is also the $r$-cube of $N_j^{p+1}$.

We note that if $\boldsymbol{k}_j^p$ is not additive, the above definition is meaningless except the special case when $\phi$ or $\phi'$ is trivial.

As we showed previously [4], there is a one to one correspondence between the semi cubical (S.Q.) mappings $T=T(x_{n_j}, \cdots, x_{n_q}): K\to N_j^q$ and the sequences $(x_{n_j}, \cdots, x_{n_q})=x_{j\cdots q}(T)$ of a cocycle $x_{n_j}\in Z^{n_j}(K; \Pi^j)$ and cochains $x_{n_r}\in C^{n_r}(K; \Pi^r)$ satisfying

$$k_j^{r-1}\, T(x_{n_j}, \cdots, x_{n_{r-1}}) = \delta x_{n_r} \qquad j<r\leqq q.$$

We shall call such a sequence $x_{j\cdots q}=(x_{n_j}, \cdots, x_{n_q})$ as *cocycloid.*

Let $x_{j\cdots q}=(x_{n_j}, \cdots, x_{n_q})$, $x'_{j\cdots q}=(x'_{n_j}, \cdots, x'_{n_q})$ be two cocyloids, we denote the sequence $(x_{n_j}-x'_{n_j}, \cdots, x_{n_q}-x'_{n_q})$ as $(y_{n_j}, \cdots, y_{n_q})$. Then, we have

LEMMA 1.2. *If* $q=j$ *or* $\mathbf{k}^j, \cdots, \mathbf{k}^{q-1}$ *are additive,* $y_{j\cdots q}$ *is also a cocycloid. Especially,* $y_{nj}=0, \cdots, y_{nh}=0$, *then* $y_{h+1\cdots q}=(y_{nh+1}, \cdots, y_{nq})$ *is a cocycloid if* $h+1=q$ *or* $\boldsymbol{k}^{h+1}, \cdots, \boldsymbol{k}^{q-1}$ *are additive.*

*Proof.* If $q=j$, $x'_{j\cdots q}\equiv x_{nj}$, $x'_{j\cdots q}\equiv x'_{nj}\in Z^{nj}(K;\Pi^j)$, then $y_{j\cdots q}\equiv y_{nj}$ is also a cocycle (cocycloid).

If $\boldsymbol{k}^j, \cdots, \boldsymbol{k}^{q-1}$ are additive,

$$\begin{aligned}\delta y_{n_r} &= \delta x_{n_r}-\delta x'_{n_r} = k_j^{r-1}T(x_{j\cdots r-1})-k_j^{r-1}T(x'_{j\cdots r-1})\\ &= k_j^{r-1}[T(y_{j\cdots r-1})\circ T(x'_{j\cdots r-1})]-k_j^{r-1}T(x'_{j\cdots r-1})\\ &= k_j^{r-1}T(y_{j\cdots r-1}) \qquad\qquad j<r\leqq q\end{aligned}$$

inductively.

The latter half is similarly recognized since

$$\delta y_{nh+1} = \delta x_{nh+1}-\delta x'_{nh+1} = k_j^h\, T(x_{j\cdots r-1})-k_j^h T(x'_{j\cdots r-1}) = 0\,.$$

Two mappings $T(x_{j\cdots q})$ and $T(x'_{j\cdots q})$ are homotopic if and only if

$x_{nj}$ *and* $x'_{nj}$ *are cohomologous,*

$x_{n_r}-x'_{j_r}-k_j^{r-1}E_j^r$ *is cohomologous zero for* $j<r\leqq q$,

*where* $E_j^r: T(x_{nj}, \cdots, x_{n_{r-1}})\cong T(x'_{nj}, \cdots, x'_{n_{r-1}})$ *are some chain homotopies whose existence are secured inductively.*

As our future convenience, we shall call $x_{j\cdots q}$ and $x'_{j\cdots q}$ being cohomologous if they satisfy the above conditions, and denote $x_{j\cdots q}\in Z^{nj\cdots nq}(K;N_j^q)$ and its cohomology class $\boldsymbol{x}_{j\cdots q}\in H^{nj\cdots nq}(K;N_j^q)$.

