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On the minimal complexes

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If a topological space Y has vanishing homotopy groups π_i for $i<n$ and $n<i<q$ $(1<n<q)$, it is a well known result[1] of Eilenberg and MacLane that the homology and cohomology groups of Y in dimensions $\leqq q$ are quite determined by the complex $K(\pi_n, n)$ and the invariant cohomology class $\boldsymbol{k}=\boldsymbol{k}_n^{q+1}(Y)$ of $H^{q+1}(\pi_n, n; \pi_q)$, where $\pi_n=\pi_n(Y)$ and $\pi_q=\pi_q(Y)$.

According to the Mathematical Reviews ***13****(1952)*, it is reported that M. M. Postnikov[2] defined the characteristic cohomology classes $\{\boldsymbol{k}_r\}$ in the more general cases, and he determined by them the homotopy type of the space Y.

In this paper, modifying a definition of $K(\Pi, n)$ a little, we shall construct a complex $K(\pi_n, \pi_q, \boldsymbol{k})$ and define a new invariant $\boldsymbol{k}'=\boldsymbol{k}_q^{q'+1}(Y)$ in the case where Y has vanishing homotopy groups π_i for $i<n$, $n<i<q$ and $q<i<q'$ $(1<n<q<q')$. Furthermore we shall determine a homotopy type of a topological space which is dominated by a CW-complex by making use of the *minimal complexes*[3].

1. Notations

We write $[m]$ for the naturally ordered set of integers $\{0, 1, \dots, m\}$. For each pair of non-negative integers (m, p), let $K_m(p)$ be a free abelian group whose free generators are the monotonic[4], degenerate[5] or non-degenerate[6] maps

$$\beta: \ [m] \longrightarrow [p].$$

Following Eilenberg and MacLane [6], we introduce the special monotonic maps

$$\varepsilon=\varepsilon_p : [p] \longrightarrow [p]$$

defined as the identity map,

$$\varepsilon^i=\varepsilon_p^i : [p-1] \longrightarrow [p] \qquad i=0, \dots, p,$$

defined as the map which covers all of $[p]$ except i, and

$$\eta^i=\eta_p^i : [p] \longrightarrow [p-1] \qquad i=0, \dots, p-1,$$

with $\eta^i(j)=j$ for $j\leq i$, $\eta^i(j)=j-1$ for $j>i$.

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1) The main theorem of [5].
2) M. M. Postnikov; Doklady Akad. Nauk. SSSR (N.S) ***76***, *359-362, 789-791* (*1951*). (***Russian***).
3) Refer [*4*].
4) $\beta(i)\leqq\beta(j)$ if $i\leqq j$.
5) If $\beta(i)=\beta(j)$ for some $i<j$.
6) If $\beta(i)<\beta(j)$ for any $i<j$.

Let Π be a (discrete) abelian group with the unit element 0, and let $F_p(\Pi, m)$ be the family of homomorphism

$$\phi : K_m(p) \longrightarrow \Pi$$

satisfying

$$\phi(\gamma) = 0 \qquad \text{if } \gamma \in K_m(p) \text{ is degenerate.}$$

For any element β of $K_r(p)$ the *β-face* ϕ_β of $\phi \in F_p(\Pi, m)$ is defined to be the element of $F_r(\Pi, m)$ determined by

$$\phi_\beta(\gamma) = \phi(\beta\gamma) \qquad \text{for any map } \gamma \text{ of } K_m(r),$$

where $\beta\gamma = \sum_{ij} \lambda_i \mu_j \beta_i \gamma_j$ if $\beta = \sum_i \lambda_i \beta_i$ and $\gamma = \sum_j \mu_j \gamma_j$.

In particular, the ε^i-face ϕ_{ε^i} of ϕ will be denoted simply by $\phi^{(i)}$ and is called the *i-th face.* We shall further denote the neutral element of $F_p(\Pi, m)$ by $\iota_{p\cdot m}$ i.e.

$$\iota_{p\cdot m}(\gamma) = 0 \qquad \text{for any map } \gamma \text{ of } K_m(p).$$

