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On coverings and continuous fuctions

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Since the publishing of the paper [14] by A. H. Stone locally finite coverings have played an increasingly important role in studies of topological spaces. A. H. Stone [14], M. Katětov [5], R. H. Bing [2], C. H. Dowker [3], K. Morita [8], [9], E. Michael [6], [7], K. Nagami [11] and the others have established relations between locally finite coverings, point-finite coverings, paracompactness, full normality and the other normalities. The investigations of metrizability by means of locally finite coverings and of similar coverings have been made by Yu. Smirnov [13], R. H. Bing [2], K. Morita [10] and the author [12].

The purpose of this paper is to study relations between continuous functions and locally finite coverings of topological spaces. Using families of continuous functions, we shall give a necessary and sufficient condition for a T_2-space to be fully normal in an analogous form to Urysohn's lemma and shall give necessary and sufficient conditions for metrizability. Furthermore, we shall generalize Hausdorff's theorem for continuous functions by using coverings.

1. Full normality

LEMMA 1. *Let R be a topological space and let $V_\alpha=\{x|f_\alpha(x)>0\}$ $(\alpha<\tau)$*[1] *for real valued continuous functions f_α on R. If $\mathfrak{V}=\{V_\alpha|\alpha<\tau\}$ covers R, and if $\underset{\beta<\alpha}{\cup} f_\beta(x)$ is continuous for every $\alpha\leqq\tau$, then $\mathfrak{V}$ has a locally finite refinement.*[2]

PROOF. Let $V_{1\alpha}=\left\{x|f_\alpha(x)>\frac{1}{2}\right\}$ and $V_{n\alpha}=\left\{x|f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}\right\}$ $(n\geqq 2)$, then obviously $\bar{V}_{i\alpha}\subseteq V_{i+1\alpha}\subseteq V_\alpha (i=1, 2, \cdots; \alpha<\tau)$. From these $V_{n\alpha}$ we define $N_{n\alpha}$ by $N_{n1}=V_{n1}$, $N_{n\alpha}=V_{n\alpha}-\overline{\underset{\beta<\alpha}{\cup} V_{n+1\beta}}$ $(1<\alpha<\tau)$. Then it is easily seen that $\{N_{n\alpha}|n=1,2\ldots; \alpha<\tau\}$ covers R. For $x\in V_1$ implies $x\in V_{n1}=N_{n1}$ for some n, and $x\in V_\alpha$, $x\notin V_\beta$ $(\beta<\alpha)$, $1<\alpha<\tau$ imply $x\in V_{n\alpha}$ for some n and $\underset{\beta<\alpha}{\cup} f_\beta(x)\leqq 0$. Since $\underset{\beta<\alpha}{\cup} f_\beta$ is continuous from the assumption of the proposition, there exists a nbd

The content of this paper is the detail of our note published in Proc. of Japan Acad., Vol. 31, No. 10 (1955). Notions and notations but recent ones in this paper are due chiefly to J. W. Tukey [15].

1) α, β, γ, τ denote ordinals in this lemma.

2) In this note coverings and refinements are open but in the proof of Lemma 2.
$\underset{\beta<\alpha}{\cup} f_\beta$ denotes the function sup $\{f_\beta(x)\mid\beta<\alpha\}(x\in R)$. $\underset{\beta<\alpha}{\cup} f_\beta(x)$ denotes the value of this function at x.

(=neighbourhood) $U(x)$ of x such that $\bigcup_{\beta<\alpha} f_\beta(U(x)) < \frac{1}{2}-\frac{1}{2^2}\cdots-\frac{1}{2^{n+1}}$ [3] and consequently $U(x)\cap(\bigcup_{\beta<\alpha} V_{n+1\beta})=\phi$. Hence $x\notin\overline{\bigcup_{\beta<\alpha} V_{n+1\beta}}$, and hence $x\in N_{n\alpha}$.

Next, we shall show $\{N_{n\alpha}|\alpha<\tau\}$ is locally finite for a fixed n. Let $V_\alpha' = \left\{x|f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{2\cdot 2^{n+1}}\right\}$, then obviously $V_\alpha'\subseteqq V_{n+1\alpha}$. If $x\in V_\alpha'$, $x\notin V_\beta'(\beta<\alpha)$, $\alpha\leqq\tau$, then $\bigcup_{\beta<\alpha} f_\beta(x)\leqq\frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2\cdot 2^{n+1}}$. Since $\bigcup_{\beta<\alpha} f_\beta$ is continuous, there exists a nbd $V(x)$ of x such that $V(x)\cap V_{n\beta}=\phi$ $(\beta<\alpha)$. Moreover, $x\in V_{n+1\alpha}$ and $V_{n+1\alpha}\cap N_{n\gamma}=\phi$ $(\gamma>\alpha)$. Hence there exists a nobd of x intersecting at most one of $N_{n\alpha}$ $(\alpha<\tau)$. Therefore $F_n=\bigcup_{\alpha<\tau}\overline{N}_{n\alpha}$ is closed for every n. Put $V'_{n\alpha}=\left\{x|f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{3\cdot 2^{n+1}}\right\}$, $V''_{n\alpha}=\left\{x|f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{2}{3\cdot 2^{n+1}}\right\}$ and put $M_{n1}=V'_{n1}$, $M_{n\alpha}=V'_{n\alpha}-\overline{\bigcup_{\beta<\alpha} V''_{n\beta}}$ $(1<\alpha<\tau)$, then $\overline{N}_{n\alpha}\subseteqq M_{n\alpha}$ for very n, $\alpha<\tau$. $\overline{N}_{n1}\subseteqq M_{n1}$ is obvious. If $n\geqq 2$, $x\notin M_{n\alpha}$, then since $\overline{N}_{n\alpha}\subseteqq\overline{V}_{n\alpha}\subseteqq V'_{n\alpha}$, $x\notin V'_{n\alpha}$ implies $x\notin\overline{N}_{n\alpha}$. Since $x\notin\bigcup_{\beta<\alpha} V_{n+1\beta}$ implies $\bigcup_{\beta<\alpha} f_\beta(x)\leqq\frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2^{n+1}}$ and consequently $\bigcup_{\beta<\alpha} f_\beta(U(x))\leqq\frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{3\cdot 2^{n+1}}$, *i.e.* $U(x)\cap(\bigcup_{\beta<\alpha} V''_{n\beta})=\phi$ for some nbd $U(x)$ of x, it holds $x\notin\overline{\bigcup_{\beta<\alpha} V''_{n\beta}}$. Hence $x\notin\overline{\bigcup_{\beta<\alpha} V''_{n\beta}}$ implies $x\notin\bigcup_{\beta<\alpha} V_{n+1}\subseteqq N^c_{n\alpha}$ [4] and $x\notin\overline{N}_{n\alpha}$. Thus we conclude $\overline{N}_{n\alpha}\subseteqq M_{n\alpha}$.