## 2. $\gamma$-operations

Given two S.Q. pairs $(K, L_i)$ $i=1, 2$ and two cocycloids $x_{j\cdots q}\in Z^{nj\cdots nq}(K, L_1; N_j^q)$, $x_{k\cdots r}\in Z^{nk\cdots nr}(K, L_2; N_k^r)$, we shall define a chain transformation

$$\gamma(x_{j\cdots q}, x_{k\cdots r}): (K, L)\to N_h^s$$

where $L$ is the union of the subcomplexes $L_1, L_2$ and $h=\min(j, k)$, $s=\max(q, r)$.

The map $\gamma(x_{j\cdots q}, x_{k\cdots r})$ is defined as the composite of the maps displayed in the following diagram

$$\begin{tikzcd}
(K, L) \arrow[d, "e"] & \\
(K, L_1)\times(K, L_2) \arrow[r, "f"] & (K, L_1)\otimes(K, L_2) \arrow[d, "R(x_{j\cdots q})\otimes R(x_{k\cdots r})"] \\
& N_j^q\otimes N_k^r \arrow[d, "i\binom{qs}{jh}\otimes i\binom{rs}{kh}"] \\
N_h^s\times N_h^s \arrow[d, "\gamma"] & N_h^s\otimes N_h^s \arrow[l, "g"'] \\
N_h^s &
\end{tikzcd}$$

Here the first map $e$ is the diagonal map. The second map $f$ is the standard map of the cartesian into the tensor product.

The third map is the tensor product of the $FD$-maps $R(x_{j\cdots q})$, $R(x_{k\cdots r})$ each of which is defined by

$$R(x_{n_s}, \cdots, x_{n_t}) = T(x_{n_s}, \cdots, x_{n_t}) - T(0, \cdots, 0).$$

The fourth map is the tensor product of the inclusion maps; for instance if $h=j<k$, $q<r=s$

$$i\binom{qs}{jh} : N_j^q \to N_h^s = N_j^s = \mathcal{P}(N_j^q, k_j^q, \cdots, k_j^{s-1}, N_j^s)$$
$$i\binom{rs}{kh} : N_k^r = N_k^s \to N_j^s \text{ induced by } M_{n_k} \subset M_{n_j},$$

here it is easily verified that the map at first case is meaningless when the dimension of cube of $K$ is greater than $n_{q+1}$ while the map at second case has no restriction on dimensions, since the image of $i\binom{qs}{jh}$ in above case does not belong to $N_h^s$ in general.

The fifth map $g$ is the standard map of the tensor into the cartesian product. Finally, the map $\gamma$ is given in terms of the internal product in $N_h^s$ which is meaningless if $k^h, \cdots, k^{s-1}$ is not additive without the special case $j<\cdots<q<k<\cdots<r$ or $k<\cdots<r<j<\cdots<q$.

The final definition may be written as

$$\gamma(x_{j\cdots q}, x_{k\cdots r}) = \gamma g[i\binom{qs}{jh} \otimes i\binom{rs}{kh}][R(x_{j\cdots q}) \otimes R(x_{k\cdots r})]\, fe\,.$$

According to the dimensional restriction which is occured by the inclusion map $i(\ )$, our maps are meaningless in the case when $\gamma(x_{j\cdots q}, x_{k\cdots r})$ operates upon the cells whose dimensions are larger than min $(n_j+n_{r+1}, n_k+n_{q+1})$.

Since $f$ is natural with respect to $R(x_{j\cdots q})$ and $gf \simeq$ *identity map*, we have

$$\begin{aligned}\gamma(x_{j\cdots q}, x_{j\cdots q}) &= \gamma g f[R(x_{j\cdots q}) \times R(x'_{j\cdots q})]e \\ &\simeq \gamma[R(x_{j\cdots q}) \times R(x'_{j\cdots q})]e\,,\end{aligned}$$

then we have

LEMMA 2.1. $\gamma(x_{j\cdots q}, x'_{j\cdots q}) \simeq R(x_{j\cdots q}) \circ R(x'_{j\cdots q})$.

