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On $GL(2, K[x])$

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1. It was pointed out by D. Livingstone that the unit group of the total matrix ring of degree n over a polynomial ring $K[x]$ (denoted by $GL(n, K[x])$) is finitely generated if $n \geq 3$ and K is a finite field. In this note, we shall first prove that Livingstone's result can not be extended to the case $n=2$. Namely, we shall prove

THEOREM 1[1]. *$GL(2, K[x])$ is not finitely generated if K is a field*[2].

This theorem is an immediate consequence of the following theorem, in which a presentation of $GL(2, K[x])$ will be determined.

THEOREM 2. *Let A be the subgroup of $GL(2, K[x])$ consisting of the matrices of the form*

$$T(f) = \begin{pmatrix} 1 & f(x) \\ 0 & 1 \end{pmatrix}$$

with $f(x)$ satisfying $f(0)=0$, and denote by $T(\alpha, \beta, \gamma)$ the matrix

$$\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix},$$

where α, β and γ are elements in K. Then $GL(2, K[x])$ is isomorphic to the factor group of the free product of $GL(2, K)$ and A by the normal subgroup generated by

$$T(\alpha, \beta, \gamma)\, T(f)\, T(\alpha, \beta, \gamma)^{-1}\, T\left(-\frac{\alpha}{\beta} f\right).$$

Next, we shall prove the following theorem 3, by the same argument as in the proof of theorem 2, and then we will apply theorem 3 to the problems discussed in the joint paper [1] of Chang, Jennings and Ree.

THEOREM 3. *Let A be the subgroup as in theorem 2, and B be the subgroup consisting of the matrices of the form*

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1) This question was suggested to me by Professor D. G. Higman and Professor J. E. McLaughlin. I express here my hearty thanks to them. Also I would like to express my gratitude to Professor R. M. Thrall who gave me the opportunity to work at the University of Michigan during 1958–1959.

2) If K is an infinite field, then the theorem is trivial. In the following theorem K is assumed to be any (commutative) field.

$$S(g)=\begin{pmatrix}1 & 0\\ g(x) & 1\end{pmatrix}$$

with $g(x)$ any polynomial. Then the subgroup generated by A and B is isomorphic to the free product of A and B.

In [1], it was proved that

$$\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & 0\\ x & 1\end{pmatrix}$$

generate a free group if x is a transcendental complex number. This result can be generalized in the following way.

THEOREM 4. *Let K be a field of characteristic p and x be a transcendental element over K. Then for any $\alpha\neq 0$ in K,*

$$\begin{pmatrix}1 & 0\\ \alpha & 1\end{pmatrix} \text{ and } \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}$$

generate the free product of two cyclic groups of order p. (Here a cyclic group of order 0 means an infinite cyclic group.)

In [1], the authors also showed that the free product of two free abelian groups each of which has rank 2 has an isomorphic matrix representation of degree 2 over the complex number field, but they were not able to obtain a matrix representation of free products of free abelian groups whose ranks are greater than 2. The following theorem will give an answer to this problem.

THEOREM 5. *If A and B are two free abelian groups or any elementary abelian groups with the same exponent p, the free product of A and B has an isomorphic matrix representation of degree 2 over a suitable field.*

2. To establish the theorems, we shall mention two evident facts as lemmas.

LEMMA 1. *Let*

$$\begin{pmatrix}h_{11}(x) & h_{12}(x)\\ h_{21}(x) & h_{22}(x)\end{pmatrix}\begin{pmatrix}f(x) & \alpha\\ \beta & 0\end{pmatrix}=\begin{pmatrix}h'_{11}(x) & h'_{12}(x)\\ h'_{21}(x) & h'_{22}(x)\end{pmatrix}.$$

If $\deg h_{11}(x)\geq \deg h_{12}(x)$, $\deg h_{11}(x)\geq n$ and $\deg f(x)>0$, then $\deg h'_{11}(x)>\deg h'_{12}(x)$ and $\deg h'_{11}(x)>n$.

Proof. This is immediate from

$$h'_{11}(x)=h_{11}(x)f(x)+\beta h_{12}(x),$$
$$h'_{12}(x)=\alpha h_{11}(x).$$

LEMMA 2. *Let*

$$\begin{pmatrix}h_{11}(x) & h_{12}(x)\\ h_{21}(x) & h_{22}(x)\end{pmatrix}\begin{pmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{pmatrix}=\begin{pmatrix}h'_{11}(x) & h'_{12}(x)\\ h'_{21}(x) & h'_{22}(x)\end{pmatrix}.$$

If $\deg h_{11}(x) > \deg h_{12}(x)$, $\deg h_{11}(x) \geqq n$ *and* $\alpha_{11} \neq 0$, *then* $\deg h'_{11}(x) \geqq \deg h'_{12}(x)$ *and* $\deg h'_{11}(x) \geqq n$.

