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On uniform topology of functional spaces

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In the study of spaces of continuous functions on a topological space the weak topology and the strong topology are useful. Indeed, every completely regular space is homeomorphic with a functional space with the weak topology, and this fact implies Tychonoff-Čech's theorem about compactification.[1] Moreover the topological space is defined up to a homeomorphism by the topological ring of all continuous functions with the weak topology.[2] It is also well known that a compact T_2-space is characterized by the topological ring of all continuous functions with the strong topology.[3] However, there does not exist such a usefulness in the weak or strong uniform topology by studying uniform spaces except particular cases. For example, a complete metric space is characterized by the topological ring of all bounded uniformly continuous functions with the strong topology or with the weak topology,[4] but in the case of a general complete uniform space this proposition is invalid.

In this paper we define a new uniform topology, m-uniform topology of functional space and give analogous theories about uniform spaces as about topological spaces. In §1 the definition of m-uniform topology is given, and it is shown that any general uniform space are uniformly homeomorphic with a functional space with this uniform topology. This fact implies that any uniform space is uniformly homeomorphic with a dense subspace of a complete uniform space. In §2 it is shown that if we introduce a suitable uniform topology in the topological ring or lattice of all bounded uniformly continuous functions with the strong topology, then this ring or lattice defines the uniform space up to a uniform homeomorphism. M-uniform topology is used as the suitable uniform topology.

§1. From now on we denote by R a uniform space and by $\{\mathfrak{U}_\alpha | \alpha \in A\}$, $\mathfrak{U}_\alpha = \{U_\alpha(x) | x \in R\}$ the uniform nbd (= neighborhood) system of R.

DEFINITION. For a real valued function $f(x)$ on R and for a subset A of R we

1) Any completely regular space is homeomorphic with a dense subset of a compact T_2-space, A. Tychonoff, Über die topologishe Erweiterung von Räumen, Math. Ann. 102. E. Čech, On Bicompact Spaces, Ann. of Math.

2) J. Nagata, On lattices of functions on topological spaces and of functions on uniform spaces, Osaka Math. Journal, Vol. 1, 1949.

3) G. Silov, Ideals and subrings of the rings of continuous functions, C. R. URSS, 22.

4) J. Nagata, loc. cit. T. Shirota, A. Generalization of a Theorem of I. Kaplansky, Osaka Math. Journal, Vol. 4, 1952.

denote by $[f(A)]$ the closed interval $[\inf_{x\in A} f(x), \sup_{x\in A} f(x)]$.

DEFINITION. $d_\alpha(g(x), f(x)) = \mathrm{Max}\,(d(g(x)\,[f(U_\alpha(x))]),\ d(f(x)\,[g(U_\alpha(x))]))$.[5]

DEFINITION. $U_{\alpha\varepsilon}(f) = \{g\,|\,d_\alpha(f(x), g(x)) < \varepsilon$ for all $x\in R\} \subseteq F(R)$, where $F(R)$ denotes a set of real valued functions on R.

THEOREM 1. *$\{\mathfrak{U}_{\alpha\varepsilon}\,|\,\alpha\in A,\ \varepsilon>0\}$ $(\mathfrak{U}_{\alpha\varepsilon} = \{U_{\alpha\varepsilon}(f)\,|\,f\in C(R)\})$ satisfies the condition of uniform nbd system, where $C(R)$ is a set of continuous functions on R.*

Proof. Let $f, g\in C(R)$, $f\neq g$, then there exists $x\in R$ such that $f(x)\neq g(x)$. Since $f(x)$ is continuous, there exist $\varepsilon>0$ and $\alpha\in A$ such that $d(g(x), [f(U_\alpha(x))]) > \varepsilon > 0$. Hence $g\notin U_{\alpha\varepsilon}(f)$.

Since for $\mathfrak{U}_\beta < \mathfrak{U}_\alpha$,[6] $0<\delta<\varepsilon$ we get $U_{\beta\delta}(f)\subseteq U_{\alpha\varepsilon}(f)$ for every $f\in C(R)$, for every $\mathfrak{U}_{\alpha\varepsilon}$ and $\mathfrak{U}_{\alpha'\varepsilon'}$ there exists $\mathfrak{U}_{\beta\delta}$ such that $U_{\beta\delta}(f)\subseteq U_{\alpha\varepsilon}(f)\cap U_{\alpha'\varepsilon'}(f)$ $(f\in C(R))$.