Replacement of $x_{j\cdots q}$ or $x_{k\cdots r}$ by a cohomologous cocycloid replace $R(x_{j\cdots q})$ or $R(x_{k\cdots r})$ by a chain homotopic map, therefore the homotopy class of the map $\gamma(x_{j\cdots q}, x_{k\cdots r})$ depends only on the cohomology classes $\mathbf{x}_{j\cdots q}, \mathbf{x}_{k\cdots r}$ of $x_{j\cdots q}, x_{k\cdots r}$ respectively; this homotopy class will be denoted by $\gamma(\mathbf{x}_{j\cdots q}, \mathbf{x}_{k\cdots r})$.

Let $\mathbf{y} \in H^t(N_h^s; G)$ is a cohomology class. The $\gamma$-*operation* $\mathbf{y}_\gamma$ is defined for cohomology classes $\mathbf{x}_{j\cdots q} \in H^{n_j\cdots n_q}(K, L_1; N_j^q)$, $\mathbf{x}_{k\cdots r} \in H^{n_k\cdots n_r}(K, L_2; N_k^r)$, $t \leqq \min(n_j+n_{r+1}, n_k+n_{q+1})$ by the formula

$$\mathbf{y}_\gamma(\mathbf{x}_{j\cdots q}, \mathbf{x}_{k\cdots r}) = \gamma(\mathbf{x}_{j\cdots q}, \mathbf{x}_{k\cdots r})^* \mathbf{y}$$

it is an element of $H^t(K, L; G)$.

If $j=k$, $q=r$, our operation is a natural prolongation of the internal operation

$\boldsymbol{y} \vdash [2]$. If $j \cdots q \equiv 1$, $k \cdots r \equiv 2$, our operation is the same as the $\gamma$-operation which we defined in the previous paper [3].

LEMMA 2.2. *If* $\boldsymbol{y} \in H^t(N_h^s; G)$ *and* $\boldsymbol{x}_{j\cdots q} \in H^{n_j \cdots n_q}(K, L_1; N_j^q)$, $\boldsymbol{x}_{k\cdots r} \in H^{n_k \cdots n_r}(K, L_2; N_k^r)$ *with* $t < n_j + n_k$, *then* $\boldsymbol{y}_\gamma(\boldsymbol{x}_{j\cdots q}, \boldsymbol{x}_{k\cdots r})$ *is zero.*

*Proof.* At the distributions $(\beta_1, \beta_2)$ of $f$-map, at least one of $\dim(\beta_1^* \sigma) - n_j$, $\dim(\beta_2^* \sigma) - n_k$ is negative since $\dim(\beta_1^* \sigma) + \dim(\beta_2^* \sigma) = t < n_j + n_k$. Then the result follows from the facts $R(x_{j\cdots q})$ is zero unless $\dim(\beta_1^* \sigma) \geqq n_j$ and $R(x_{k\cdots r})$ is zero unless $\dim(\beta_2^* \sigma) \geqq n_k$.

We consider next the case where $t = n_j + n_k$. The same argument shows that

$$\gamma(x_{j\cdots q}, x_{k\cdots r})\sigma = \gamma g[i(\tfrac{qs}{jh}) \otimes i(\tfrac{rs}{kh})][R(x_{j\cdots q})(F^{0 \cdot n_k}\sigma) \otimes R(x_{k\cdots r})({}^{n_j}F^1\sigma)]$$

And,

$$R(x_{j\cdots q})(F^{0\cdot n_k}\sigma) = x_{n_j}(F^{0\cdot n_k}\sigma) \in \Pi^j = \pi_{n_j}(Y)$$
$$R(x_{k\cdots r})({}^{n_j}F^1\sigma) = x_{n_k}({}^{n_j}F^1\sigma) \in \Pi^k = \pi_{n_k}(Y)$$

where the left equalities are equalities modulo norms, then we have

$$\gamma(x_{j\cdots q}, x_{k\cdots r})\sigma = \gamma g([x_{n_j}(F^{0\cdot n_k}\sigma)] \otimes [x_{n_k}({}^{n_j}F^1\sigma)])$$

where each bracketed element denotes the corresponding cube of $N_h^s$.