Let $T^p : \Delta_p \to Y$ be a singular simplex of a space Y. Given an element β of $K_m(p)$, we define T^p_β, the *β-face* of T^p, as the singular m-chain

$$T^p_\beta : \Delta_m \longrightarrow Y$$

obtained by the composition of the barycentric (order preserving) map

$$\bar{\beta} : \Delta_m \longrightarrow \Delta_p$$

determined by β and the map $T^p : \Delta_p \longrightarrow Y$. The ε^i-face T_{ε^i} of a singular simplex T will be denoted simply by $T^{(i)}$ and is called the *i-th face.*

2. The complexes $K(\Pi, n)$ and $K(\Pi, n, \Pi', q, k)$

i) Let Π be a (discrete) abelian group with the unit element 0, and let n be a positive integer greater than 1. We shall then define an R-complex[7] $K(\Pi, n)$ as follows:

A p-cell ϕ of $K(\Pi, n)$ is an element of $F_p(\Pi, n)$ satisfying the condition:

$$\sum_{i=0}^{n+1}(-1)^i \phi(\gamma\varepsilon^i_{n+1}) = 0 \qquad \text{for any map } \gamma \text{ of } K_{n+1}(p). \tag{2.1}$$

Then we can introduce an FD-structure[8] in $K(\Pi, n)$ by defining the homomorphisms

$$\beta^* : K_p(\Pi, n) \longrightarrow K_r(\Pi, n)$$

for each monotonic map $\beta : [r] \longrightarrow [p]$ as $\beta^*(\phi) = \phi_\beta$. Especially we denote ε^{i*}_p, η^{i*}_p as F^p_i, D^p_i respectively. Note that $\iota_{p\cdot p}$ is the identity of $K_p(\Pi, n)$, and our requirements on the homomorphisms F_i and D_i include

$$F_i \iota_{p\cdot n} = \iota_{p-1\cdot n}, \quad D_i \iota_{p\cdot n} = \iota_{p+1\cdot n}.$$

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7) For the definition of the R-complex, see [6].

8) Refer [6].

ii) Let further Π' be a (discrete) abelian group with the unit element 0, and let q be a positive integer greater than n, and k be a fixed cocycle of $Z^{q+1}(\Pi, n\,;\,\Pi')$. We shall then define an R-complex $K(\Pi, n, \Pi', q, k)$ as follows:

A p-cell of $K(\Pi, n, \Pi', q, k)$ is a pair (ϕ, ψ), where ϕ is a p-cell of $K(\Pi, n)$, and ψ is an element of $F_p(\Pi'\, q)$ subject to the condition:

(2.2) $$\textstyle\sum_{i=0}^{q+1}(-1)^i\psi(\gamma\varepsilon_{q+1}^i)+k(\phi_\gamma)=0 \quad \text{for any map } \gamma \text{ of } K_{q+1}(p).$$

It is obvious that $(\iota_{p\cdot n},\ \iota_{p\cdot q})$ is the p-identity[9] of $K(\Pi, n, \Pi', q, k)$. Further $K(\Pi, n, \Pi', q, k)$ becomes an *FD*-complex[10] if we define for any monotonic map $\beta : [r] \longrightarrow [p]$ a homomorphism

$$\beta^* : K_p(\Pi, n, \Pi', q, k) \longrightarrow K_r(\Pi, n, \Pi', q, k)$$

by the formula

$$\beta^*(\phi, \psi) = (\phi, \psi)_\beta = (\phi_\beta, \psi_\beta)\,.$$

Then, by the above definitions, we have obiously

$$K_i(\Pi, n, \Pi', q, k) = \{(\phi, \psi)\,;\, \psi = \iota_{i\cdot q}\} \simeq K_i(\Pi, n) \ \text{for}\ i < q\,,$$
$$K_q(\Pi, n, \Pi', q, k) \supset \{(\phi, \psi)\,;\, \psi = \iota_{q\cdot q}\} \simeq K_q(\Pi, n)\,.$$

iii) Uniqueness of $K(\Pi, n, \Pi', q, k)$.

Let $K(\Pi, n, \Pi', q, k_1)$, $K(\Pi, n, \Pi', q, k_2)$ be the complexes corresponding to any cocycles k_1, k_2 which is cohomologous to each other. Let h be a q-cochain of $C^q(\Pi, n\,;\,\Pi')$ such that $k_1 - k_2 = \delta h$.