Now we denote $W_{1\alpha}=M_{1\alpha}$, $W_{n\alpha}=M_{n\alpha}-\bigcup_{i=1}^{n-1}F_i$ $(n\geqq 2)$. Then $\mathfrak{W}=\{W_{n\alpha}|n=1,2\cdots;\ \alpha<\tau\}$ is a locally finite refinement of $\mathfrak{V}$. Firstly, we prove that $\mathfrak{W}$ covers R. Since $\cup\{\overline{N}_{n\alpha}|n=1,2\cdots;\ \alpha<\tau\}=R$, for every $x\in R$ there exists n such that $x\in\overline{N}_{n\alpha}$ for some $\alpha<\tau$ and $x\notin\overline{N}_{m\beta}$ $(m<n,\ \beta<\tau)$. From $\overline{N}_{n\alpha}\subseteqq M_{n\alpha}$ we get $x\in M_{n\alpha}$ and $x\notin\bigcup_{i=1}^{n-1}F_i$, and hence $x\in W_{n\alpha}$. Since $\mathfrak{W}<\mathfrak{V}$ is obvious, we show lastly that $\mathfrak{W}$ is locally finite. If $x\in N_{k\alpha}\subseteqq F_k$, then $N_{k\alpha}\cap W_{m\beta}=\phi$ $(m>k,\ \beta<\tau)$. Next, we denote $V_\alpha'=\left\{x|f_\alpha(x)>\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^n}-\frac{1}{2\cdot 2^{n+1}}\right\}(\alpha<\tau)$ for a fixed $n\leqq k$. If $x\in V_\alpha'$ and $x\notin V_\beta'$ $(\beta<\alpha)$, $\alpha\leqq\tau$, then since $\bigcup_{\beta<\alpha} f_\beta(x)\leqq\frac{1}{2}-\cdots-\frac{1}{2^n}-\frac{1}{2\cdot 2^{n+1}}$, there exists a nbd $V(x)$ of x such that $V(x)\cap V'_{n\beta}=\phi$ $(\beta<\alpha)$. Hence $V(x)\cap M_{n\beta}=\phi$ and consequently $V(x)\cap W_{n\beta}=\phi$ $(\beta<\alpha)$. Moreover, $x\in V''_{n\alpha}$ and $V''_{n\alpha}\cap M_{n\gamma}=\phi(\gamma>\alpha)$. Therefore there exists a nbd $V_n(x)$ of x intersecting at most one of $M_{n\alpha}$ $(\alpha<\tau)$. Hence the nbd $\bigcap_{n=1}^{k} V_n(x)\cap N_{k\alpha}$ of x intersects only finitely many $W_{n\alpha}$.

In fact, we have no need to assume that $\mathfrak{V}$ covers R, that is to say

COROLLARY 1. *Let R be a topological space and let* $V_\alpha=\{x|f_\alpha(x)>0\}\,(\alpha<\tau)$

3) $f(U)\leqq k$ means $f(x)\leqq k$ for every $x\in U$.

4) We denote by N^c or $C(N)$ the complement of N.

for real valued continuous functions f_α on R. If $\bigcup_{\beta<\alpha} f_\beta$ is continuous for every $\alpha \leqq \tau$, then there exists a locally finite collection $\mathfrak{W}=\{W_\gamma|\gamma \in C\}$ of open sets such that $W_\gamma \subseteqq V_\alpha$ for every $\gamma \in C$ and some $\alpha \in A$, and such that $\bigcup_{\gamma \in C} W_\gamma = \bigcup_{\alpha \in A} V_\alpha$.

Considering A. H. Stone's theorem[5] "full normality and paracompactness are equivalent for T_2-spaces", we get the following theorems from this lemma.

THEOREM 1. *In order that a T_2-space R is fully normal or paracompact it is necessary and sufficient that for every open covering $\{V_\alpha|\alpha \in A\}$, there exists a family $\{f_\alpha|\alpha \in A\}$ of real valued functions on R such that $f_\alpha(V_\alpha^c)=0$, $\bigcup_{\alpha \in A} f_\alpha=1$, $\bigcup_{\beta \in B} f_\beta$ is continuous for every $B \subseteqq A$.*

Proof. Sufficiency is directly deduced from the lemma.

Conversely, if $\mathfrak{V}=\{V_\alpha|\alpha \in A\}$ is an open covering of a fully normal T_2-space R, then there exist a locally finite refinement $\mathfrak{U}=\{U_\beta|\beta\}$ of $\mathfrak{V}$ and a covering $\mathfrak{W}=\{W_\beta|\beta \in B\}$ such that $\overline{W}_\beta \subseteqq U_\beta$ $(\beta \in B)$. Defining continuous functions $g_\beta(\beta \in B)$ on R such that $g_\beta(W_\beta)=1$, $g_\beta(U_\beta^c)=0$ and $0 \leqq f_\beta \leqq 1$, from the locally finiteness of $\mathfrak{U}$ we see obviously that $\bigcup_{\gamma \in C} g_\gamma$ is continuous for every $C \subseteqq B$. If we put $f_\alpha = \cup\{g_\beta|U_\beta \subseteqq V_\alpha\}$, then $\bigcup_{\alpha \in A} f_\alpha=1$ from $\mathfrak{U}<\mathfrak{V}$, and the continuity of $\bigcup_{\alpha \in A'} f_\alpha$ $(A' \subseteqq A)$ is deduced from the continuity of $\bigcup_{\gamma \in C} g_\gamma$. Since $f_\alpha(V_\alpha^c)=0$ is obvious, the necessity is proved.