For $\alpha\in A$ and $\varepsilon>0$ we take $\beta\in A$ such that $\cup\{U_\beta(y)\,|\,y\in U_\beta(x)\}\subseteq U_\alpha(x)$. If $g, h\in U_{\beta\frac{\varepsilon}{2}}(f)$, then since $d(g(x), [f(U_\beta(x))]) < \frac{\varepsilon}{2}$, there exist $y, z\in U_\beta(x)$ such that $d(g(x), [f(y), f(z)]) < \frac{\varepsilon}{2}$. Since $d(f(y), [h(U_\beta(y))]) < \frac{\varepsilon}{2}$, $d(f(z), [h(U_\beta(z))]) < \frac{\varepsilon}{2}$, there exist y', y'' such that $h(y') - \frac{\varepsilon}{2} < f(y)$, $h(y'') + \frac{\varepsilon}{2} > f(z)$. Hence we get $h(y') - \varepsilon < f(y) - \frac{\varepsilon}{2} < g(x)$, $h(y'') + \varepsilon > f(z) + \frac{\varepsilon}{2} > g(x)$. Since $y', y''\in U_\alpha(x)$, $d(g(x), [h(U_\alpha(x))]) < \varepsilon$ holds. In a similar way we get $d(h(x), [g(U_\alpha(x))]) < \varepsilon$ and accordingly $d_\alpha(g(x), f(x)) < \varepsilon$. Therefore $g\in U_{\alpha\varepsilon}(h)$.

DEFINITION. We call this uniform topology *m-uniform topology* and denote by $C_u(R)$, $C'(R)$ and $C_u'(R)$ the uniform spaces with the m-uniform topology consist of the bounded uniformly continuous functions, of the continuous functions taking values between 0 and 1 and of the uniformly continuous functions taking values between 0 and 1 respectively.

REMARK. Generally, the m-topology, the topology defined by the m-uniform topology is weaker than the strong topology and it is stronger than the weak topology in $C(R)$.

DEFINITION. We define $M(x)$ to mean the mapping which maps $x\in R$ to the function $x(f) = f(x)$ $(f\in C_u'(R))$ on $C_u'(R)$.

LEMMA 1. $M(R)\subseteq C'(C_u'(R))$.

Proof. For any $f\in C_u'(R)$ and $\varepsilon>0$ there exists $\alpha\in A$ such that $f(U_\alpha(x))\subseteq S_\varepsilon(f(x))$ for every $x\in R$.[7] Since $g\in U_{\alpha\varepsilon}(f)$ implies $d(g(x), [f(U_\alpha(x))]) < \varepsilon$,

5) $d(p, M)$ denotes the distance between a real number p and a subset M of the space of real numbers.

6) We denote $\mathfrak{U}_\beta < \mathfrak{U}_\alpha$ to mean $U_\beta(x)\subseteq U_\alpha(x)$ for every $x\in R$.

7) $S_\varepsilon(p) = \{q\,|\,d(p, q) < \varepsilon\}$.

there exist $y, z \in U_\alpha(x)$ such that $(f(y)-\varepsilon, f(z)+\varepsilon) \ni g(x)$. Hence $g(x) \in (f(x)-2\varepsilon, f(x)+2\varepsilon)$, *i.e.* $|x(f)-x(g)| = |f(x)-g(x)| < 2\varepsilon$. Therefore $x(f) \in C'(C_u'(R))$.

THEOREM 2. *$M(x)$ is a uniformly homeomorphic mapping between R and $M(R) \subseteq C'(C_u'(R))$.*

Proof. It is obvious that $M(x)$ is one-to-one. $M(x)$ is uniformly continuous. For given $\varepsilon > 0$ and $\alpha \in A$, we take $\beta \in A$ such that $\mathfrak{U}_\beta^* < \mathfrak{U}_\alpha$.[8] If $y \in U_\beta(x)$ and if $f \in C_u'(R)$, $f(y) > f(x)$, then there exist nbds $V_1(x)$, $V_2(y)$ of x, y respectively such that $V_1(x) \cap V_2(y) = \phi$, $V_1(x) \cup V_2(y) \subseteq U_\beta(x)$; $f(V_1(x)) \subseteq (f(x)-\varepsilon, f(x)+\varepsilon)$, $f(V_2(y)) \subseteq (f(y)-\varepsilon, f(y)+\varepsilon)$. From g_1, $g_2 \in C_u'(R)$ such that $g_1(x) = f(y)$, $g_1(V_1^c(x)) = 0$, $f(y) \geqq g_1 \geqq 0$;[9] $g_2(y) = f(x)$, $g_2(V_2^c(y)) = 1$, $1 \geqq g_2 \geqq f(x)$, we get an element $g = (f \vee g_1) \wedge g_2$ of $C_u'(R)$. If $z \in V_1(x)$, then $f(y)+\varepsilon > g(z) \geqq f(z) > f(x)-\varepsilon = g(y)-\varepsilon$. Since $y \in U_\alpha(z)$, we get $d_\alpha(f(z), g(z)) < \varepsilon$. In a similar way we get $d_\alpha(f(z), g(z)) < \varepsilon$ for $z \in V_2(y)$. For $z \notin V_1(x) \cup V_2(y)$ $f(z) = g(z)$ holds. Hence $g \in U_{\alpha\varepsilon}(f)$. Since $x(f) = y(g)$, $x(g) = y(f)$, we see $M(y) \in U_{\alpha\varepsilon\varepsilon}(M(x))$.[10] Thus $M(x)$ is uniformly continuous.