For each p-cell (ϕ_1, ψ_1) of $K(\Pi, n, \Pi', q, k_1)$, we shall define a p-cell (ϕ_2, ψ_2) of $K(\Pi, n, \Pi'\, q, k_2)$ such that

$$\phi_2 = \phi_1\,,$$
$$\psi_2(\gamma) = \psi_1(\gamma) + h(\phi_\gamma) \quad \text{for any map } \gamma \text{ of } K_q(p).$$

Setting $(\phi_2, \psi_2) = \eta(\phi_1, \psi_1)$ we have the natural isomorphism

(2.3) $$\eta : K(\Pi, n, \Pi', q, k_1) \longrightarrow K(\Pi, n, \Pi', q, k_2).$$

Moreover we have

$$\beta^*\eta(\phi_1, \psi_1) = (\phi_{2\cdot\beta}, \psi_{2\cdot\beta}) = \eta\beta^*(\phi_1, \psi_1)\,,$$

where

$$\psi_{2\cdot\beta}(\gamma) = \psi_{1\cdot\beta}(\gamma) + h(\phi_{\beta\gamma})\,,$$

and therefore η is an FD-map[11].

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9) The identity of K_p (Π, n, Π', q, k).
10) Refer [6].
11) Refer [6].

Thus, in the following we shall denote $K(\Pi, n, \Pi', q, k)$ simply as $K(\Pi, \Pi', \boldsymbol{k})$ if no confusion is occured, where $\boldsymbol{k}$ is the fixed cohomology class reprsented by k. Strictly speaking, $K(\Pi, \Pi', \boldsymbol{k})$ is the family of the complexes $K(\Pi, n, \Pi', q, k)$ being isomorphic onto one another.

3. The main theorem

Let Y be a topological space with vanishing homotopy groups π_i for $i<n$ and $n<i<q$ $(1<n<q)$, and let M be a minimal subcomplex of $S_n(Y)$. Then there is a pair of FD-maps $(\kappa, \bar{\kappa})$ satisfying the following conditions:

(3.1) $\kappa : M \longrightarrow K(\pi_n, n)$,

(3.2) For any cocycle k of the invariant $\boldsymbol{k} = \boldsymbol{k}_n^{q+1}(Y)$, there is an FD-map

$$\bar{\kappa} = \bar{\kappa}_k : K(\pi_n, n) \longrightarrow M$$

such that

(3.2.1) For each p-cell ϕ of $K(\pi_n, n)$ $\bar{\kappa}_k(\phi) = \bar{\kappa}_{k'}(\phi)$ and

$$\kappa(\bar{\kappa}(\phi)) = \phi \qquad (p<q),$$

(3.2.2) $\bar{\kappa}\kappa(T) = T$ for any singular simplex T of $M^{(q-1)}$,[12]

(3.2.3) For each q-cell ϕ of $K(\pi_n, n)$, $\bar{\kappa}_k(\phi)$ depends upon k and $\kappa(\bar{\kappa}_k(\phi)) = \phi$. For any $(q+1)$-cell ϕ of $K(\pi_n, n)$, if we define a map $f_\phi : \Delta_{q+1,q} \longrightarrow Y$ such that[13] $f_\phi \bar{\varepsilon}^{\,i}_{q+1} = \bar{\kappa}_k(\phi^{(i)})$ $i=0, \dots, q+1$, then we have[14] $k(\phi) = c(f_\phi)$.

Since arbitrary minimal complexes are isomorphic onto one another, we shall consider a fixed minimal complex in the following.

THEOREM I. *If a topological space Y has vanishing homotopy groups π_i for $i<n$, $n<i<q$ and $q<i$ $(1<n<q)$, $\pi_n(Y) = \pi_n$, $\pi_q(Y) = \pi_q$ and $\boldsymbol{k}_n^{q+1}(Y) = \boldsymbol{k}$, then the complex $K(\pi_n, \pi_q, \boldsymbol{k})$ is isomorphic onto the minimal subcomplex M of $S(Y)$*[15].

THEOREM II. *If a topological space Y has vanishing homotopy groups π_i for $i<n$, $n<i<q$ and $p<i<q'$ $(1<n<q<q')$, $\pi_n(Y) = \pi_n$, $\pi_q(Y) = \pi_q$, $\boldsymbol{k}_n^{q+1}(Y) = \boldsymbol{k}$ and $\pi_{q'}(Y) = \pi_{q'}$, then there is a unique cohomology class $\boldsymbol{k}_q^{q'+1}(Y)$ of $H^{q'+1}(\pi_n, \pi_q, \boldsymbol{k}; \pi_{q'})$, and it is a topological invariant* (if we pay no heed to the identification of the complexes $K(\pi_n, n, \pi_q, q, k)$).