COROLLARY 2. *In order that a completely regular space R is fully normal or paracompact it is necessary and sufficient that if $\{\varphi_\alpha|\alpha \in A\}$ is a family of real valued continuous functions on R such that $\bigcup_{\alpha \in A}\varphi_\alpha$ is continuous, then for every $\varepsilon>0$ there exists a family $\{f_\alpha|\alpha \in A\}$ of functions on R such that $f_\alpha \leqq \varphi_\alpha$ $(\alpha \in A)$, $|\bigcup_{\alpha \in A} f_\alpha - \bigcup_{\alpha \in A}\varphi_\alpha| \leqq \varepsilon$, and $\bigcup_{\beta \in B} f_\beta$ is continuous for every $B \subseteqq A$.*

Proof. Necessity. If $\bigcup_{\alpha \in A}\varphi_\alpha=\varphi$ is continuous on a fully normal T_1-space R, then for a given $\varepsilon>0$ we put $V_\alpha=\{x|\varphi_\alpha(x)>\varphi(x)-\varepsilon\}$ $(\alpha \in A)$. Since $\mathfrak{V}=\{V_\alpha|\alpha \in A\}$ is an open covering of R, we may choose locally finite coverings $\mathfrak{U}=\{U_\beta|\beta \in B\}$, $\mathfrak{W}=\{W_\beta|\beta \in B\}$ of R such that $\mathfrak{U}<\mathfrak{V}$, $\overline{W}_\beta \subseteqq U_\beta$ $(\beta \in B)$. Define continuous functions g_β for every $\beta \in B$ such that $g_\beta(x)=\varphi(x)-\varepsilon$ $(x \in W_\beta)$, $g_\beta(U_\beta^c)=-\infty$, $g_\beta(x) \leqq \varphi(x)-\varepsilon (x \in R)$, then $g_\beta \leqq \varphi_\alpha$ $(U_\beta \subseteqq V_\alpha)$, $\bigcup_{\beta \in B} g_\beta=\varphi(x)-\varepsilon$, and $\bigcup_{\gamma \in C} g_\gamma$ is obviously continuous for every $C \subseteqq B$. $f_\alpha=\cup\{g_\beta|U_\alpha \subseteqq V_\alpha\}$ $(\alpha \in A)$ have all the necessasry properties.

Sufficiency. Let $\{V_\alpha|\alpha \in A\}$ is an arbitrary open covering of R, then for every $x \in R$ there exist $V_\alpha \ni x$ and a continuous function φ_x such that $\varphi_x(x)=1$, $\varphi_x(V_\alpha^c)=0$, $0 \leqq \varphi_x \leqq 1$.

Since $\bigcup_{x \in R}\varphi_x=1$, there exists a family $\{f_x|x \in R\}$ of continuous functions on R such that $f_x \leqq \varphi_x$ $(x \in R)$, $|\bigcup_{x \in R} f_x - 1| \leqq \frac{1}{2}$, and $\bigcup_{x \in S} f_x$ is continuous for every $S \subseteqq R$. Hence $U_x=\{y|f_x(y)>0\} \subseteqq \{y|\varphi_x(y)>0\} \subseteqq V_\alpha$ for some $\alpha \in A$, and $\{U_x|x \in R\}$ covers

5) See [14].

R. Since from the lemma $\{V_\alpha | \alpha \in A\}$ has a locally finite refinement, we conclude the full normality of R.

Furthermore, we give three already known theorems as direct consequences of Theorem 1 and of Lemma 1.

COROLLARY 3. (*E. Michael*)[6] *A regular space R is paracompact if and only if every open covering of R has an open refinement $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$, where each $\mathfrak{V}_n$ is a locally finite collection of open sets.*

COROLLARY 4. (*K. Nagami*)[7] *Let R be a topological space and $V_n = \{x | f_n(x) > 0\}$ $(n=1,2,\cdots)$, where f_n are real-valued continuous functions on R. If $\mathfrak{V} = \{V_n | n = 1,2,\cdots\}$ covers R, then $\mathfrak{V}$ has a locally finite refinement.*

COROLLARY 5. (A. H. Stone)[5] Every fully normal T_2-space is paracompact.

2. Metrizability

THEOREM 2. *In order that a T_1-space R is metrizable it is necessary and sufficient that there exists a family $\{f_\alpha | \alpha \in A\}$ of real valued continuous functions on R such that $\bigcup_{\beta \in B} f_\beta$ and $\bigcap_{\beta \in B} f_\beta$ are continuous for every $B \subseteqq A$, and such that for every $x \in R$ and every nbd $U(x)$ of x there exists $f_\alpha \in \{f_\alpha | \alpha \in A\} : f_\alpha(x) < \varepsilon$ and $f_\alpha(U^c(x)) \geqq \varepsilon$ for some $\varepsilon > 0$.*

Proof. We shall prove the sufficiency. Let $\{f_\alpha | \alpha \in A\}$ be a family satisfying the condition of this proposition and put $V_{r'\alpha} = \{y | f_\alpha(y) < r'\}$, $W_{r\alpha} = \{y | f_\alpha(y) > r\}$ for rational numbers $r' > r > 0$ and $U_{rr'}(B) = (\bigcap_{\alpha \in B} V_{r'\alpha} \bigcap_{\alpha \in C(B)} W_{r\alpha})^\circ$ for $B \subseteqq A$,[8] where we put $\bigcap_{\alpha \in B} V_{r'\alpha} = R$ for $B = \phi$, $\bigcap_{\alpha \in c(B)} W_{r\alpha} = \phi$ for $C(B) = \phi$. Moreover, we define a collection $\mathfrak{U}_{rr'} = \{U_{rr'}(B) | B \subseteqq A\}$ of sets. Letting $A(x) = \left\{\alpha | f_\alpha(x) < \frac{r+r'}{2}\right\}$ for a definite point x of R, we get $\bigcup_{\alpha \in A(x)} f_\alpha(x) \leqq \frac{r+r'}{2}$ and $M(x) = \{y | \bigcup_{\alpha \in A(x)} f_\alpha(y) < r'\}$ $\subseteqq \bigcap_{\alpha \in A(x)} V_{r'\alpha}$, $N(x) = \{y | \bigcap_{\alpha \in C(A(x))} f_\alpha(y) > r\} \subseteqq \bigcap_{\alpha \in C(A(x))} W_{r\alpha}$, where $M(x) = R$ for $A(x) = \phi$, $N(x) = R$ for $C(A(x)) = \phi$. Since $\bigcup_\alpha f_\alpha$ and $\bigcap_\alpha f_\alpha$ are continuous, $M(x)$ and $N(x)$ are open nbd of x such that $M(x) \cap N(x) \subseteqq U_{rr'}(A(x))$. Hence $\{M(x) \cap N(x) | x \in R\}$ $= \mathfrak{N} < \mathfrak{U}_{rr'}$.