Next we show that the inverse mapping $M^{-1}(x(f))$ is also uniformly continuous. For a given $\alpha \in A$ we take $\beta \in A$ such that $\overline{U_\beta(x)} \subseteq U_\alpha(x)$ for every $x \in R$. If $y \notin U_\alpha(x)$, then for an element f of $C_u'(R)$ such that $f(U_\beta(x)) = 1$, $f(y) = 0$, $g \in U_{\beta\frac{1}{2}}(f)$ implies $|g(x)-1| < \frac{1}{2}$, *i.e.* $x(g) = g(x) > \frac{1}{2}$. Since $y(f) = 0$, we get $d_{\beta\frac{1}{2}}(y(f), x(f)) \geqq \frac{1}{2}$, *i.e.* $M(y) \notin U_{\beta\frac{1}{2}\frac{1}{2}}(M(x))$. Hence $M(y) \in U_{\beta\frac{1}{2}\frac{1}{2}}(M(x))$ implies $y \in U_\alpha(x)$, and hence $M^{-1}(x(f))$ is uniformly continuous.

Let $\{F_\gamma | \gamma \in C\}$ be a cauchy filter of R, then for each $f \in C_u'(R)$ $\{f(F_\delta) | \gamma \in C\}$ converges to a real value $p = u(f)$. Hence $u(f)$ is a real valued function defined on $C_u'(R)$.

LEMMA 2. $u(f) \in C'(C_u'(R))$.

Proof. For an arbitrary $\varepsilon > 0$ we take $\alpha, \beta \in A$ such that $|f(x)-f(y)| < \varepsilon$ $(y \in U_\alpha(x))$, $\mathfrak{U}_\alpha > \mathfrak{U}_\beta^*$. Let $g \in U_{\beta\varepsilon}(f)$ and let $U_\beta(x) \supseteq F$, then since for each $y \in F_\gamma$ there exist $z, z' \in U_\beta(y)$ such that $f(z)-\varepsilon < g(y) < f(z)+\varepsilon$, $f(z) > f(x)-\varepsilon$ and $f(z') < f(x)+\varepsilon$ hold for $z, z' \in U_\alpha(x)$. Hence $f(x)-2\varepsilon < g(y) < f(x)+2\varepsilon$, *i.e.* $|f(x)-g(y)| < 2\varepsilon$. Since $|f(x)-u(f)| \leqq \varepsilon$, we get $|u(f)-g(y)| < 3\varepsilon$ for every $y \in F_\gamma$. Therefore $|u(f)-u(g)| \leqq 3\varepsilon$, *i.e.* $u(f) \in C'(C_u'(R))$.

LEMMA 3. *For every diverging cauchy filter $\{F_\gamma | \gamma \in C\}$ of R $\{M(F_\gamma) | \gamma \in C\}$ converges to $u(f)$ in $C'(C_u'(R))$.*

Proof. For simplicity we can restrict $\{F_\gamma\}$ to a cauchy filter of closed sets.

8) We define $\mathfrak{U}_\beta^* < U_\alpha$ to mean $y, z \in U_\beta(x)$ implies $z \in U_\alpha(y)$ for every $x \in R$.

9) $V_1^c(x)$ means the complement of $V_1(x)$. $g_1(V_1^c(x)) = 0$ means that $g_1(z) = 0$ for every $z \in V_1^c(x)$.

10) $U_{\alpha\varepsilon\varepsilon}(a)$ denotes the uniform nbd of $a \in C'(C'_u(R))$ defined by $\mathfrak{U}_{\alpha\varepsilon}$ and ε.

For given $\alpha' \in A$ and $\varepsilon > 0$, we take $\alpha, \beta \in A$ such that $\mathfrak{U}_{\alpha'} > \mathfrak{U}_\alpha^* > \mathfrak{U}_\alpha > \mathfrak{U}_\beta^*$ and x, F_γ such that $x \in F_\gamma \subseteq U_\beta(x_\beta)$. Now we shall prove $x \in U_{\alpha'\varepsilon\varepsilon}(u)$. For $f \in C_u'(R)$ we take $F \in \{F_\gamma\}$ such that $x \notin F \subseteq U_\beta(x_\beta)$, $|f(y) - f(z)| < \frac{\varepsilon}{2}$ $(g, z \in F)$. If $d(f(x), [f(F)]) < \frac{\varepsilon}{3}$, then $|x(f) - u(f)| < \varepsilon$, *i.e.* $d_{\alpha'\varepsilon}(x(f)\, u(f)) < \varepsilon$. If $f(x) > \sup_{y \in F} f(y)$, then there exist a nbd V_1 of F and a nbd V_2 of x such that $V_1 = S(F\, \mathfrak{U}_{\beta'})$[11] for some $\beta' \in A$; $V_1 \cap V_2 = \phi$, $V_1 \cup V_2 \subseteq U_\alpha(x_\beta)$; $f(V_1) \subseteq \left(a - \frac{\varepsilon}{4},\ a + \frac{\varepsilon}{2}\right)$, $f(V_2) \subseteq (f(x) - \varepsilon, f(x) + \varepsilon)$. For V_1, V_2 we define $g_1, g_2, g \in C_u'(R)$ such that $g_1(F) = f(x)$, $g_1(V_1^c) = 0$, $0 \leqq g_1 \leqq f(x)$; $g_2(x) = u(f)$, $g_2(V_2^c) = 1$, $1 \geqq g_2 \geqq u(f)$; $g = (f \vee g_1) \wedge g_2$. In a similar way as in the proof of Theorem 2 we can show $g \in U_{\alpha'\varepsilon}(f)$. Since $x(f) = u(g)$, $x(g) = u(f)$, we get $x \in U_{\alpha'\varepsilon\varepsilon}(u)$. Hence for any $\alpha' \in A$, $\varepsilon > 0$ there exists $\gamma_0 \in C$ such that $\gamma \geqq \gamma_0$ implies $F \cap U_{\alpha'\varepsilon\varepsilon}(u) \neq \phi$. Therefore $\{F_\gamma\}$ converges to $u(f)$.

