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On homological theorems of algebras

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Recently Rosenberg and Zelinsky [7], [8] have obtained a sufficient condition that an algebra is of finite degree over a field K. We shall show, in section 1, a some generalized sufficient (and necessary) condition of [8], Corollary to Theorem 3. In section 2 we shall consider the dimension of $K[[x]]$. Auslander and Buchsbaum [1] has shown if R is commutative Noetherian, then $gl.\dim R[[x_1, \cdots, x_n]] = n + gl.\dim R$. We shall show, however, K-$\dim K[[x]] = \infty$ if K is an integral domain and that if further K is Noetherian, $K[[x]] \underset{K}{\otimes} K[[y]]$ can be identified with a proper sub-ring of $K[[x, y]]$.

1. THEOREM 1. *Let A be an algebra over a field K. If for the algebraic closure L_1 of K and the rational function L_2 in one indeterminant over K, $A \otimes L_i$'s satisfy the left minimum condition, then A is of finite degree over K and conversely.*

Proof. From the assumption A satisfies the left minimum condition, hence A has the nilpotent radical N $(N^n = 0)$ and $A/N = (D_1)_{n_1} \oplus \cdots \oplus (D_r)_{n_r}$ where $(D_i)_{n_i}$'s are the ring of total matrices over division rings D_i's respectively. If we can show $[D : K] < \infty$, then $[A/N : K] < \infty$ and since N^i/N^{i+1}'s are A/N-modules with finite composition length and $N^n = 0$, we can obtain $[A : K] < \infty$. Furthermore since $(D_i)_{n_i}$'s are homomorphic images of A, D_i's satisfy the same conditions in Theorem 1. Therefore we may assume A is a division ring. Let Z be the center of A. We shall show Z is of finite degree over K. If $\alpha (\in Z)$ is not algebraic over K, then $K(\alpha) \underset{K}{\otimes} K(\alpha)$ does not satisfy the minimum condition. But by the assumption $A \underset{K}{\otimes} K(\alpha)$ satisfies one and if $\{u_i\}$ is a right basis of A over $K(\alpha)$, then $\sum u_i \mathfrak{l}$ is a left ideal of $A \underset{K}{\otimes} K(\alpha)$ for any ideal $\mathfrak{l}$ of $K(\alpha) \otimes K(\alpha)$. Hence we have a contradiction. Therefore since Z is algebraic over K, regarding Z as a sub-field of the algebraic closure L of K, we obtain $Z \underset{K}{\otimes} Z$ satisfies the minimum condition as above. Hence by [8] Theorem 3 $[Z : K] < \infty$. Furthere for an indeterminant x over K we have $A \underset{K}{\otimes} Z(x) = (A \underset{K}{\otimes} K(x)) \underset{K(x)}{\otimes} Z(x)$ and observing $[Z(x) : K(x)] < \infty$ we can easily obtain that $A \underset{K}{\otimes} Z(x)$ satisfies the left minimum condition. Hence we may assume that A is a central division algebra. By the same argument as above we can show A is algebraic over K. If L is an algebraic closure of K, $A \underset{K}{\otimes} L = (D)_m$ where D is a division ring. Let α' be an element of A. Then as above $K(\alpha')\underset{K}{\otimes}L$ has a finite composition length for left ideals, not exceeding m. Let N' be the radical of $K(\alpha')\underset{K}{\otimes}L$. $(K(\alpha')\underset{K}{\otimes}L)/N'=L_1\oplus\cdots\oplus L_t$, $L_i\simeq L$. Therefore $[K(\alpha'):K]=[K(\alpha')\underset{K}{\otimes}L:L]=$the composition length for left ideals of $K(\alpha')\underset{K}{\otimes}L)$ $(\leqq m)$. Hence, since A is a algebraic algebra of bounded degree by [5] Theorem 16, A is locally finite. Let $\alpha\in D, \alpha=\sum a_i\otimes l_i, a_i\in A$ and $l_i\in L$, then $\alpha\in K(\cdots a_i\cdots)\underset{K}{\otimes}L$. Since $[K(\cdots a_i\cdots):K]<\infty$, $[L(\alpha):L]<\infty$, hence $\alpha\in L$ and $D=L$. Therefore $[A:K]=n$.

REMARK. We shall give examples which show that if we drop one of the assumptions in Theorem 1, A is not necessarily of finite degree.

Let A be the rational function field in x, then $A\underset{K}{\otimes}L=L(x)$ satisfies the left minimum condition for the algebraic closure L of K.