Now we shall show that $\mathfrak{N}$ has a locally finite refinement. Obviously it holds $\bigcup_\alpha f_\alpha(x) < r'$, if and only if $\bigcap_\alpha \ (r+r'-f_\alpha(x)) > r$. Therefore $M(x) \cap N(x)$ $= \{y | \bigcap_{\alpha \in C(A(x))} f_\alpha(y) \bigcap_{\alpha \in A(x)} (r+r'-f_\alpha(y)) > r\}$. To prove the continuity of the function $\bigcup \{ \bigcap_{\alpha \in C(B)} f_\alpha(y) \bigcap_{\alpha \in B} (r+r'-f_\alpha(y)) | B \subseteqq \mathfrak{B}\} = F(y)$ for an arbitrary $\mathfrak{B} \subseteqq 2^A$, we denote by a the value of this function at a definite point y of R. For an arbitrary $\varepsilon > 0$ there exists $\alpha \in C(B) : f_\alpha(y) < a + \frac{\varepsilon}{2}$ or $\alpha \in B : r+r'-f_\alpha(y) < a + \frac{\varepsilon}{2}$ for every $B \in \mathfrak{B}$. We

6) See [6].
7) See [11].
8) A° denotes the interior of A.

denote by B' the totality of α such that $\alpha \in C(B)$, $f_\alpha(y) < a+\frac{\varepsilon}{2}$ for some $B \in \mathfrak{B}$ and by B'' the totality of α such that $\alpha \in B$, $r+r'-f_\alpha(y) < a+\frac{\varepsilon}{2}$ for some $B \in \mathfrak{B}$. Then $\bigcup_{\alpha \in B'} f_\alpha(y) \leqq a+\frac{\varepsilon}{2}$, $\bigcup_{\alpha \in B''}(r+r'-f_\alpha(y)) \leqq a+\frac{\varepsilon}{2}$, and $C(B) \cap B' \neq \phi$ or $B \cap B'' \neq \phi$ for every $B \in \mathfrak{B}$. Hence from the continuity of $\bigcup_{\alpha \in B'} f_\alpha$ there exists a nbd $U(y)$ of y such that $\bigcup_{\alpha \in B'} f_\alpha(U(y)) < a+\varepsilon$. Since $\bigcup_{\alpha \in B''}(r+r'-f_\alpha) = r+r'-\bigcap_{\alpha \in B''} f_\alpha$ and is continuous, there exists a nbd $V(y)$ of y such that $\bigcup_{\alpha \in B''}(r+r'-f_\alpha(V(y))) < a+\varepsilon$. Therefore $F(U(y) \cap V(y)) \leqq a+\varepsilon$ from the definition of $F(y)$. Since $\bigcap_{\alpha \in C(B)} f_\alpha \bigcap_{\alpha \in B}(r+r'-f_\alpha)$ $= \bigcap_{\alpha \in C(B)} f_\alpha \cap (r+r'-\bigcup_{\alpha \in B} f_\alpha)$ is continuous for every $B \subseteqq A$, there exists a nbd $W(y)$ of y such that $F(W(y)) > a-\varepsilon$. Thus from the firstly proved lemma $\mathfrak{N}$ has a locally finite refinement.

Lastly, let $U(x)$ be a nbd of x, then there exist a positive rational number r' and $\alpha \in A$ such that $x \in V_{r'\alpha} \subseteqq U(x)$ from the property of $\{f_\alpha | \alpha \in A\}$. Taking a rational number $r > 0 : f_\alpha(x) < r < r'$, we obtain $S(x, \mathfrak{U}_{rr'}) \subseteqq U(x)$.[9] For if $x \in U_{rr'}(B)$, then since $f_\alpha(x) < r$ and consequently $x \notin W_{r\alpha}$, it must be $\alpha \in B$. Hence $U_{rr'}(B) \subseteqq V_{r'\alpha} \subseteqq U(x)$, and hence $S(x, \mathfrak{U}_{rr'}) \subseteqq U(x)$. Take a locally finite refinement $\mathfrak{V}_{rr'}$ for each $\mathfrak{U}_{rr'}$, then $\{\mathfrak{V}_{rr'} | r, r'$ are rational positive numbers$\}$ is an enumerable family of locally finite coverings, and the totality of sets contained in some $\mathfrak{V}_{rr'}$ makes an open basis of R. Since R is obviously regular, we conclude the metrizability of R from the theorem by Yu. Smirnov and the author "in order that a regular space is metrizable it is necessary and sufficient that there exists an open basis which is an enumerable sum of locally finite collections of open sets."[10]

Conversely, if R is metrizable, then $\{\rho(x,y) | x \in R\}$ satisfies the condition of this theorem, where $\rho(x,y)$ denotes a bounded distance of R.

We give here only two of various possible corollaries of this theorem.

Corollary 6. *In order that a T_1-space R is metrizable it is necessary and sufficient that we can assign a function $\varphi(x,y)$ on $R \times R$ having non-negative* (*bounded*) *values such that* 1) $\{y | \varphi(x,y) < \varepsilon\}$ $(\varepsilon > 0)$ *is a nbd basis of x for every $x \in R$,* 2) $\bar{d}(A,x) = sup$ $\{\varphi(y,x) | y \in A\}$ *and* $\underline{d}(A,x) = inf$ $\{\varphi(y,x) | y \in A\}$ *are continuous functions of x for every subset A of R.*

Proof. It is obvious.

From this corollary, we get easily the following

Corollary 7. *In order that a T_1-space R is metrizable it is necessary and sufficient that we can assign a function $\varphi(x,y)$ on $R \times R$ having non-negative* (*bounded*) *values such that* 1) $\{y | \varphi(x,y) < \varepsilon\}$ $(\varepsilon > 0)$ *is a nbd basis of x for every $x \in R$,* 2) $\bar{d}(F,x)$ *and* $\underline{d}(F,x)$ *are continuous functions of x for every closed set F of R.*

Corollary 8. *If R is a completely regular space, then in order that R is*

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9) $S(A, \mathfrak{U}) = \cup \{U | U \cap A \neq \phi,\ U \in \mathfrak{U}\}$.

10) See [13], [12].

metrizable it is necessary and sufficient that there exists a sequence L_n $(n=1,2,\cdots)$ *of families of continuous functions on* R *such that* $\bigcup_{\alpha\in A} f_\alpha$ *and* $\bigcap_{\alpha\in A} f_\alpha$ *are continuous for every* $f_\alpha \in L_n$ $(\alpha\in A)$ *for a definite* n, *and such that for every bounded continuous function* f *and every* $\varepsilon>0$ *there exist* $f_\beta \in \bigcup_{n=1}^{\infty} L_n$ $(\beta\in B)$: $|\bigcap_{\beta\in B} f_\beta - f| < \varepsilon$.