From this lemma we get

THEOREM 3. $\overline{M(R)}(\subseteq C'(C_u'(R)))$ *is complete.*[12]

§2. The topological ring of the bounded uniformly continuous functions with the strong topology characterizes the complete metric space. Generally, a uniform homeomorphism between two complete uniform spaces implies the topological isomorphism between their topological rings of the bounded uniformly continuous functions with the strong topology, but the inverse is not valid.[13] Hence we use that topological ring with a suitable uniform topology to characterize R.

LEMMA 4. *M-uniform topology agrees with the strong topology in* $C_u(R)$.

Proof. For given $f \in C_u(R)$ and $\varepsilon > 0$ we choose $\alpha \in A$ such that $y, z \in U_\alpha(x)$ implies $|f(y) - f(z)| < \frac{\varepsilon}{2}$. Since $g \in U_{\alpha\frac{\varepsilon}{2}}(f)$ implies $d(g(x), [f(U_\alpha(x))]) < \frac{\varepsilon}{2}$ for every $x \in R$, $|g(x) - f(x)| < \varepsilon$ holds for $g \in U_{\alpha\frac{\varepsilon}{2}}(f)$. Thus this lemma is established.

DEFINITION. We use the notation $L(R)$ to mean $C_u(R)$ with the natural lattice order.

DEFINITION. If a non-vacous subset J of $L(R)$ satisfies the conditions,

i) $f \leqq g \in J$ implies $f \in J$,

ii) if there exists $\bigcap_{\gamma \in C} f_\gamma$ for $f_\gamma \notin J$, then $\cap f_\gamma \notin J$,

then we call J an *i-set*.

11) $S(F, \mathfrak{U}_\beta) = \cup\{U_\beta(x) \mid U_\beta(x) \cap F\} \neq \phi$
12) $\overline{M(R)}$ means the closure in $C'(C_u'(R))$.
13) Let R be a non-compact complete uniform space and let S be the totally bounded uniform space having the same topology as R and with the uniform subbasis $\{\mathfrak{N}_f \mid f \in C_u(R)\}$, $\mathfrak{N}_f = \{N_k \mid k = \pm 1, \pm 2, \cdots\}$, $N_k = \{x \mid \frac{k}{n} < f(x) < \frac{k+2}{n}\}$, then $C_u(R) = C_u(S)$. If we denote by $\bar{S}$ the completion of S, then $C_u(R) = C_u(\bar{S})$, but R and $\bar{S}$ are not uniformly homeomorphic.

We call a non-vacous subset satisfying the dual condition an *s-set*.

DEFINITION. We mean by an *i-ideal* a subset I of $L(R)$ satisfying

1) $I = \cap\{J_\lambda | \lambda \in M\}$, where J_λ are *i-sets*, and for every $\lambda, \mu \in M$ there exists $\nu \in M$ such that $J_\nu \subseteq J_\lambda \cap J_\mu$,

2) if $f_\gamma \in I$ $(\gamma \in C)$ and if $\{f_\gamma | \gamma \in C\}$ are upper bounded,[14] then for every $\mathfrak{U}_{\alpha\varepsilon}$ there exist $g \in I$ and $f_\gamma'(\gamma \in C)$ such that $g \in U_{\alpha\varepsilon}(f_\gamma')$, $f_\gamma' \geqq f_\gamma$,

3) I is a non-trivial ideal.[15]

We call a subset satisfying the dual condition an *s-ideal*.

LEMMA 5. *For an open set* V $\{f | \exists x \in V : f(x) < a\} = J_a(V)$ *is an i-set.* $\{f | \exists x \in V : f(x) > a\} = S_a(V)$ *is an s-set.*

Proof. It is obvious.