Now observe that the given cohomology class $\boldsymbol{y} \in H^t(N_h^s; G)$ may be used to define a homomorphism

$$\smile : \Pi_{n_j} \otimes \Pi_{n_k} \to G$$

according to the formula

$$z_j \smile z_k = y\gamma g([z_j] \otimes [z_k]) \qquad z_j \in \Pi_{n_j},\ z_k \in \Pi_{n_k}.$$

This implies that

$$\boldsymbol{y}_\gamma(\boldsymbol{x}_{j\cdots q}, \boldsymbol{x}_{k\cdots r})\sigma = (\boldsymbol{x}_{n_j} \smile \boldsymbol{x}_{n_k})\sigma,$$

where $x_{n_j} \smile x_{n_k}$ is the cup product of the cocycles $x_{n_j}, x_{n_k}$ relative to the pairing just defined. We have proved

LEMMA 2.3. *If* $\boldsymbol{y} \in H^t(N_h^s; G)$ *and* $\boldsymbol{x}_{j\cdots q} \in H^{n_j \cdots n_q}(K, L_1; N_j^q)$, $\boldsymbol{x}_{k\cdots r} \in H^{n_k \cdots n_r}(K, L_2; N_k^r)$ *with* $t = n_j + n_k$, *then*

$$\boldsymbol{y}_\gamma(\boldsymbol{x}_{j\cdots q}, \boldsymbol{x}_{k\cdots r}) = \boldsymbol{x}_{n_j} \smile \boldsymbol{x}_{n_k}$$

*where the cup product on the right is taken relative to the above pairing determined by* $\boldsymbol{y}$.

We note that if $\boldsymbol{y}$ is a Postnikov invariant $\boldsymbol{k}_h^s \in H^{n_{s+1}+1}(N_h^s; \Pi^{s+1})$ the cup product is paired by the Whitehead product

$$[\pi_{n_j}(Y), \pi_{n_k}(Y)] \subset \pi_{n_{s+1}}(Y) \quad \textit{where} \quad n_j + n_k = n_{s+1} + 1.$$

Let $\boldsymbol{y} \in H^t(N_j^q; G)$ and $\boldsymbol{x}_{j\cdots q} \in H^{n_j \cdots n_q}(K, L; N_j^q)$, and we shall denote $R(\boldsymbol{x}_{j\cdots q})^* \boldsymbol{y}$ by $\boldsymbol{y}_\gamma \boldsymbol{x}_{j\cdots q}$ in the following discussion, it is also a natural prolongation of the internal operation $\boldsymbol{y} \vdash [2]$.

If $x_{j\cdots q}=(x_{n_j}, \cdots x_{n_h}, x_{n_{h+1}}, \cdots, x_{n_q})$ where $x_{n_j}=0, \cdots, x_{n_h}=0$, then

$$\begin{aligned} R(x_{j\cdots q}) &= T(0, \cdots, 0, x_{n_{h+1}}, \cdots, x_{n_q})-T(0, \cdots, 0) \\ &= \gamma[i(\begin{smallmatrix} hq \\ jj \end{smallmatrix})\times i(\begin{smallmatrix} q & q \\ h+1 & j \end{smallmatrix})][T(0, \cdots, 0)\times R(x_{h+1\cdots q})]e \\ &\cong \gamma g[i(\begin{smallmatrix} hq \\ jj \end{smallmatrix})\otimes i(\begin{smallmatrix} q & q \\ h+1 & j \end{smallmatrix})][T(0, \cdots, 0)\otimes R(x_{h+1\cdots q})]fe\,. \end{aligned}$$

Hence we have

LEMMA 2.4. *If* $\boldsymbol{y}\in H^t(N^q_j; G)$ *and* $\boldsymbol{x}_{h+1\cdots q}\in H^{n_{h+1}\cdots n_q}(K, L; N^q_{h+1})$ *then*

$$\boldsymbol{y}\gamma g[i(\begin{smallmatrix} hq \\ jj \end{smallmatrix})\otimes i(\begin{smallmatrix} q & q \\ h+1 & j \end{smallmatrix})][T(0, \cdots, 0)\otimes R(\boldsymbol{x}_{h+1\cdots q})]fe = [i(\begin{smallmatrix} q & q \\ h+1 & j \end{smallmatrix})^*\boldsymbol{y}]_\gamma \boldsymbol{x}_{h+1\cdots q}\,.$$

