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Correction to the paper "Some general properties of Markov processes"

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1. The proof of Theorem 2.1 on page 12 is incomplete. In fact, it asserts only the fact that (2.1) holds for any $\tilde{\mathfrak{B}}$-measurable function, while what we want to show is that (2.1) is true for any $\mathfrak{B}$-measurable function. For the completeness of the proof, the condition (P.3) on page 10 should be replaced by the following stronger condition:

(P.3)$'$ Let $\{\tau_n\}$ be any sequence of $\tilde{\mathfrak{B}}$-measurable Markov times*) increasing monotonely with P_x-probability 1 and τ_∞ denote $\lim_{n\to\infty} \tau_n$. Then there holds

$$(1) \qquad P_x(\lim_{n\to\infty} x_{\tau_n} = x_{\tau_\infty} \,|\, \tau_\infty < \sigma_\infty) = 1\,.$$

The first remark is that (P.3)$'$ implies (P.3). For this it is enough to show that the completion $\mathfrak{B}_x$ of $\tilde{\mathfrak{B}}$ relative to P_x contains $\mathfrak{B}$. We shall prove that, for the passage time τ for a closed subset F of E and for any $t \in [0, +\infty)$, the set $\{w\,;\, \tau < t\}$ belongs to $\mathfrak{B}_x$. Consider G_n a sequence of open subsets of E such that $G_n \supset F$ and $\bar{G}_n \downarrow F$. As was noted on page 10, the passage time τ_n for the set G_n is a $\tilde{\mathfrak{B}}$-measurable Markov time. It is evident that $\tau_\infty \leqq \tau$. If $\tau_\infty < \sigma_\infty$, $\lim_{n\to\infty} x_{\tau_n}$ exists and belongs to F (from (W.2)), and hence

$$(2) \qquad \{w\,;\, \tau_\infty < \sigma_\infty,\ \tau_\infty \neq \tau\} \subset \{w\,;\, \tau_\infty < \sigma_\infty,\ \lim_{n\to\infty} x_{\tau_n} \neq x_{\tau_\infty}\}\,.$$

Since ∞ is an isolated point, σ_∞ is $\tilde{\mathfrak{B}}$-measurable. Therefore the set of the right side in (2) belongs to $\tilde{\mathfrak{B}}$ and according to (1), has P_x-probability 0. Thus the set $\{\tau_\infty < \sigma_\infty,\ \tau_\infty \neq \tau\}$ is an element of $\mathfrak{B}_x$ with P_x-probability 0. On the other hand, if $\tau < t < +\infty$, $\tau_\infty < \sigma_\infty$. In fact, if $\tau_\infty \geqq \sigma_\infty$, $\tau \geqq \sigma_\infty$. But since $x_\tau \in F$ when τ is finite, this is impossible from (W.2) except for $\tau = +\infty$. Consequently we have

$$A = \{\tau < t\} = \{\tau_\infty < \sigma_\infty,\ \tau < t\} \subset \{\tau_\infty < \sigma_\infty,\ \tau_\infty < t\} = A' \in \tilde{\mathfrak{B}}_t\,,$$

and

$$A' - A \subset \{\tau_\infty < \sigma_\infty,\ \tau_\infty \neq \tau\}\,.$$

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*) A random time $\tau(w)$ is called a $\tilde{\mathfrak{B}}$-measurable Markov time if $(w\,;\, \tau(w) \geqq t) \in \tilde{\mathfrak{B}}_t$ for every $t \geqq 0$, where $\tilde{\mathfrak{B}}_t$ is the Borel field consisting of all the sets $(w\,;\, w_t^- \in B)$ for $B \in \tilde{\mathfrak{B}}$.

This means that $A \in \mathfrak{B}_x$.

The second remark is that (P.3)$'$ is also true for a sequence $\{\sigma_n\}$ of $\mathfrak{B}$-measurable Markov times. For this purpose it is enough to show that, for any $\mathfrak{B}$-measurable Markov time σ there exists a $\tilde{\mathfrak{B}}$-measurable Markov time τ which coincides with σ up to P_x-probability 0. This is shown by approximating σ by a sequence of Markov times taking only finitely many values, for which our statement is easily reduced to the first argument.

Proof of Theorem 2.1. Since $\mathfrak{B}$ is generated by $\tilde{\mathfrak{B}}$ and the passage time τ for any closed subset of E, it is enough to prove (2.1) when f is the indicator function $\chi_A(w)$ of a set A, where A is an element of $\tilde{\mathfrak{B}}$ or the set $\{\tau < t\}$.

(i) For χ_A, $A \in \tilde{\mathfrak{B}}$, we proved on page 12 as was remarked at the beginning of this note.