LEMMA 6. $\{f | f(x) \leqq k\} = J_k(x)$ *is an i-ideal.*
$\{f | f(x) \geqq k\} = S_k(x)$ *is an s-ideal.*

Proof. $I_k(x) = \cap\{J_a(V) | a > k,\ V$ is an open nbd of $x\}$. Since Condition 2) is obviously valid for an isolated point x, we prove 2) for an accumulating point x. Let $f_\gamma \in I_k(x)$ $(\gamma \in C)$ and let $f_\gamma \leqq q$ for a real number q, then for a given $\mathfrak{U}_{\alpha\varepsilon}$ and for $\mathfrak{U}_\beta$ such that $\mathfrak{U}_\beta{}^* < \mathfrak{U}_\alpha$ we can define $f, f_\gamma' \in L(R)$ such that $f(x) = k$, $f(U_\beta{}^c(x)) = q$; $f(y) = q$, $y \in U_\beta(x)$; $k \leqq f \leqq q$; $f_\gamma' = f_\gamma \vee f$. For a point $z \notin U_\beta(x)$ $f_\gamma'(z) = f(z)$ holds. For a point $z \in U_\beta(x)$, $f(y) = q \geqq f_\gamma'(z) \geqq f(z) \geqq k = f_\gamma'(x)$ holds. Since $x, y \in U_\alpha(z)$, $f \in U_{\alpha\varepsilon}(f_\gamma')$ and $f \in I_k(x)$ hold.

LEMMA 7. *For every i-set* J *there exists a real number* a *and an open set* P *such that* $f(x) \leqq a$ *for some* $x \in P$ *implies* $f \in J$.[16]

Proof. If we assume the contrary, then from the property of *i-set* $a \notin J$ holds for every a. Hence $J = \phi$, which is a contradiction.

LEMMA 8. *If* $I = \cap\{J_\lambda | \lambda \in M\}$ *is an i-ideal and if* $\sup\{a |$ *there exists* P *such that* $x \in P$, $f(x) \leqq a$ *imply* $f \in J_\lambda\} - \varepsilon = a_\lambda (\varepsilon > 0)$, *then* $\inf_{\lambda \in M} a_\lambda \neq -\infty$.

Proof. Assume that $\inf_{\lambda \in M} a_\lambda = -\infty$, then for every $f \in L(R)$ there exists a real number k and a_λ such that $\varepsilon + a_\lambda < k < f$. Now let us show $f \notin J_\lambda$. Since $k > \varepsilon + a_\lambda$, for every open set P there exists $f_P(x) \in L(R)$ such that $f_P(x) \leqq k$, $x \in P$; $f_P \notin J_\lambda$. Hence $f_P \cup k \notin J_\lambda$, and hence $\inf_P (f_P \cup k) = k \notin J$. Thus we get $f \notin J_\lambda \supseteq I$ and $I = \phi$, which is a contradiction.

Let us put $\inf_{\lambda \in M} a_\lambda = a$ $(\neq -\infty)$, then for every J_λ there exists some open set P

14) There exists f such that $f \geqq f_\gamma$ $(\gamma \in C)$.

15) $I \neq L(R)$, ϕ.

16) Lemmas 7-14 admit the dual propositions.

such that $f(x) \leqq a$ and $x \in P$ imply $f \in J_\lambda$. For this a we give the following

DEFINITION. $\cup\{P | f(x) \leqq a$ and $x \in P$ (open) imply $f \in J_\lambda\} = P_\lambda$.

$$\{f | \exists x \in P_\lambda : f(x) \leqq a\} = A_\lambda \subseteqq J_\lambda.$$

LEMMA. 9. *$J_\mu \subseteqq J_\lambda$ $(\mu, \lambda \in M)$ implies $P_\mu \subseteqq P_\lambda$.*

Proof. If $P_\mu \nsubseteqq P_\lambda$, then there exist $f \in L(R)$ and $x \in P_\mu$ such that $f(x) \leqq a$, $f \notin J_\lambda$. Since $f \in J_\mu$, $J_\mu \nsubseteqq J_\lambda$.

LEMMA 10. *$\{P_\lambda | \lambda \in M\}$ is a cauchy filter.*

Proof. $\{P_\lambda\}$ is a filter from Lemma 9.

Assume that $\{P_\lambda\}$ is not cauchy, then there exists $\mathfrak{U}_\alpha$ such that $U_\alpha(x) \nsupseteq P_\lambda$ for every x, λ. Let $\mathfrak{U}_\beta{}^* < \mathfrak{U}_\alpha$ and let b be an arbitrary large real number, then taking $f_x \in L(R)$ such that $f_x(U_\beta(x)) = b$, $f_x(U_\alpha{}^c(x)) = a$; $a \leqq f_x \leqq b$, we get $f_x \in \cap A_\lambda \subseteqq \cap J_\lambda = I$ for every $x \in R$. Since $\{f_x | x \in R\}$ is bounded, from Condition 2) of an i-ideal there exist $f \in I$, $f_x' \gtrsim f_x$ such that $f \in U_{\beta\varepsilon}(f_x')$. Since $f_x'(U_\beta(x)) \gtrsim b$, it must be $f(x) \geqq b - \varepsilon$ for every $x \in R$, and hence $b - \varepsilon \in I$. Therefore $I = L(R)$ holds, but this is impossible.

Since R is complete, $\{P_\lambda\}$ converges to a point x of R. Then $\{f | f(x) \leqq a - \varepsilon\} = J_{a-\varepsilon}(x) \subseteqq I$ is obvious.

LEMMA 11. *$\{f | f(x) < c\} = J_c'(x) \subseteqq I$ holds for $c = \sup\{k | J_k(x) \subseteqq I\}$.*

Proof. It is obvious.