If A is an algebraic field with $[A:K]=\infty$ and L a finitely generated field over K; $L=K(y_1,\cdots,y_s,z_1,\cdots,z_t)$ where y_i's are independent indeterminantes over K and z_i's are algebraic over $K(y_1,\cdots,y_s)$, then $A\underset{K}{\otimes}L=(A\underset{K}{\otimes}K(y_1,\cdots,y_s))\underset{K(y_1,\cdots,y_s)}{\otimes}L=A(y_1,\cdots,y_s)\underset{K(y_1,\cdots,y_s)}{\otimes}L$ satisfies the minimum condition.

It is clear that Theorem 1 implies Corallary to Theorem 3 of [8]. We shall show that Theorem 1 implies Theorem 1 of [2] in the special case where K is a field.

If K-dim $A=0$, L-dim $A\underset{K}{\otimes}L=0$ for any field $L(\supseteqq K)$. Since L-dim $A\underset{K}{\otimes}L\geqq l.gl.\dim A\underset{K}{\otimes}L$, $A\underset{K}{\otimes}L$ satisfies minimum conditions, hence $[A:K]<\infty$.

2. We shall consider the dimension of the ring of formal power series in one variable x.

THEOREM 2. *Let K be an integral domain. Then*

$$\dim K[[x]]=w.\dim K[[x]]=\infty.$$

Proof. Let Q and Q' be the rings of quotients of K and $K[[x]]$ respectively. Then from [3] Theorem 5 we have

$$K\text{-}\dim K[[x]]\geqq K\text{-}\dim Q'.$$

From Lemma 1 (below) there exist n algebraic independent element y_i's for any integer n, hence we have

$$K\text{-}\dim K[[x]]\geqq K\text{-}\dim Q'\geqq K\text{-}\dim Q(y_1,\cdots,y_n).$$

If we can prove $Q(y_1,\cdots,y_n)\underset{Q}{\otimes}Q(y_1,\cdots,y_n)\approx Q(y_1,\cdots,y_n)\underset{K}{\otimes}Q(y_1,\cdots,y_n)$, since any $Q(y_1,\cdots,y_n)$ module as Q-algebra is a $Q(y_1,\cdots,y_n)$ module as K-algebra, we obtain

$$K\text{-}\dim Q(y_1,\cdots,y_n)\geqq Q\text{-}\dim(y_1,\cdots,y_n)=n$$

by [7] Theorem 7 and observing the standard complex of $Q(y_1,\cdots,y_n)$ as

Q-algebra and K-algebra respectively. Since we have a natural epimorphism $\varphi : Q(y_1, \cdots, y_n) \underset{K}{\otimes} Q(y_1, \cdots, y_n) \to Q(y_1, \cdots, y_n) \underset{Q}{\otimes} Q(y_1, \cdots, y_n)$, it is enough to prove that φ is monomorphism. For the sake of brief we shall prove it in the case $n=1$. Let

$$\sum_{i=1}^{m} \frac{f_i(x)}{g_i(x)} \underset{Q}{\otimes} \frac{t_i(x)}{h_i(x)} = \frac{\sum_{i=1}^{m} f_i'(x) \otimes t_i'(x)}{\Pi\, g_i(x) \otimes h_i(x)} = 0$$

where f_i, g_i, t_i, h_i, f_i', and t_i' are in $K[x]$. If $f_1', \cdots, f_n'$ are linearly independent over Q and

$$f_{n+j}'(x) = \sum_{s=1}^{n} f_s'(x)\, a_{s,j}, \qquad a_{s,j} \in Q,$$

then we have

$$t_s'(x) + \sum_j a_{s,j} t_{n+j}(x) = 0 \qquad \text{for all } s.$$

Hence if we take $\mu (\neq 0)$ in K such that $\mu \cdot a_{s,j} \in K$ for all s, then

$$\begin{aligned}
\sum \frac{f_i(x)}{g_i(x)} \underset{K}{\otimes} \frac{t_i(x)}{h_i(x)} &= \frac{1}{\Pi\, g_i(x) \otimes h_i(x)} \left(\sum f_i'(x) \underset{K}{\otimes} t_i'(x)\right) \\
&= \frac{1}{(\mu \otimes 1)(\Pi\, g_i(x) \otimes h_i(x))} \left(\sum \mu \cdot f_i'(x) \underset{K}{\otimes} t_i'(x)\right) \\
&= \frac{1}{(\mu \otimes 1)(\Pi\, g_i(x) \otimes h_i(x))} \left(\sum_{s=1}^{n} f_s'(x) (\mu \cdot t_s'(x) + \sum a_{s,j} \cdot \mu \cdot t_{n+j}(x)\right) \\
&= 0.
\end{aligned}$$

By the same method as above and [4] we have the theorem for weak dimension.