Proof. If R is metric, then $L_n=\{\varphi \mid |\varphi(x)-\varphi(y)| \leqq n\rho(x,y)\ (x,y\in R)\}$ satisfies the condition of the proposition. Assume that f is an arbitrary bounded continuous function of R such that $f(x)\leqq A$ $(x\in R)$. Let $F_n=\{x \mid n\varepsilon \leqq f(x) \leqq (n+1)\varepsilon\}$ $(n=0, \pm1, \pm2, \cdots)$ for a given $\varepsilon>0$ and let $\frac{1}{m(x)}=\max\left\{\frac{1}{m}\middle| S_{\frac{1}{m}}(x)\frown F_{n+2}=\phi\right\}$[11] for $x\in R: n\varepsilon < f(x) \leqq (n+1)\varepsilon$. We take the integer $p(x)$ such that $p(x)-1<m(x)(A-n\varepsilon) \leqq p(x)$ and put $f_x(y)=p(x)\rho(x,y)+n\varepsilon$. Then $f_x\in L_{p(x)}$, $f_x(x)=n\varepsilon$, and $f_x(S^c_{1/m(x)}(x))\geqq A$. Since $y\in F_n$ implies $(n-1)\varepsilon \leqq \bigcap_{x\in R} f_x(y) \leqq n\varepsilon$, $|\bigcap_{x\in R} f_x(y)-f(y)| \leqq 2\varepsilon$ $(y\in R)$.

Conversely, the sufficiency is easily checked. Assume that L_n $(n=1,2,\cdots)$ satisfies the conditions of this proposition. We define a continuous function f for a given point x of R and nbd $U(x)$ such that $f(x)=0$, $f(U^c(x))=1$, $0\leqq f\leqq 1$. Then it must be $|\bigcap_{\beta\in B} f_\beta - f| < \frac{1}{2}$ for some $f_\beta \in \bigcup_{n=1}^{\infty} L_n$ $(\beta\in B)$. Hence there exists $\beta\in B$: $f_\beta(x)<\frac{1}{2}$ and $f_\beta(U^c(x))>\frac{1}{2}$. Hence, as is easily seen from the proof of Theorem 2, R is metrizable.

We give the following lemma in the form of an extension of Chittenden's theorem.

LEMMA 2. *A* T_1*-space* R *is metrizable, if and only if a function* $\varphi(x,y)$ *on* $R\times R$ *having non-negative values can be defined so that* 1) $\{y \mid \varphi(x,y)<\varepsilon\}$ $(\varepsilon>0)$ *is a nbd basis of* x *for every* $x\in R$, 2) *for every* $\varepsilon>0$ *and* $x\in R$ *there exist nbds* $S_1(\varepsilon\ x)$, $S_2(\varepsilon\ x)$, $S_3(\varepsilon\ x)$ *of* x : $\varphi(x,y)\geqq\varepsilon$ *and* $z\in S_3(\varepsilon\ y)$ *imply* $z\notin S_1(\varepsilon\ x)$, $z\notin S_1(\varepsilon\ x)$ *and* $u\in S_3(\varepsilon\ x)$ *imply* $u\notin S_2(\varepsilon\ x)$.

Proof. Since the necessity is clear, we prove only the sufficiency. The proof is analogous to that of Lemma 1. We put $S_n(x)=\left\{y \mid \varphi(x,y)<\frac{1}{n}\right\}$, $S^1_n(x)=S_1\left(\frac{1}{n}\ x\right)$, $S^2_n(x)=S_2\left(\frac{1}{n}\ x\right)$, $S^3_n(x)=S_3\left(\frac{1}{n}\ x\right)$. To prove firstly the full normality of R, we take an arbitrary open covering $\mathfrak{V}=\{V_\alpha \mid \alpha<\tau\}$ of R and put $V_{n\alpha}=\{x \mid S_n(x)\subseteqq V_\alpha\}$, $V'_{n\alpha}=\cup\{S^1_n(x) \mid x\in V_{n\alpha}\}$, $V''_{n\alpha}=\cup\{S^2_n(x) \mid x\in V_{n\alpha}\}$; $M_{n\alpha}=(V''_{n\alpha}-\bigcup_{\beta<\alpha} V'_{n\beta})^\circ$, $F_n=\bigcup_{\alpha<\tau} M_{n\alpha}$, $W_{n\alpha}=M_{n\alpha}-\bigcup_{i=1}^{n-1} F_i$ and $\mathfrak{W}=\{W_{n\alpha} \mid n=1,2,\cdots,\ \alpha<\tau\}$. Then it is easily proved that $\mathfrak{W}$ is a locally finite, not necessarily open refinement of $\mathfrak{V}$. $\mathfrak{W}<\mathfrak{V}$ is obvious. To prove that $\mathfrak{W}$ covers R, we assume $x\in V_\alpha$, $x\notin V_\beta(\beta<\alpha)$, $\alpha<\tau$ for a given $x\in R$. Since $x\in V_\alpha$, $S_n(x)\subseteqq V_\alpha$ for some n, and hence $S^2_n(x)\subseteqq V''_{n\alpha}$. Since $x\notin V_{n\beta}(\beta<\alpha)$, from the property of φ it hold $S^3_n(x)\frown(\bigcup_{\beta<\alpha} V'_{n\beta})=\phi$ $(\beta<\alpha)$. Hence $x\in M_{n\alpha}$, and hence $\{M_{n\alpha} \mid n=1,2,\cdots;\ \alpha<\tau\}$ covers R. Now let $x\in M_{n\alpha}$, $x\notin M_{m\beta}$ $(m<n,\ \beta<\tau)$, then $x\in W_{n\alpha}$; hence $\mathfrak{W}$ covers R. Finally we show the locally finiteness of $\mathfrak{W}$. For every point x of R there exists some $M_{n\alpha}: x\in M_{n\alpha}$. Then $M_{n\alpha}$ is a nbd of x,

11) $S_{1/m}(x)=\{y \mid \rho(x,y)<\frac{1}{m}\}$, where $\rho(x,y)$ is the distance between x and y.

and $M_{n\alpha} \frown W_{m\beta} = \phi$ $(m \geqq n+1, \beta < \tau)$. For a definite natural number $m \leqq n$, we assume $x \in V'_{m\alpha}$, $x \notin V'_{m\beta} (\beta < \alpha)$, $\alpha \leqq \tau$. From the property of φ it hold $S_n^3(x) \frown (\bigcup_{\beta < \alpha} V''_{m\beta}) = \phi$ and $V'_{m\alpha} \frown M_{m\gamma} = \phi$ $(\gamma > \alpha)$, hence $S_n^3(x) \frown W_{n\beta} = \phi$ $(\beta < \alpha)$, $V'_{m\alpha} \frown W_{m\gamma} = \phi$ $(\gamma > \alpha)$. Therefore the nbd $S_n^3(x) \frown V'_{m\alpha}$ of x intersects at most one of $W_{m\alpha} (\alpha < \tau)$. Thus we conclude the full normality of R from E. Michael's theorem "regular space is fully normal, if and only if every open covering of R has a locally finite not necessarily open refinement"[12] combining the regularity of R, which is easily to be checked.