LEMMA 12. *$f(x) > c$ implies $f \notin I$ for the same c in lemma 11.*

Proof. If we assume that $f(x) > c$ and $f \in I$, then there exists a real number k such that $f(x) > k > c$. If $g(x) \leqq k$, then there exists $h \in L(R)$ such that $h(x) < c$, $f \cup h > g$. Since $h \in I$, we get $f \cup h \in I$ and accordingly $g \in I$. Hence $J_k(p) \subseteqq I$, which contradicts the definition of c.

DEFINITION. We denote by $I(x, c)$ an ideal I satisfying Lemmas 12, 11.

Every i-ideal is represented uniquely by the form $I(x, c)$.

DEFINITION. For two i-ideals I_1, I_2 we define $I_1 \sim I_2$ to mean that there exists some s-ideal S such that $S \cap I_1 = \phi$, $S \cap I_2 = \phi$.

LEMMA 13. *$I(x, c) \sim I(y, d)$, if and only if $x = y$.*

Proof. It is obvious.

DEFINITION. For an i-ideal I and an s-ideal S, we define $S \sim I$ to mean that there exist some i-ideal I_1 and s-ideal S_1 such that $S \sim S_1$, $I \sim I_1$; $S_1 \cap I_1 = \phi$.

LEMMA 14. *$I(x, c) \sim S(y, d)$, if and only if $x = y$.*

Proof. It is obvious.

Hence we can classify all the i-ideals and all the s-ideals by $\sim$. We denote by $L(R)$ the totality of such classes and by $L(x)$ the one-to-one mapping from R onto $\mathfrak{L}(R)$, which maps x to the classes consisting of $I(x, a)$ and $S(x, b)$.

DEFINITION. If for a family $\{I(x, a(x))\,|\,x\in A\}$ of i-ideals there exists $f\in\bigcap_{x\in A} I(x, a(x))$, then we call this family *lower bounded*.
"*Upper bounded*" is defined as the dual.

LEMMA 15. *$\{I(x, a(x))\,|\,x\in A\}$ is lower bounded, if and only if $\inf_{x\in A} a(x) \neq -\infty$.*

Proof. It is obvious.

DEFINITION. $\mathfrak{L}(U)$ and $\mathfrak{L}(A)$ are called *u-disjoint*, if and only if for every lower bounded $\{I(x, a(x))\,|\,x\in U\}$ and upper bounded $\{S(x, b(x))\,|\,x\in A\}$ there exists $f\in\bigcap_{x\in U} I(x, a(x))\bigcap_{x\in A} S(x, b(x))$.

LEMMA 16. *$\mathfrak{L}(U)$ and $\mathfrak{L}(A)$ are u-disjoint, if and only if there exists a uniformly continuous function f such that $f(U)=0$, $f(A)=1$, $0\leqq f\leqq 1$.*

Proof. It is obvious.

DEFINITION. A family $\{\mathfrak{L}(V(x))\,|\,x\in R\}$ of non-void subsets of $\mathfrak{L}(R)$ is called a *uniform nbd* of $\mathfrak{L}(R)$, if and only if there exists $\{\mathfrak{L}(U(x))\,|\,x\in R\}$ such that

1) $\mathfrak{L}(U(x))$ and $\mathfrak{L}(V^c(x))$ are u-disjoint,

2) if $\{I(y, a(y))\,|\,y\in U(x)\}$ is lower bounded for every $x\in R$, then there exist $b(x)$, α, ε such that $f_x\in\bigcap\{I^c(y, a(y))\,|\,y\in U(x)\}$ and $g_x\in I(x, b(x))$ imply $g_x\notin U_{\alpha\varepsilon}(f_x)$ for every $x\in R$.

LEMMA 17. *$\{\mathfrak{L}(V(x))\,|\,x\in R\}$ is a uniform nbd of $\mathfrak{L}(R)$, if and only if $\{V(x)\,|\,x\in R\}$ is a uniform nbd of R.*

Proof. If $\{V(x)\,|\,x\in R\}$ is a uniform nbd of R, then there exists a uniform nbd $\mathfrak{U}_\alpha$ such that $U_\alpha(x)$ and $V^c(x)$ are u-disjoint for each $x\in R$. Since for a lower bounded family $\{I(y, a(y)\,|\,y\in U_\alpha(x)\}$ $\inf\{a(y)\,|\,y\in U_\alpha(x)\}=c(x)\neq-\infty$, we put $b(x)=c(x)-1$. If $g_x\in I(x, b(x))$, $f_x\in\bigcap\{I^c(y, a(y))\,|\,y\in U_\alpha(x)\}$, then $g_x(x)\leqq b(x)$ and $f_x(y)\geqq b(x)+1$ $(y\in U_\alpha(x))$ hold. Hence $g_x\notin U_{\alpha 1}(f_x)$. Therefore $\{\mathfrak{L}(V(x))\,|\,x\in R\}$ is a uniform nbd of $\mathfrak{L}(R)$.