LEMMA 1. *Let K be a commutative ring. Then $K[[x]]$ has infinite many mutually algebraic independent elements over K.*

If K is a field with cardinal number $\leqq \aleph_0$, we can prove this by the method of Cantor. We shall prove Lemma 1 by the elementary calculation.

Proof of Lemma 1. We shall show that $y_0 = \sum_n x^{n^n}$ is algebraic independent over $K[x]$. Assume that y_0 is algebraic over $K[x]$. Then there exists a non zero polynomial such that

$$(*) \qquad f_0(x) + f_1(x) y_0 + \cdots + f_m(x) y_0^m = 0, \quad f_m(x) \neq 0, \quad f_i(x) \in K[x].$$

Let N be the highest degree of $f_i(x)$'s. We can find n such that $mn^n < (n+1)^{n+1}$ and $n^n - (n-1)^{n-1} > N$. If $f_m(x) = a_0 + a_1 x + \cdots + a_t x^t$, $a_i \in K$, $a_t \neq 0$, then the coefficient of x of degree $\{(m-1)n^n + (n-1)^{n-1} + t\}$ in the left side of (*) is not zero. Hence we have a contradiction. Let $K_1 = K[x]$ [1], then $y_1 = \sum y_0^{n^n}$ is algebraic independent over $K_1[y_0]$ as above. Hence we can prove Lemma 1 by induction.

1) My first proof was more complicated, but this method was kindly pointed out to me by Prof. Asano and Dr. Nobusawa.

Next we shall study the identification of $K[[x]]\otimes K[[y]]$ with a sub-ring of $K[[x, y]]$.

LEMMA 2. *Let K be a field and* $K[[x]]$ *the ring of formal power series in x. Then*

$$0 \longrightarrow K[[x]]\underset{K}{\otimes}K[[y]] \xrightarrow{\varphi} K[[x, y]]$$

is exact, where $\varphi(\sum a_i x^i \otimes \sum b_j y^j) = \sum a_i b_j x^i y^j$, a_i, $b_j \in K$.

It is clear.

LEMMA 3. *Let K be a Noetherian and commutative integral ring. Then*

$$0 \longrightarrow K[[x]]\underset{K}{\otimes}K[[y]] \xrightarrow{\varphi} K[[x, y]]$$

is exact, where φ *is the same as in Lemma 2.*

We can easily prove Lemma 3 by using the same method as in the proof of Theorem 2 and the fact that $w.\dim K[[x]]=0$, (see [2] II Exer. 2).

THEOREM 3. *Let K be a Noetherian and commutative integral ring. Then*

$$0 \longrightarrow K[[x]]\underset{K}{\otimes}K[[y]] \xrightarrow{\varphi} K[[x, y]]$$

is exact, but not epimorphic, where $\varphi(\sum a_i x^i \otimes \sum b_j y^j) = \sum a_i b_i x^i y^j$.

We can easily show that $\sum_i x^i y^i$ is not contained in $\varphi(K[[x]]\otimes K[[y]])$. For if $\sum x^i y^i = \varphi(\sum_k f_k(x) \otimes g_k(y))$, $f_k(x) = \sum_i a_{k,i} x^i$, $g_k(y) = \sum_i b_{k,i} y^i$, then $\sum_{k=1}^{m} a_{k,i}\cdot b_{k,j} = \delta_{i,j}$ for all i, j ($\delta_{i,j}$ Kronecker delta). From the well known theorem we have no solutions of them.

REMARK. If K is a field, $\mathfrak{m} = (x)\otimes K[[y]]+K[[x]]\otimes(y)$ is a maximal ideal of $K[[x]]\otimes K[[y]]$ and $\bigcap_n \mathfrak{m}^n = \bigcap_n ((x)^{[\frac{n}{2}]}$[2]$\otimes K[[y]]+K[[x]]\otimes(y)^{[\frac{n}{2}]})$. By Lemma 2 the intersection is mapped isomorphically into $\bigcap_n (x, y)^{[\frac{n}{2}]}$ by φ. By the well known theorem (cf. [6] III Th. 3) we have $\bigcap_n (x, y)^{[\frac{n}{2}]}=0$, hence $\bigcap_n \mathfrak{m}^n=0$. We can easily see by observing an element $\sum_{i=1}^{n} x^i\cdot y^i$ that $K[[x]]\otimes K[[y]]$ is neither a local ring nor a complete ring in the topological space induced by $\mathfrak{m}$. Therefore in general $K[[x]]\otimes K[[y]]$ is neither ring-isomorphic nor homeomorphic onto $K[[x, y]]$ in this sense.

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2) $[\frac{n}{2}]$ means the largest integer $\leqq \frac{n}{2}$.

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