Let $U(x)$ be an arbitrary nbd of $x \in R$, then $S_n(y) \not\ni x$ for some n and every $y \notin U^c(x)$. Hence $S(x, \mathfrak{U}_n) \subseteqq U(x)$, where $\mathfrak{U}_n = \{(S_n(x))^\circ | x \in R\}$. Therefore R is metrizable from Urysohn's theorem.[13]

From this lemma we get the following extensions of P. Alexandroff and P. Urysohn's theorem.

THEOREM 3. *A T_1-space R is metrizable if and only if there exists an enumerable collection $\{\mathfrak{U}_n | n = 1, 2, \cdots\}$ of open coverings of R such that*

1) *$\{S(x, \mathfrak{U}_n) | n = 1, 2, \cdots\}$ makes a nbd basis of x for every point x of R,*

2) *every $\mathfrak{U}_n$ has an open refinement $\mathfrak{U}_n'$ such that $U_1, U_2, U_3 \in \mathfrak{U}_n'$, $U_1 \frown U_2 \neq \phi$, $U_2 \frown U_3 \neq \phi$ imply $U_1 \smile U_2 \smile U_3 \subseteqq U$ for some $U \in \mathfrak{U}_n$, where $\mathfrak{U}_n'$ does not necessarily belong to $\{\mathfrak{U}_n | n = 1, 2, \cdots\}$.*

Proof. Since the necessity is obvious, we show only the sufficiency. We define the function $\varphi(x, y)$ on $R \times R$ by $\varphi(x, y) = \mathrm{Min}\left\{\frac{1}{n+1} \middle| y \in S(x, \mathfrak{U}_n)\right\}$, $\varphi(x, y) = 1$ if $y \notin S(x, \mathfrak{U}_n)$ $(n = 1, 2, \cdots)$. Since we can assume $\mathfrak{U}_{n+1} < \mathfrak{U}_n$ without loss of generality, it is clear that $\left\{y | \varphi(x, y) < \frac{1}{n}\right\}$ $(n = 1, 2, \cdots)$ makes a nbd basis of x. For every $\varepsilon > 0$ we take $n \geqq \frac{1}{\varepsilon}$. Then $\varphi(x, y) \geqq \varepsilon$ implies $S(x, \mathfrak{U}_n) \not\ni y$ and consequently $S^2(x, \mathfrak{U}_n') \frown S(y, \mathfrak{U}_n') = \phi$.[14] $y \notin S^2(x, \mathfrak{U}_n')$ implies $S(y, \mathfrak{U}_n') \frown S(x, \mathfrak{U}_n') = \phi$, and hence $S_1(\varepsilon, x) = S^2(x, \mathfrak{U}_n')$, $S_2(\varepsilon, x) = S_3(\varepsilon, x) = S(x, \mathfrak{U}_n')$ satisfy the condition of the lemma. Therefore this theorem is valid.

The following proposition is a direct consequence of this theorem.

COROLLARY 9. *A T_1-space R is metrizable, if and only if there exists a collection $\{\mathfrak{U}_n | n = 1, 2, \ldots\}$ of open coverings such that*

1) *$\{S(x, \mathfrak{U}_n) | n = 1, 2, \cdots\}$ makes a nbd basis of x,*

2) *every $\mathfrak{U}_n$ has a star-refinement.*[15]

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12) See [6].

13) Although Michael's theorem is essentially unnecessary in this proof, his theorem has various uses for simplifying proofs of theorems for coverings.

14) $S^2(x, \mathfrak{U}) = S(S(x, \mathfrak{U}), \mathfrak{U})$.

15) We call $\mathfrak{V}$ a star-refinement of $\mathfrak{U}$ if $\mathfrak{V}^* < \mathfrak{U}$, where $\mathfrak{V}^* = \{S(V, \mathfrak{V}) | V \in \mathfrak{V}\}$. We don't know whether "star-refinement" in this proposition may be changed with "delta-refinement." We call $\mathfrak{V}$ a delta-refinement of $\mathfrak{U}$ if $\mathfrak{V}^\Delta = \{S(x, \mathfrak{V}) | x \in R\} < \mathfrak{U}$.

3. Extension theorems of continuous functions on uniform spaces

THEOREM 4. *Let R be a fully normal uniform space with the uniform topology defined by the unform coverings $\{\mathfrak{M}_{\alpha'}|\alpha'\in A'\}$ and S a uniform space with the uniform topology defined by the uniform coverings $\{\mathfrak{N}_\alpha|\alpha\in A\}$ such that $|A'|=|A|=\mathfrak{m}$.*[16] *If f is a continuous mapping defined on a closed set F of R and having values in S, then S can be imbedded in a uniform space T which uniform topology is defined by a system of uniform coverings with the cardinal number $\mathfrak{m}$, such that f can be continuously extended to R with values in T, such that the extension is a homeomorphism of $R-F$ with $T-S$, and such that S is a closed sub-uniform space of T. If f is a homeomorphism, then the extension is also a homeomorphism.*

Proof. Since, as it is easily seen, $\mathfrak{N}_\alpha^*<\mathfrak{N}_\beta$ implies $\mathfrak{U}_\alpha^*<\mathfrak{U}_\beta=f^{-1}(\mathfrak{N}_\beta)=\{f^{-1}(N)|N\in\mathfrak{N}_\beta\}$, we can choose $\mathfrak{U}_{i\alpha}(i=1,2,\cdots)$ from $\{\mathfrak{U}_\alpha|\alpha\in A\}$ such that $\mathfrak{U}_{1\alpha}=\mathfrak{U}_\alpha$, $\mathfrak{U}_{1\alpha}>\mathfrak{U}_{2\alpha}^*>\mathfrak{U}_{2\alpha}>\mathfrak{U}_{3\alpha}^*>\mathfrak{U}_{3\alpha}>\cdots$. Putting $\mathfrak{V}'_{1\alpha}=\{(R-F)\cup U|U\in\mathfrak{U}_{2\alpha}\}$, we get a covering $\mathfrak{V}_{1\alpha}=\mathfrak{V}'_{1\alpha}\wedge\mathfrak{M}_{\alpha'}$ of R such that $\mathfrak{V}_{1\alpha}\wedge F=\{V\cap F|V\in\mathfrak{V}_{1\alpha}\}<\mathfrak{U}_{2\alpha}$, $\mathfrak{V}_{1\alpha}<\mathfrak{M}_{\alpha'}$. Since R is fully normal, we can choose further a covering $\mathfrak{V}_{2\alpha}$ of R such that $\mathfrak{V}_{2\alpha}\wedge F<\mathfrak{U}_{3\alpha}$, $\mathfrak{V}_{2\alpha}^*<\mathfrak{V}_{1\alpha}$. We can obtain successively in the same way a sequence of coverings of R $\mathfrak{M}_{\alpha'}>\mathfrak{V}_{1\alpha}>\mathfrak{V}_{2\alpha}^*>\mathfrak{V}_{2\alpha}>\mathfrak{V}_{3\alpha}^*>\cdots$ such that $\mathfrak{V}_{n\alpha}\wedge F<\mathfrak{U}_{n+1\alpha}$ $(n=1,2,\cdots)$.