### 3. Obstruction theorems

Let $K$ be a geometric complex. We shall be interested in continuous maps $f: K\to Y$. Such a map induces a cubical map $K\to Q(Y)$ which is also denoted by $f$. Conversely, every cubical map $K\to Q(Y)$ arises in this fashion form a unique continuous map $K\to Y$. The map $f$ is called minimal if it maps $K$ into $M$. In the theory of the minimal complex we shall assume without loss of generality that the maps $K\to Y$ are minimal.

Therefore, a map $f: K^{n_1}\to Y$ determines a cochain $a_{n_1}(f)\in C^{n_1}(K; \Pi^1)$ which is a cocyle if and only if $f$ admits an extension $f_2: K^{n_2}\to Y$. This extension $f_2$ presents an obstruction cocyle $c^{n_2+1}(f_2)\in Z^{n_2+1}(K; \Pi^2)$ which is represented by

$$c^{n_2+1}(f_2) = k^1_1 T(a_{n_1}(f))-\delta(b_2 f_2)$$

where $b_2$ is a basic cochain of $C^{n_2}(M; \Pi^2)$ determined by setting

$$b_2\sigma = d(\tilde{\kappa}\kappa\sigma, \sigma) \qquad \text{for any } n_2\text{-cube of } M,$$

we shall denote $b_2 f_2$ by $a_{n_2}(f)$. This obstruction is zero if and only if the map $f_2$ admits an extension $f_3: K^{n_3}\to Y$, and presents a third obstruction cocycle

$$c^{n_3+1}(f_3) = k^2_1 T(a_{n_1}(f), a_{n_2}(f))-\delta(b_3 f_3)$$

and so on.

If $f: K^{n_1}\to Y$ admits an extension $f'=f_{i+1}: K^{n_{i+1}}\to Y$, there is a cocycloid $a_{1\cdots i}(f')=(a_{n_1}(f'), a_{n_2}(f'), \dots, a_{n_i}(f'))\in Z^{n_1\cdots n_i}(K; N^i_1)$ and presents an obstruction cocycle

$$c^{n_{i+1}+1}(f') = k^i_1 T(a_{1\cdots i}(f'))-\delta(a_{n_{i+1}}(f')).$$

Let $f, g: K^{n_{i+1}}\ L\to Y$ be two maps which agree on $L$. Then they induce a cocycloid $a_{1\cdots i}(f, g)\in Z^{n_1\cdots n_2}(K, L; N^i_1)$ in which $a_{n_j}(f, g)=a_{n_j}(f)-a_{n_j}(g)$, $1\leq j\leq i$, if $\boldsymbol{k}^1, \dots, \boldsymbol{k}^{i-1}$ are additive. In general, if $f$ and $g$ agree on $K^{n_h}\ L$ they induce a cocycloid $a_{h+1\cdots i}(f, g)\in Z^{n_{h+1}\cdots n_i}(K, L; N^i_{h+1})$ if $\boldsymbol{k}^{h+1}, \dots, \boldsymbol{k}^{i-1}$ are additive. We shall denote in the following the cohomology class of $c^{n_j+1}(f)$, $c^{n_j+1}(g)$, $a_{1\cdots i}(f)$, $a_{1\cdots i}(g)$, $a_{1\cdots i}(f, g), \dots$ as $\boldsymbol{z}^{n_j+1}(f)$, $\boldsymbol{z}^{n_j+1}(g)$, $\boldsymbol{a}_{1\cdots i}(f)$, $\boldsymbol{a}_{1\cdots i}(g)$, $\boldsymbol{a}_{1\cdots i}(f, g), \dots$ respectively.

In the following we assume that the Postnikov invariants $\boldsymbol{k}^1, \boldsymbol{k}^2, \dots, \boldsymbol{k}^i$ are additive.