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12) $M^{(q-1)}$ is the $(q-1)$-dimensional skeleton of M.

13) $\bar{\varepsilon}^{\,i}_{q+1}$ is the barycentric map induced by ε^i_{q+1}.

14) c is the characteristic function defined in [1].

15) Cf. the main theorem of [5], p. 529.

4. Proof of the theorem I

To prove the theorem I, it is sufficient only to show the existence of a pair of FD-maps $(\kappa', \bar{\kappa}')$

$$\kappa' : \quad M \longrightarrow K(\pi_n, n, \pi_q, q, k),$$
$$\bar{\kappa}' : \quad K(\pi_n, n, \pi_q, q, k) \longrightarrow M$$

satisfying the following conditions:

(4.1) For each p-cell (ϕ, ψ) of $K(\pi_n, n, \pi_q, q, k)$ there is a unique p-simplex in the minimal complex M such that

$$\kappa'\{\bar{\kappa}'(\phi, \psi)\} = (\phi, \psi),$$

(4.2) $\bar{\kappa}' \kappa'(T) = T$ for any singular simplex T of M.

First, let $p < q$. Since every map in $K_q(p)$ is degenerate, there is only one neutral element $\iota_{p\cdot q}$ in $F_p(\pi_q, q)$. Therefore the p-cell (ϕ, ψ) is quite determined only by ϕ, and also $\bar{\kappa}'$, κ' can be determined by[16]

$$\kappa'(T) = (\kappa(T), \iota_{p\cdot q}),$$
$$\bar{\kappa}'(\phi, \iota_{p\cdot q}) = \bar{\kappa}_k(\phi).$$

For any $p \geq q$, we shall define $\bar{\kappa}'$ as follows:

Let $p = q$. For each q-cell (ϕ, ψ) we first choose a q-singular simplex $\bar{\kappa}_k(\phi)$ as in (3.2), and we choose a singular simplex

$$T^q : \Delta_q \longrightarrow Y$$

of M compatible[17] with $\bar{\kappa}_k(\phi)$ and satisfying[18]

$$d(\bar{\kappa}_k(\phi), T^q) = \psi(\varepsilon_q) \in \pi_q(Y).$$

Then, we define

$$\bar{\kappa}'(\phi, \psi) = T^q.$$

Let $p = q+1$. For each $(q+1)$-cell (ϕ, ψ), we define the map

$$f_\phi : \Delta_{q+1\cdot q} \longrightarrow Y$$

as in (3.2.3), and for any face $(\phi, \psi)^{(i)}$ of (ϕ, ψ) we define a singular simplex of M

$$T^{q\cdot i} : \Delta_q \longrightarrow Y$$

as in above. Since these mappings are compatible with $\bar{\kappa}_k(\phi^{(i)})$, the set of mappings $T^{q\cdot i}$ $(i = 0, 1, \dots, q+1)$ define a map

$$\dot{T} : \Delta_{q+1\cdot q} \longrightarrow Y.$$

16) The existence of FD-maps κ, $\bar{\kappa}_k$ follows from (3.1) and (3.2).

17) Two singular q-simplexes T_1, T_2 are called compatible if their faces coincide: $T_1^{(i)} = T_2^{(i)}$ for $0 \leq i \leq q$.

18) d is the difference-function defined in [1].

From (2.2) and (3.2.5) we have

$$c(\dot{T}) = \sum_{i=0}^{q+1}(-1)^i d(\bar{\kappa}_k(\phi^{(i)}),\ T^{(i)}) + c(f_\phi)$$
$$= \sum_{i=0}^{q+1}(-1)^i \psi(\varepsilon_{q+1}^i) + k(\phi) = 0\,.$$