Conversely, if $\{V(x)\,|\,x\in R\}$ is no uniform nbd of $\mathfrak{L}(R)$, then we can show that Condition 2) is not valid for any $\{U(x)\}$ such that $U(x)$ and $V^c(x)$ are u-disjoint. Take the lower bounded family $\{I_1(y)\,|\,y\in U(x)\}$ and any $I(x, b(x))$, α, ε, then since there exist $\beta\in A$ and $x\in R$ such that $\mathfrak{U}_\beta{}^*<\mathfrak{U}_\alpha$, $U_\beta(x)\subsetneqq V(x)\supseteqq U(x)$, we get $f, g\in L(R)$ such that $f(U(x))=2$, $f(V^c(x))=b'(x)=\mathrm{Min}\ (b(x)-1, 2)$, $b'(x)\leqq f\leqq 2$; $g(x)=b'(x)$, $g(y)=2$, $y\in U_\beta(x)-V(x)$, $g(z)=f(z)$ (for $z\notin U_\beta(x)$), $b'(x)\leqq g\leqq 2$ as in the proof of Theorem 2. It is obvious that $f\in\bigcap\{I_1^c(y)\,|\,y\in U(x)\}$,

$g \in I(x, b(x))$ and $f \in U_{\alpha\varepsilon}(g)$. Therefore $\{V(x) \mid x \in R\}$ is not a uniform nbd of $\mathfrak{L}(R)$.

From this lemma we get

THEOREM 4. *In order that two compete uniform spaces R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $L(R_1)$ and $L(R_2)$ are uniformly isomorphic.*[17]

Next, by $C(R)$ we denote $C_u(R)$ with the natural ring-operation. Since $f \geqq g$ in $L(R)$, if and only if there exists h such that $h^2 = f - g$ in $C(R)$, a ring-isomorphism between $C(R_1)$ and $C(R_2)$ generates a lattice-isomorphism between $L(R_1)$ and $L(R_2)$. Hence we get the following corollary.

COROLLARY. *In order that two complete uniform spaces R_1 and R_2 are uniformly homeomorphic, it is necessary and sufficient that $C(R_1)$ and $C(R_2)$ are uniformly isomorphic.*

Finally, let us denote by $L'(R)$ $C_u'(R)$ with the natural lattice-order, then the analogous theory is simpler.

DEFINITION We call a subset I of $L(R)$ an *i′-ideal*, if and only if I satisfies conditions 1), 2) in the definition of i-ideal and 3)′ I is a non-trivial ideal and closed, 4)′ I is a minimum set satisfying 1), 2), 3).

We call a subset S satisfying the dual condition an *s′-ideal*.

LEMMA 18. $I(x) = \{f \mid f(x) = 0\}$ *is an i′-ideal.*

LEMMA 19. *For any i-set J there exists an open set P and $a > 0$ such that $f(x) \leqq a$, $x \in P$ imply $f \in J$.*

Let I be an i'-ideal and let $I = \bigcap_{\lambda \in M} J_\lambda$, where J_λ are i-sets, then we use the following notations, $\cup\{P \mid f(x) \leqq a$ for some $x \in P$ implies $f \in J_\lambda$, and P is open$\}$ $= P_{\lambda\alpha}$, $\bigcup_{a>0} P_{\lambda\alpha} = P_\lambda$; $\{f \mid \exists x \in P_{\lambda\alpha} : f(x) \leqq a\} = A_{\lambda\alpha}$, $\bigcup_{a>0} A_{\lambda\alpha} = A_\lambda \subseteq J_\lambda$.

LEMMA 20. $J_\mu \subseteq J_\lambda$ *implies* $P_\mu \subseteq P_\lambda$.

Proof. $P_{\mu a} \subseteq P_{\lambda\alpha}$ is proved for every $a > 0$ as in Lemma 9, and hence $P_\mu \subseteq P_\lambda$.

LEMMA 21. $\{P_\lambda \mid \lambda \in M\}$ *is a cauchy filter.*

Proof. $\{P_\lambda\}$ is a filter by Lemma 20. If we assume that it is not cauchy, then for some $\mathfrak{U}_\alpha$ and for every λ, x $U_\alpha(x) \not\supseteq P_\lambda$ holds. Choose β such that $\mathfrak{U}_\beta^* < \mathfrak{U}_\alpha$ and $f_x \in L'(R)$ such that $f_x(U_\beta(x)) = 1$, $f_x(U_\alpha^c(x)) = 0$, then $f_x \in \cap A_\lambda \subseteq \cap J_\lambda = I$. Hence there exist f_x', g such that $f_x \leqq f_x'$, $g \in U_{\beta\varepsilon}(f_x')$, $g \in I$ for every $\varepsilon > 0$. Therefore $g(x) \geqq 1 - \varepsilon$ for every $x \in R$, and hence $1 - \varepsilon \in I$. Thus it must be $I = \bar{I} = L'(R)$, but this is a contradiction.

From this lemma any $\{P_\lambda\}$ converges to a point x.

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17) A uniform isomorphism means a uniform homeomorphism preserving the lattice-order.

LEMMA 22. *If* $\{P_\lambda\}$ *converges to* x, *then* $I = I(x)$.