THEOREM 3.1. *Let $f, g: K^{n_i}\cup L\rightarrow Y$ be two maps which agree on $L$ and which are extensible to $K^{n_i+1}\cup L$, then*

$$z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g)=\boldsymbol{k}_{1}^{i}\gamma a_{1\cdots i}(f,g)+\boldsymbol{k}_{1}^{i}\gamma(a_{1\cdots i}(f,g),a_{1\cdots i}(g)),$$

*where the last term vanishes if $n_{i+1}+1<2n_1$.*

*Proof.* $z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g)$ is representable by cochain

$$c^{n_{i+1}+1}(f)-c^{n_{i+1}+1}(g)=k_{1}^{i}T(a_{1\cdots i}(f))-\delta(a_{n_{i+1}}(f))-k_{1}^{i}T(a_{1\cdots i}(g))+\delta(a_{n_{i+1}}(g)).$$

Since $f$ and $g$ coincide on $L$, it follows that $a_{n_{i+1}}(f)-a_{n_{i+1}}(g)=(f^{\#}-g^{\#})b_{i+1}$ is zero on $L$; this yields the cohomology

$$c^{n_{i+1}+1}(f)-c^{n_{i+1}+1}(g)\sim k_{1}^{i}[T(a_{1\cdots i}(f))-T(a_{1\cdots i}(g))].$$

However,

$$\begin{aligned}T(a_{1\cdots i}(f))-T(a_{1\cdots i}(g)) &= T(a_{1\cdots i}(f,g))+T(a_{1\cdots i}(f,g)+a_{1\cdots i}(g))\\ &\quad -T(a_{1\cdots i}(f,g))-T(a_{1\cdots i}(g))\\ &= R(a_{1\cdots i}(f,g))+R(a_{1\cdots i}(f,g)+a_{1\cdots i}(g))\\ &\quad -R(a_{1\cdots i}(f,g))-R(a_{1\cdots i}(g))\\ &= R(a_{1\cdots i}(f,g))+R(a_{1\cdots i}(f,g))\circ R(a_{1\cdots i}(g))\end{aligned}$$

then our result follows from Lemma 2.1. and Lemma 2.2., since $n_{i+1}+1<n_1+n_{i+1}$.

THEOREM 3.2. *Let $f, g: K^{n_i}\cup L\rightarrow Y$ be two maps which agree on $K^{n_1}\cup L$ and which are extensible to $K^{n_i+1}\cup L$ and $n_{i+1}+1<2n_2$, then*

$$z^{n_{i+1}}(f)-z^{n_{i+1}}(g)=\boldsymbol{k}_{2}^{i}\gamma\boldsymbol{a}_{2\cdots i}(f,g)+\boldsymbol{k}_{1}^{i}\gamma(\boldsymbol{a}_{n_1}(f),\boldsymbol{a}_{2\cdots i}(f,g)),$$

*where the last term vanishes if $n_{i+1}+1<n_1+n_2$.*

This theorem is a special case of the next theorem.

THEOREM 3.3. *Let $f, g: K^{n_i}\cup L\rightarrow Y$ be two maps which agree on $K^{n_h}\cup L$ and which are extensible to $K^{n_i+1}\cup L$ and $n_{i+1}+1<2n_{h+1}$, then*

$$z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g)=\boldsymbol{k}_{h+1}^{i}\gamma\boldsymbol{a}_{h+1\cdots i}(f,g)+k_{1}^{i}\gamma(\boldsymbol{a}_{1\cdots h}(f),\boldsymbol{a}_{h+1\cdots i}(f,g)),$$

*where the last term vanishes if $n_{i+1}+1<n_1+n_{h+1}$.*

*Proof.* $z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g)$ is representable by a cochain

$$c^{n_{i+1}+1}(f)-c^{n_{i+1}+1}(g)=k_{1}^{i}[T(a_{1\cdots i}(f))-T(a_{1\cdots i}(g))]-\delta[a_{n_{i+1}}(f)-a_{n_{i+1}}(g)]$$

where $a_{n_{i+1}}(f)-a_{n_{i+1}}(g)$ is zero on $L$. And

$$T(a_{1\cdots i}(f))=\gamma[i(\tbinom{h\,i}{1\,1})\times i(\tbinom{i\quad i}{h+1\ 1})][T(a_{1\cdots h}(f))\times T(a_{h+1\cdots i}(f))]e\,,$$