Therefore we can extend the map T all over the Δ_{q+1} such that we obtain a singular simplex

$$T^{q+1}\ :\ \Delta_{q+1} \longrightarrow Y$$

of M, then, we define

$$\bar{\kappa}'(\phi,\ \psi) = T^{q+1}.$$

Let $p > q+1$. For each p-cell (ϕ, ψ) we shall define $\bar{\kappa}'(\phi, \psi)$ inductively as follows: For any i-th face of (ϕ, ψ) we may determine the singular simplex of M

$$T^{p-1\cdot i}\ :\ \Delta_{p-1} \longrightarrow Y$$

by the inductive assumption, and hence these maps define a map

$$\dot{T}\ :\ \Delta_{p\cdot p-1} \longrightarrow Y.$$

Since $\pi_{p-1}(Y) = 0$, we may extend it all over the Δ_p such that we obtain a singular simplex

$$T^p\ :\ \Delta_p \longrightarrow Y$$

of M, then, we define

$$\bar{\kappa}'(\phi,\ \psi) = T^p.$$

These constructions are uniquely determined, because, if T is a singular simplex all of whose faces are in M, then M contains a unique singular simplex compatible with and homotopic to T.

Conversely, for any singular simplex $T^p\ :\ \Delta_p \longrightarrow Y$ of $S_n(Y)$, we may construct a p-cell $\kappa'(T) = (\phi, \psi)$ of $K(\pi_n, n, \pi_q, q, k)$ as follows:

At first, ϕ is determined by $\kappa(T^p)$ as in (3.1). Any element β of $K_q(p)$ determines a q-dimentional (degenerate or non-degenerate) face T_β of T^p and ϕ_β of just determined ϕ, also an element

$$d\,(\bar{\kappa}_k(\phi_\beta),\ T_\beta) \in \pi_q\,,$$

where $d(\bar{\kappa}_k(\phi_\beta),\ T_\beta) = \sum_i m_i d(\bar{\kappa}_k(\phi_{\beta_i}),\ T_{\beta_i})$ if $\beta = \sum_i m_i \beta_i$.

Consequently we may define ψ by

$$\psi(\beta) = d(\bar{\kappa}_k(\phi_\beta),\ T_\beta) \quad \text{for any map } \beta \text{ of } K_q(p)\,.$$

Then it is easily seen that, these constructions satisfy the conditions (2.1), (2.2) and also (4.1), (4.2). The proof is complete.

5. Proof of the theorem II.

In this section we shall consider a topological space Y with vanishing homotopy groups π_i for $i<n$, $n<i<q$ and $q<i<q'$ $(1<n<q<q')$, and corresponding minimal subcomplex M of $S(Y)$.

In the above section, if we attempt to continue the definition of $\bar{\kappa}'$ for $(q'+1)$ -cells $\Phi=(\phi, \psi)$ of $K(\pi_n, n, \pi_q, q, k)$, we can only go as far as to define a map

$$f_\Phi : \Delta_{q'+1\cdot q'} \longrightarrow Y$$

such that

$$f_\Phi \bar{\varepsilon}^i_{q'+1} = \bar{\kappa}'(\Phi^{(i)}) \qquad i=0, \dots, q+1 .$$

Since $\pi_{q'}$ is not assumed to vanish, the map f_Φ in general will not be extendable to a map $\Delta_{q'+1} \longrightarrow Y$. We define $k'=k^{q'+1}$ writing

$$k'(\Phi)=c(f_\Phi) \in \pi_{q'} .$$

Then k' is a cochain of $C^{q'+1}$ $(\pi_n, n, \pi_q, q, k ; \pi_{q'})$,[19] and it follows from

$$(\delta k')(\Phi)=k'(\partial\Phi)=c(\partial f_\Phi)=(\delta c)(f_\Phi)=0$$

that k' is a cocycle, where Φ is any $(q'+2)$-cell of $K(\pi_n, n, \pi_q, q, k)$ and f_Φ is an induced map $\Delta_{q'+2\cdot q'} \longrightarrow Y$ corresponding to Φ.

The cohomology class of the cocycle k' will be denoted by $\boldsymbol{k}'$ or $\boldsymbol{k}_q^{q'+1}(Y)$. It is an element of the cohomology group $H^{q'+1}$ $(\pi_n, n, \pi_q, q, k ; \pi_{q'})$, and it depends upon the choice of the minimal complex M and after that upon the choice of $\bar{\kappa}'$ seemingly.