Now we define a sequence of coverings of R from the above sequence by $\mathfrak{P}_{i\alpha}=\{\mathfrak{U}_{i\alpha},\ \mathfrak{V}_{i\alpha}\}=\{N(U,\mathfrak{V}_{i\alpha}),\ V\cap(R-F)|U\in\mathfrak{U}_{i\alpha},\ V\in\mathfrak{V}_{i\alpha}\}$, where $N(U,\mathfrak{V}_{i\alpha})$ denotes the open set $\cup\{V|\phi\neq V\cap F\subseteqq U,\ V\in\mathfrak{V}_{i\alpha}\}$ of R. Let us show $\mathfrak{P}_{i\alpha}>\mathfrak{P}_{i+1\alpha}^\Delta$ $(i=1,2,\cdots)$. We denote by x an arbitrary point of R. If $S(x,\ \mathfrak{P}_{i+1\alpha})\cap F=\phi$, then there exists $V\in\mathfrak{V}_{i\alpha}$ such that $S(x,\ \mathfrak{P}_{i+1\alpha})=S(x,\ \mathfrak{V}_{i+1\alpha})\subseteqq V$ from $\mathfrak{V}_{i+1\alpha}^*<\mathfrak{V}_{i\alpha}$ and the definition of $\mathfrak{P}_{i+1\alpha}$. Hence $S(x,\ \mathfrak{P}_{i+1\alpha})\subseteqq V\cap(R-F)\in\mathfrak{P}_{i\alpha}$. If $x\in R-F$ and $S(x,\mathfrak{P}_{i+1\alpha})\cap F\neq\phi$, then since $\mathfrak{V}_{i+1\alpha}^*<\mathfrak{V}_{i\alpha}$ and $\mathfrak{V}_{i\alpha}\wedge F<\mathfrak{U}_{i+1\alpha}$, there exist $V\in\mathfrak{V}_{i\alpha}$ and $U_0\in\mathfrak{U}_{i+1\alpha}$ such that $S(x,\mathfrak{V}_{i+1\alpha})\subseteqq V$, $V\cap F\subseteqq U_0\in\mathfrak{U}_{i+1\alpha}$. From $\mathfrak{U}_{i+1\alpha}^*<\mathfrak{U}_{i\alpha}$ it holds $S(U_0,\ \mathfrak{U}_{i+1\alpha})\subseteqq U'$ for some $U'\in\mathfrak{U}_{i\alpha}^*$. Since $V\cap F\neq\phi$, $V\in\mathfrak{V}_{i\alpha}$, $V\cap F\subseteqq U'$, we get $S(x,\mathfrak{V}_{i+1\alpha})\subseteqq V\subseteqq N(U',\mathfrak{V}_{i\alpha})\in\mathfrak{P}_{i\alpha}$. If $x\in N(U,\mathfrak{V}_{i+1\alpha})$ and $U\in\mathfrak{U}_{i+1\alpha}$, then $S(x,\ \mathfrak{V}_{i+1\alpha})\cap F\cap N(U,\mathfrak{V}_{i+1\alpha})=S(x,\mathfrak{V}_{i+1\alpha})\cap F\cap U\neq\phi$, and hence $U_0\cap U\neq\phi$ from $S(x,\mathfrak{V}_{i+1\alpha})\cap F\subseteqq V\cap F\subseteqq U_0$. Therefore $U\subseteqq S(U_0,\ \mathfrak{U}_{i+1\alpha})\subseteqq U'$ and consequently. $N(U,\mathfrak{V}_{i+1\alpha})\subseteqq N(U',\mathfrak{V}_{i\alpha})$. Thus we get $S(x,\mathfrak{P}_{i+1\alpha})\subseteqq N(U',\mathfrak{V}_{i\alpha})\in\mathfrak{P}_{i\alpha}$. In the case that $x\in F$ we get $S(x,\mathfrak{U}_{i+1\alpha})\subseteqq U$ for some $U\in\mathfrak{U}_{i\alpha}$ and consequently $S(x,\mathfrak{P}_{i+1\alpha})\subseteqq N(U,\mathfrak{V}_{i\alpha})\in\mathfrak{P}_{i\alpha}$. Therefore $\mathfrak{P}_{i\alpha}>\mathfrak{P}_{+1\alpha i}^\Delta$ is established.

Putting $(R-F)\cup S=T$, we define a mapping f^* from R into T by $f^*(z)=z$ $(z\in R-F)$, $f^*(x)=f(x)$ $(x\in F)$. Defining coverings $\mathfrak{Q}_{i\alpha}$ of T by $f^*(\mathfrak{P}_{i\alpha})=\mathfrak{Q}_{i\alpha}$, we see easily $\mathfrak{Q}_{i\alpha}>\mathfrak{Q}_{i+1\alpha}^\Delta(i=1,2,\ldots;\ \alpha\in A)$. In fact, $y\in T$, $f^*(P)\ni y$ and $P\in\mathfrak{P}_{i+1\alpha}$ imply $P\ni x$ for any point x of R such that $f^*(x)=y$. Since $S(x,\mathfrak{P}_{i+1\alpha})\subseteqq P'$ for some $P'\in\mathfrak{P}_{i\alpha}$, it holds $S(f^*(x),\ f^*(\mathfrak{P}_{i+1\alpha}))=S(y,\mathfrak{Q}_{i+1\alpha})\subseteqq f^*(P')\in\mathfrak{Q}_{i\alpha}$. In addition

16) We denote by $|A|$ the cardinal number of the set A. We denote by α' the image of α by a one-to-one mapping from A onto A'.

we note that $\{\mathfrak{Q}_{i\alpha}\wedge S|\alpha\in A,\ i=1,2,\ldots\}=\{\mathfrak{N}_\alpha|\alpha\in A\}$ is obvious from the definition of $\mathfrak{Q}_{i\alpha}$.