Proof. This is evident from

$$h'_{11}(x) = \alpha_{11} h_{11}(x) + \alpha_{21} h_{12}(x)\ ,$$
$$h'_{12}(x) = \alpha_{12} h_{11}(x) + \alpha_{22} h_{12}(x)\ .$$

Proof of Theorem 2. Denote by W the matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

anb by $T(\alpha, \beta, \gamma)$ a triangular matrix

$$\begin{pmatrix} \alpha & \gamma \\ 0 & \beta \end{pmatrix}.$$

Any constant matrix C can be written as a product of the form

(1) $\qquad WT(\alpha, \beta, \gamma)$

(2) or $\qquad WT(\alpha_1, \beta_1, \gamma_1)\, WT(\alpha_2, \beta_2, \gamma_2)$

acccording as the $(1, 1)$ element of C is equal to 0 or not. Namely

$$\begin{pmatrix} 0 & \gamma \\ \delta & \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \delta & \beta \\ 0 & \gamma \end{pmatrix}$$

and

$$\begin{pmatrix} \alpha & \gamma \\ \delta & \beta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \beta - \frac{\gamma\delta}{\alpha} & \delta \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{\gamma}{\alpha} \\ 0 & 1 \end{pmatrix}$$

if $\alpha \neq 0$. If the $(1, 1)$ element of $T(\alpha_1, \beta_1, \gamma_1)\ W$ in (2), namely γ_1, is 0, then

$$T(\alpha_1, \beta_1, 0)\, W = WT(\beta_1, \alpha_1, 0)\ .$$

Therefore $C = T(\beta_1, \alpha_1, 0)\ T(\alpha_2, \beta_2, \gamma_2)$ is a triangular matrix.

Denote by E the identity matrix and suppose that a relation between elements in $GL(2, K)$ and elements in A

(3) $\qquad T(f_1)\, C_1\ T(f_2)\, C_2 \cdots T(f_r)\, C_r = E$

with $C_i \neq E$ $(i = 1, 2, \cdots, r-1)$ is given. We shall first show that some of C_i, $i = 1, 2, \cdots, r-1$ must be triangular matrices of the form $T(\alpha, \beta, \gamma)$. Suppose that any C_i $(1 \leqq i < r)$ is not a triangular matrix, and denote by R_j a matrix of the form

$$\begin{pmatrix} f(x) & \alpha \\ \beta & 0 \end{pmatrix} = \begin{pmatrix} \alpha & f(x) \\ 0 & \beta \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

where $f(x)$ may be a constant or a polynomial of degree greater than 0. Then from the remark above it is easy to see that (3) can be written as

(4) $\qquad R_1 R_2 \cdots R_s = C\ ,$

where

$$R_1 = T(f_1)\, W = \begin{pmatrix} f_1(x) & 1 \\ 1 & 0 \end{pmatrix},$$

C is some constant matrix, and if R_i is constant its $(1, 1)$ element is not 0. Applying lemmas 1 and 2 repeatedly we see that the $(1, 1)$ element of the left hand side in (4) has a degree greater than 0. This is a contradiction. Thus some C_i $(1 \leq i < r)$ is a triangular matrix $T(\alpha, \beta, \gamma)$. Then using the relation

$$(5) \qquad T(\alpha, \beta, \gamma)\ T(f_{i+1})\ T(\alpha, \beta, \gamma)^{-1}\ T\left(-\frac{\alpha}{\beta} f_{i+1}\right) = E$$

we have a relation

$$T(f_1)\, C_1 \cdots C_{i-1}\, T\left(f_i + \frac{\alpha}{\beta} f_{i+1}\right)\ (C_i C_{i+1}) \cdots C_r = E\,.$$

In this way we can make the length of the relation shorter. Thus by using relations such as (5) the relation (3) can be reduced to a relation of the form

$$T(f_1)\, C_1\ T(f_2)\, C_2 = E$$

and C_1 must be a triangular metrix. Then C_2 is also a triangular matrix and, as is easily seen, this is a relation of the form (5). This shows that relations between the elements in $GL(2, K)$ and the elements in A are generated by the relations of the form (5).

Proof of Theorem 1. $GL(2, K[x])$ is generated by constant matrices and matrices of the form

$$T(x^r) = \begin{pmatrix} 1 & x^r \\ 0 & 1 \end{pmatrix}, \quad r = 1, 2, \cdots\cdots.$$

If $GL(2, K[x])$ is finitely generated, then there is an n such that $GL(2, K[x])$ is generated by certain constant matrices and matrices $T(x^r)$, $r = 1, 2, \cdots, n-1$. Since $T(x^n)$ must be a product of constant matrices and matrices $T(x^r)$, $r = 1, 2, \cdots, n-1$, we have a relation

$$T(f_1)\, C_1\ T(f_2)\, C_2 \cdots T(f_t)\, C_t = E\,.$$

such that $f_i(0) = 0$ for $1 \leq i \leq t$, $deg\, f_1 = n$, $0 < deg\, f_i < n$ for $i \neq 1$ and $C_i \neq E$ for $i \neq t$. On the other hand, it is an immediate consequence of theorem 2 that the number of $T(f_i)$ with $deg\, f_i = n$ must be even. This is a contradiction.

Proof of Theorem 3. Suppose that there is a relation of the form

$$T(f_1)\ S(g_1)\ T(f_2)\ S(g_2) \cdots T(f_r)\ S(g_r) = E\,,$$

where $g_i \neq 0$ for $i = 1, 2, \cdots, r-1$. Denote by $R(f)$ the matrix

$$\begin{pmatrix} f(x) & 1 \\ 1 & 0 \end{pmatrix} = T(f)\, W\,.$$

Since $S(g_i) = WT(g_i)\, W$, we have

$$R(f_1)\ R(g_1) \cdots R(f_r)\ R(g_r) = E\,.$$

Since $g_i \neq 0$ for $1 \leq i < r$, we see, by applying lemmas 1 and 2 repeatedly, that the $(1, 1)$ element or the $(1, 2)$ element of the left hand side has a degree greater than 0 according as $g_r \neq 0$ or $g_r = 0$.

Proof of Theorems 4 and 5. Theorem 4 is immediate from theorem 3, and theorem 5 is also easily seen from theorem 5 and the fact that A and B are both isomorphic to the additive group of $K[x]$.

Reference

[1] B. Chang, S. A. Jennings and R. Ree: On certain pairs of matrices which generate free groups, Can. J. of Math. 10 (1958), 279-284.

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