We shall first examine the effect of altering $\bar{\kappa}'$ without altering M. Let $\bar{\kappa}_1'$, $\bar{\kappa}_2'$ be the different FD-maps satisfying (4.1) and (4.2). For each q'-cell Φ of $K(\pi_n, n, \pi_q, q, k)$, the q'-simplexes $\bar{\kappa}_1'(\Phi)$ and $\bar{\kappa}_2'(\Phi)$ are compatible. Let h' be a cochain of $C^{q'}(\pi_n, n, \pi_q, q, k ; \pi_{q'})$ defined by

$$h'(\Phi)=d(\kappa_1'(\Phi), \kappa_2'(\Phi)) \in \pi_{q'}(Y) ,$$

and let k_1', k_2' be the cocycles defined by making use of $\bar{\kappa}_1'$ and $\bar{\kappa}_2'$ respectively. Then we can easily obtain.

$$k_1'-k_2'+\delta h'=0$$

This shows that the cohomology class $\boldsymbol{k}'$ is independent of the choice of $\bar{\kappa}'$ for a fixed M (and fixed complex $K(\pi_n, n, \pi_q, q, k)$).

We shall next examine the effect of altering k without altering M. Let k_1, k_2 be the different cocycles of $\boldsymbol{k}_n^{q+1}$, and $K_i (i=1, 2)$ be the corresponding complex $K(\pi_n, n, \pi_q, q, k_i)$. Then there is the natural FD-map

$$\eta : K_1 \longrightarrow K_2$$

as in (2.5).

19) We shall denote C $(K(\pi_n, n, \pi_q, q, k); \pi_{q'})\dots$ simply by $C(\pi_n, n, \pi_q, q, k ; \pi_{q'})\dots$.

For each q-cell (ϕ_i, ψ_i), let us define the simplex $T_i^q = \bar{\kappa}_i'(\phi_i, \psi_i)$ as in §4. If $\eta(\phi_1, \psi_1) = (\phi_2, \psi_2)$ we have

$$d(\bar{\kappa}_1(\phi_1), T_1^q) = \psi_1(\varepsilon_q) = \psi_2(\varepsilon_q) - h(\phi) = d(\bar{\kappa}_2(\phi_2), T_2^q) - d(\bar{\kappa}_2(\phi), \bar{\kappa}_1(\phi))$$
$$= d(\bar{\kappa}_1(\phi_1), T_2^q) .$$

Thus T_1^q and T_2^q are homotopic, hence we have $T_1^q = T_2^q$, since M is minimal.

For each p-simplex (ϕ_i, ψ_i) where $q < p < q'$, let us define the simplex

$$T_i^p = \bar{\kappa}_i'(\phi_i, \psi_i) \qquad (i = 1, 2)$$

as in §4 inductively. Then we have that T_1^p and T_2^p are compatible, and so that $T_1^p = T_2^p$ since $\pi_p(Y) = 0$ and M is minimal.

While, for each q'-cell (ϕ_1, ψ_1) of K_1 there is at least one q'-simplex T_1 of M such that $\kappa'(T_1) = (\phi_1, \psi_1)$. Any two such simplexes are compatible. One of these simplexes T_1 will be selected and denoted by $\bar{\kappa}_1'(\phi_1, \psi_1)$. And, for each q'-cell (ϕ_2, ψ_2) we shall select the q'-simplex $\bar{\kappa}_1'(\eta^{-1}(\phi_2, \psi_2))$ and denote by $\bar{\kappa}_2'(\phi_2, \psi_2)$.

Further, if we attempt to continue the definitions of $\bar{\kappa}_1'$ and $\bar{\kappa}_2'$ for $(q'+1)$-cells of K_1 and K_2, we obtain two mappings

$$f_{\Phi_1}, f_{\Phi_2} : \Delta_{q'+1, q'} \longrightarrow Y ,$$

and it follows from our definition that

$$c(f_{\Phi_1}) = c(f_{\eta\Phi_1}) \quad \text{for any } (q'+1)\text{-cell } \Phi_1 \text{ of } K_1 .$$

This shows that

$$k_1'(\Phi_1) = k_2'(\eta\Phi_1)$$

and also that

$$\mathbf{k}_{q(1)}^{q'+1} = \eta^* \mathbf{k}_{q(2)}^{q'+1} \tag{5.6}$$

where $\mathbf{k}_{q(i)}^{q'+1}$ is the cohomology class of the cocycle k_i' and η^* is the natural homomorphism

$$\eta^* : H^{q'+1}(K_2, \pi_{q'}) \longrightarrow H^{q'+1}(K_1, \pi_{q'}) .$$

Finally, we shall examine the effect of altering M without altering k. Let M_1, M_2 be the different minimal subcomplexes relative to the same base point $y_0 \in Y$. Then there is a chain homotopy

$$\varphi_t : S(Y) \longrightarrow S(Y)$$

satisfying the following properties:

(5.1) φ_t is continuous,

(5.2) φ_0 is identity,

(5.3) φ_1 maps M_1 isomorphically onto M_2,

(5.4) $\varphi_t\, T = T$ if T is collapsed.