If $y\in T-S$, then since $f^{*-1}(y)=x\in R-F$ and F is closed, we get $S^2(x,\mathfrak{V}_{1\alpha})\frown F \subseteqq S^2(x,\mathfrak{M}_{\alpha'})\frown F=\phi$ for some $\alpha'\in A'$. Therefore $S(x,\mathfrak{P}_{1\alpha})\frown F=\phi$, and hence $S(y,\mathfrak{Q}_{1\alpha})\frown S=\phi$. Hence S is a closed subset of T. Furthermore, for every distinct points y_1, y_2 of T there exists $\mathfrak{Q}_{i\alpha}$ such that $y_2\notin S(y_1,\mathfrak{Q}_{i\alpha})$. Thus we can define a uniform topology in $(R-F)\smile S$ by the uniform covering system consisting of all the finite intersection coverings of $\mathfrak{Q}_{i\alpha}$ $(\alpha\in A,\ i=1,2,\cdots)$, and obtain the uniform space T and the extension f^* of f satisfying conditions in this theorem.

The following Hausdorff's theorem is the direct consequence of this theorem for $\mathfrak{m}=\aleph_0$.

Hausdorff's theorem.[17] *If R and S are metric spaces, F is a closed set of R and if f is a continuous mapping from F into S, then S can be imbedded isometrically in a metric space T such that f can be continuously extended to R with values in T, such that the extension is a homeomorphism of R-F with T-S, and such that S is a closed sub-space of T. If f is a homeomorphism, then the extension is also a homeomorphism.*

Finally, let us discuss extension theorems in the case that R is not fully normal.

THEOREM 5. *Theorem 4 is valid when R is normal and F satisfies the second countability axiom or when R is normal and S satisfies the second countability axiom.*

Proof. We assume that R is normal and F satisfies the second countability axiom and that $\{\mathfrak{N}_\alpha|\alpha\in A\}$ and $\{\mathfrak{M}_{\alpha'}|\alpha'\in A'\}$ are uniformities of S and R respectively. If we denote by f a continuous mapping on F having values in S, then $f^{-1}(\mathfrak{N}_\alpha)=\mathfrak{U}_\alpha$ is a normal covering of F.[18] We choose a sequence $\mathfrak{U}_\alpha=\mathfrak{U}_{1\alpha}>\mathfrak{U}^*_{2\alpha}>\mathfrak{U}_{2\alpha}>\mathfrak{U}^*_{3\alpha}>\cdots$ of coverings from $\{f^{-1}(\mathfrak{N}_\alpha)|\alpha\in A\}$ in the same way as in the proof of Theorem 4. Since F satisfies the second countability axiom, there exists a locally finite enumerable refinement $\mathfrak{U}=\{U_n|n=1,2,\cdots\}$ of $\mathfrak{U}_{2\alpha}$. Let us denote by $\mathfrak{U}_0=\{U_{0n}|n=1,2,\cdots\}$ a covering of F such that $\overline{U}_{0n}\subseteqq U_n$ $(n=1,2,\cdots)$, and consider continuous functions φ_n on R such that $\varphi_n(U_{0n})=1$, $\varphi_n(F-U_n)=0$, $0\leqq\varphi_n\leqq 1$. If we put $W_n=\{x|\varphi_n(x)>0\}$, then $\bigcup_{n=1}^{\infty}W_n=W\supseteqq F$ and $\mathfrak{W}\wedge F<\mathfrak{U}_{2\alpha}$ for $\mathfrak{W}=\{W_n|n=1,2,\cdots\}$. Now we take a continuous function φ_0 on R such that $\varphi_0(W^c)=1$, $\varphi_0(F)=0$, $0\leqq\varphi_0\leqq 1$, and define $U_0=\{x|f_0(x)>0\}$. Then we have an enumerable covering $\mathfrak{W}_{1\alpha}=\{U_0, U_1, U_2,\cdots\}$ of R such that $\mathfrak{W}_{1\alpha}=\{U_0, U_1, U_2,\cdots\}$ of R such that $\mathfrak{W}_{1\alpha}\wedge F<\mathfrak{U}_{2\alpha}$. Since $\mathfrak{W}_{1\alpha}$ is a normal covering from Corollary 4, we have a normal covering $\mathfrak{V}_{1\alpha}=\mathfrak{W}_{1\alpha}\wedge\mathfrak{M}_{\alpha'}$, of R such that $\mathfrak{V}_{1\alpha}\wedge F<\mathfrak{U}_{2\alpha}$.

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17) See [4]. R. Arens, [1] gives some extensions and a brief proof of Hausdorff's theorem by a different method from us.

18) A covering $\mathfrak{N}$ of R is called normal, when there exists a sequence $\{\mathfrak{N}_i\,|\,i=1,2,\cdots\}$ of coverings such that $\mathfrak{N}^*_{i+1}<\mathfrak{N}_i<\mathfrak{N}\,(i=1,2,\cdots)$.

Next, we take a normal covering $\mathfrak{T}_{2\alpha}$ of R such that $\mathfrak{T}_{2\alpha}^* < \mathfrak{V}_{1\alpha}$ and a normal covering $\mathfrak{W}_{2\alpha}$ of R such that $\mathfrak{W}_{2\alpha} \wedge F > \mathfrak{U}_{3\alpha}$ in the same way as in the case of $\mathfrak{W}_{1\alpha}$. Putting $\mathfrak{V}_{2\alpha} = \mathfrak{W}_{2\alpha} \wedge \mathfrak{T}_{2\alpha}$, we have a normal covering such that $\mathfrak{V}_{2\alpha}^* < \mathfrak{V}_{1\alpha}$, $\mathfrak{V}_{2\alpha} \wedge F < \mathfrak{U}_{3\alpha}$. Repeating such processes, we obtain a sequence of uniform covering $\mathfrak{V}_{1\alpha} > \mathfrak{V}_{2\alpha}^* > \mathfrak{V}_{2\alpha} > \mathfrak{V}_{3\alpha}^* > \cdots$ of R such that $\mathfrak{V}_{1\alpha} < \mathfrak{M}_{\alpha'}$, $\mathfrak{V}_{i\alpha} \wedge F < \mathfrak{U}_{i+1\alpha}$ $(i=1, 2 \cdots)$ for every $\alpha \in A$. The remainder of the proof is the same as the proof of Theorem 4 and is omitted.

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