Proof. Let $f(x) = 0$, then for any $\alpha \in A$ and $\varepsilon > 0$ there exist $\beta, \gamma \in A$ such that $\mathfrak{U}_\gamma^* < \mathfrak{U}_\beta < \mathfrak{U}_\beta^* < \mathfrak{U}_\alpha$, $f(y) < \varepsilon$ $(y \in U_\beta(x))$.

Taking $g' \in L'(R)$ such that $g'(U(x)) = 0$, $g'(U_\beta^c(x)) = 1$, we get $g = g' \cap f \in I$. For $U(x) \cap P_\lambda \neq \phi$ for every λ, and this implies $g \in \cap A_\lambda \subseteq I$. Since $g \in U_{\alpha\varepsilon}(f)$ is obvious, we get $f \in \bar{I} = I$. Therefore $I \supseteq I(x) = \{f | f(x) = 0\}$, and hence from Condition 4) of an i-ideal it must be $I = I(x)$.

LEMMA 23. *An arbitrary* s'*-ideal* S *is represented by the form* $S(x) = \{f | f(x) = 1\}$.

DEFINITION. By $I \sim S$ we denote that $I \cap S = \phi$ holds for the i'-ideal I and for the s'-ideal S.

LEMMA 24. $I(x) \sim S(y)$, *if and only if* $x = y$.

Hence we can classify all the i'- and s'-ideals by $\sim$. We denote by $\mathfrak{L}'(R)$ the totality of such pairs and by $\mathfrak{L}'(x)$ the one-to-one mapping from R onto $\mathfrak{L}'(R)$.

DEFINITION. We denote by $\overline{\mathfrak{L}'(A)} \ni \mathfrak{L}'(x)$ the fact that $\cap\{I(y) | y \in A\} \subseteq I(x)$.

LEMMA 25. $\overline{\mathfrak{L}'(A)} \ni \mathfrak{L}'(x)$ *if and only if* $x \in \bar{A}$.

DEFINITION. By *a star-uniform nbd* we mean a family $\{\mathfrak{L}'(V(x)) | x \in R\}$ of open nbds of $\mathfrak{L}'(x)$ such that for some α, ε and for every $f \in S(x)$ and $g \in \cap\{I(y) | y \in U_\alpha(x)\}$, $g \notin U_{\alpha\varepsilon}(f)$ holds.

LEMMA 26. *If* $\{U_\alpha(x) | x \in R\}$ *is a uniform nbd of* R, *then* $\{\mathfrak{L}'(U_\alpha(x)) | x \in R\}$ *is a star-uniform nbd of* $\mathfrak{L}'(R)$.

Proof. If $\{U_\alpha(x)\}$ is a uniform nbd, then $f \in S(x)$ and $g \in \cap\{I(y) | y \in U_\alpha(x)\}$ imply $f(x) = 1$, $g(U_\alpha(x)) = 0$ and accordingly $g \notin U_{\alpha 1}(f)$.

LEMMA 27. *If* $\{S(V(x), \mathfrak{V}) | x \in R\}$ *is no uniform nbd of* R, *then* $\{\mathfrak{L}'(V(x)) | x \in R\}$ *is no star-uniform nbd of* $\mathfrak{L}'(R)$, *where* $\mathfrak{V} = \{V(y) | y \in R\}$ *is a family of nbds.*

Proof. For a given $\mathfrak{U}_\alpha$ we choose $\beta \in A$ such that $\mathfrak{U}_\beta^* < \mathfrak{U}_\alpha$. Since $U_\beta(x) \subsetneq S(V(x), \mathfrak{V})$ for some $x \in R$, for $y \in U_\beta(x) - S(V(x), \mathfrak{V})$ we get nbds $V_1(x)$, $V_2(y)$ and $f \in \mathfrak{L}'(R)$ such that $V_1(x) \cap V_2(y) = \phi$, $V_1(x) \cup V_2(y) \subseteq U_\beta(x)$; $f(V(x)) = 0$, $f(y) = 1$. Moreover we get $g', g'', g \in L'(R)$ such that $g'(x) = 1$, $g'(V_1^c(x)) = 0$; $g''(y) = 0$, $g''(V_2^c(y'')) = 1$; $g = (g' \vee f) \wedge g''$. It is obvious that $f \in \cap\{I(y) | y \in V(x)\}$, $g \in S(x)$ and $g \in U_{\alpha\varepsilon}(f)$ for every $\varepsilon > 0$. Therefore $\{\mathfrak{L}'(V(x))\}$ is no star-uniform nbd.

If we define uniform topology of $\mathfrak{L}'(R)$ by the uniform nbds $\{S(\mathfrak{L}'(V(x)), \mathfrak{L}'(\mathfrak{V})) | x \in R\}$ for star-uniform nbds $\mathfrak{L}'(\mathfrak{V}) = \{\mathfrak{L}'(V(x))\}$, then R and $\mathfrak{L}'(R)$ are uniformly homeomorphic from Lemmas 26, 27.

THEOREM 5. *In order that two complete uniform spaces* R_1 *and* R_2 *are uniformly homeomorphic it is necessary and sufficient that* $L'(R_1)$ *and* $L'(R_2)$ *are uniformly isomorphic.*