(ii) Define G_n, τ_n, τ_∞ as before and put $A_1 = \{\tau < t\}$ and $A_2 = \{\tau_\infty < \sigma_\infty, \tau_\infty < t\}$. From the first remark we have

$$E_y(\chi_{A_1}(w)) = E_y(\chi_{A_2}(w)) \quad \text{for every } y \in \bar{E}.$$

Therefore, for any Markov time σ and $B \in \mathfrak{B}_{\sigma+}$, we have

$$E_x(E_{x_\sigma}(\chi_{A_1}); B) = E_x(E_{x_\sigma}(\chi_{A_2}); B). \tag{3}$$

(iii) We shall show that

$$P_x(w; \chi_{A_1}(w_\sigma^+) = \chi_{A_2}(w_\sigma^+)) = 1. \tag{4}$$

Define $\theta(w) = \sigma(w) + \tau(w_\sigma^+)$, $\theta_n(w_\sigma^+) = \sigma(w) + \tau_n(w_\sigma^+)$ and $\theta_\infty(w) = \sigma(w) + \tau_\infty(w_\sigma^+)$. These random times are Markovian because of the property of Markov times to be proved by K. Itô and H. P. McKean [6]. Therefore, in the same way as before, $\{\theta_\infty < \sigma_\infty, \theta_\infty \neq \theta\}$ is a set of P_x-probability 0. Noting that $\sigma_\infty(w) = \sigma(w) + \sigma_\infty(w_\sigma^+)$ for $x_\sigma \in E$, we can see that

$$\begin{aligned}
\{w; \chi_{A_1}(w_\sigma^+) \neq \chi_{A_2}(w_\sigma^+)\} &= \{w; w_\sigma^+ \notin A_1, w_\sigma^+ \in A_2\} \\
&= \{w; \tau(w_\sigma^+) \geqq t, \tau_\infty(w_\sigma^+) < t, \tau_\infty(w_\sigma^+) < \sigma_\infty(w_\sigma^+)\} \\
&= \{w; \theta(w) \geqq t + \sigma(w), \theta_\infty(w) < t + \sigma(w), \theta_\infty(w) < \sigma_\infty(w)\} \\
&\subset \{w; \theta_\infty(w) < \sigma_\infty(w), \theta(w) \neq \theta_\infty(w)\}.
\end{aligned}$$

This implies (4).

(iv) It is clear from (4) that

$$E_x(\chi_{A_1}(w_\sigma^+); B) = E_x(\chi_{A_2}(w_\sigma^+); B). \tag{5}$$

Since $A_2 \in \tilde{\mathfrak{B}}$, it results from (i) that the right side of (3) is equal to the right of (5). Hence (2.1) is also true for χ_{A_1}. This completes the proof of our theorem.

2. The hypothesis (H.2) on page 20 seems to be too much stronger. In fact it is not satisfied even by the Brownian motion process. Hence we shall revise it in the following way. Let $\mathfrak{C}(A)$ denote the set of all bounded functions being continuous over A, where A is an open or closed subset of E.

(H.2) For any closed subset F of E, $h_F(x, \cdot)$ maps $\mathfrak{C}(F)$ into $\mathfrak{C}(E-F)$.

Under this hypothesis, the proof of the theorems in §4 is true with no change. Similarly, read $\mathfrak{C}(E-F)$ for $\mathfrak{C}$ in (H.2)$'$ on page 27.

3. Finally we shall list some trivial errors.

Page	Line	For	Read
12	↓11	E_t	E_x
12	↑ 3	$V\in y$	$V\ni y$
13	↓ 3	E_ξ	E_x
13	↑ 5	(1.12)	(2.12)
16	↓11	$P(t_0, x, E)$	$P(t, x, E)$
18	↑ 4	$=$	$\leqq$
19	↑12	$\sum_{y\ni A-T}$	$\sum_{y\in A-T}$
20	↓ 1	$P_x(x_t = y, \sigma_n \leqq t < \sigma_{n+1})$	$P_x(x_t = y, \sigma_n \leqq t < \sigma_{n+1}, \sigma_{A_c} > t)$
20	↓ 3	$\left(\int_0^t e^{-Pt_1} P\Pi_A dt \cdots dt_n\right)(x, y)$	$\left(\int_{t=t_1+\cdots+t_{n+1}} e^{-Pt_1} P\Pi_A \cdots e^{-Pt_{n+1}} dt_1 \cdots dt_n\right)(x, y)$
21	↓ 3	$\leqq$	$\geqq$
22	↑ 1	$G_\alpha v\chi(x)$	$G_\alpha \chi v(x)$
25	↑11	$S' \subset S \not\subset S'$	$S' \subset S \not\subset \bar{S}'$
26	↓ 3	A	A_x
26	↓ 7	$\int_0^\infty P(t, x, K)$	$\int_0^\infty P(t, x, K)dt$
28	footnote	though (H.2) is always satisfied.	dropped