then

$$\begin{aligned} &T(a_{1\cdots i}(f))-T(a_{1\cdots i}(g)) \\ &\quad =\gamma[i\binom{hi}{11}\times i\binom{i\ \ i}{h+1\ 1}][T(a_{1\cdots h}(f))\times[T(a_{h+1\cdots i}(f)) \\ &\qquad\qquad -T(a_{h+1\cdots i}(g))]]e \\ &\quad \simeq \gamma g[i\binom{hi}{11}\otimes i\binom{i\ \ i}{h+1\ 1}][T(a_{1\cdots h}(f))\otimes[T(a_{h+1\cdots i}(f)) \\ &\qquad\qquad -T(a_{h+1\cdots i}(g))]]fe \end{aligned}$$

in dimensions $\leqq 2n_{h+1}$, since $a_{1\cdots h}(f)=a_{1\cdots h}(g)$ in our case.

Now, as $FD$-maps: $K\to\times_{j=h+1}^{i}F(\Pi^j, n_j)$

$$\begin{aligned} &T(a_{h+1\cdots i}(f))-T(a_{h+1\cdots i}(g)) \\ &\quad = R(a_{h+1\cdots i}(f,g))+R(a_{h+1\cdots i}(f,g))\circ R(a_{h+1\cdots i}(g)) \end{aligned}$$

and

$$\begin{aligned} &R(a_{h+1\cdots i}(f,g))\circ R(a_{h+1\cdots i}(g)) \\ &\quad =\gamma[R(a_{h+1\cdots i}(f,g))\times R(a_{h+1\cdots i}(g))]e \\ &\quad \simeq \gamma g[R(a_{h+1\cdots i}(f,g))\otimes R(a_{h+1\cdots i}(g))]fe \end{aligned}$$

then

$$T(a_{h+1\cdots i}(f))-T(a_{h+1\cdots i}(g))\simeq R(a_{h+1\cdots i}(f,g))$$

in dimensions less then $2n_{h+1}$.

Hence,

$$\begin{aligned} &c^{n_{i+1}+1}(f)-c^{n_{i+1}+1}(g) \\ &\quad \sim k_1^i\gamma g[i\binom{hi}{11}\otimes i\binom{i\ \ i}{h+1\ 1}][[R(a_{1\cdots h}(f))+T(0,\dots,0)]\otimes R(a_{h+1\cdots i}(f,g))]fe \\ &\quad = k_{h+1}^i\gamma a_{h+1\cdots i}(f,g)+k_1^i\gamma(a_{1\cdots h}(f),\ a_{h+1\cdots i}(f,g)) \end{aligned}$$

by Lemma 2.4. if $n_{i+1}+1<2n_{h+1}$. The rest of the theorem is due to Lemma 2.2.

Combining Lemma 2.3. and the above theorems, we can get various formulas. Namely, we have

COROLLARY 3.4. 1) *If* $n_{i+1}+1=2n_1$ *in Theorem* 3.1., *then*

$$z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g) = \boldsymbol{k}_{1\gamma}^i a_{1\cdots i}(f,g)+\boldsymbol{a}_{n_1}(f,g)\smile\boldsymbol{a}_{n_1}(g)$$

2) *If* $n_{i+1}+1=n_1+n_2$ *in Theorem* 3.2., *then*

$$z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g) = \boldsymbol{k}_{2\gamma}^i\boldsymbol{a}_{2\cdots i}(f,g)+\boldsymbol{a}_{n_1}(f)\smile\boldsymbol{a}_{n_2}(f,g)$$

3) *If* $n_{i+1}+1=n_1+n_{h+1}$ *in Theorem* 3.3., *then*

$$z^{n_{i+1}+1}(f)-z^{n_{i+1}+1}(g) = \boldsymbol{k}_{h+1\gamma}^i\boldsymbol{a}_{h+1\ \cdots i}(f,g)+\boldsymbol{a}_{n_1}(f)\smile\boldsymbol{a}_{n_{h+1}}(f,g).$$

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