It follows that for $T \in M_1(Y)$ we have $\kappa_1'(T) = \kappa_2'(\varphi_1 T)$. Now, select the FD-map $\bar{\kappa}_1'$ of the q'-skeleton of $K(\pi_n, n, \pi_q, q, k)$ into M_1 such that $\kappa_1' \bar{\kappa}_1' =$ identity as in §4, and also define an FD-map $\bar{\kappa}_2'$ for M_2 by setting $\bar{\kappa}_2' = \varphi_1 \kappa_1'$. For each $(q'+1)$-cell $\Phi = (\phi, \psi)$ of $K(\pi_n, n, \pi_q, q, k)$ we then have maps

$$f_\Phi^{(1)}, f_\Phi^{(2)} : \Delta_{q'+1 \cdot q'} \longrightarrow Y,$$

and it follows from the conditions on φ_t that $f_\Phi^{(1)}$ and $f_\Phi^{(2)}$ are homotopic. Thus $c(f_\Phi^{(1)}) = c(f_\Phi^{(2)})$ and also $k_{q\,(1)}^{q'+1} = k_{q\,(2)}^{q'+1}$. Consequently, the cohomology class $\boldsymbol{k}_q^{q'+1}(Y)$ does not depend on the choice of M. The proof is complete.

6. Homotopy type of a space Y

In this section we shall deal with the homotopy type of a topological space Y which is dominated by a CW-complex[20] and has vanishing homotopy groups π_i for $i < n$, $n < i < q$ and $q < i$, And we shall make use of the notations and results of the J. B. Giver's paper [3].

For each minimal subcomplex M of $S(Y)$, we shall construct the singular sub-polytope $P_M(Y)$ of $\bar{P}(Y)$ by the same method as in [3]. Then since $M(Y)$ is a deformation retract of $S(Y)$,[21] it follows that

(6.1) $P_M(Y)$ and $\bar{P}(Y)$ are of the same homotopy type.

While, following J. B. Giever [3], there is a mapping

$$f : \bar{P}(Y) \longrightarrow Y$$

which induce the isomorphisms

$$f_n : \pi_n(\bar{P}(Y)) \longrightarrow \pi_n(Y)$$

for every n. And following J. H. C. Whitehead [2], this implies that

(6.2) f is a homotopy equivalence.

THEOREM III, *Let Y be a topological spoce with vanishing homotopy groups π_i for $i < n$, $n < i < q$ and $q < i$ and is dominated by a CW-complex. Then the homotopy type of Y is quite determined by π_n, π_q and $\boldsymbol{k}_n^{q+1}(Y)$.*[22]

Proof. Let Y_1 and Y_2 be topological spaces satisfying the assumption of the theorem III. Then by the theorem I, there is the FD-maps κ_i', $\bar{\kappa}_i'$ such that

$$M(Y_1) \underset{\bar{\kappa}_1'}{\overset{\kappa_1'}{\rightleftarrows}} K(\pi_n, \pi_q, \boldsymbol{k}) \underset{\kappa_2'}{\overset{\bar{\kappa}_2'}{\rightleftarrows}} M(Y_2).$$

20) Refer [2].
21) Refer [4], p. 505.
22) The systems $(\pi_n, \pi_q, \boldsymbol{k})$ and $(\pi'_n, \pi'_q, \boldsymbol{k}')$ are identified if there are isomorphisms $f_n : \pi_n \cong \pi'_n$, $f_q : \pi_q \cong \pi'_q$ such that $f_n^* \boldsymbol{k}' = f_q^{\#} \boldsymbol{k}$ where $f_n^* : H^{q+1}(\pi'_n, n; \pi'_q) \to H^{q+1}(\pi_n, n; \pi'_q)$ and $f_q^{\#} : H^{q+1}(\pi_n, n; \pi_q) \to H^{q+1}(\pi_n, n; \pi'_q)$ are the induced homomorphisms.

For the fixed cocycle k of $\boldsymbol{k}$, $\bar{\kappa}_i'$ is uniquely determined for the fixed minimal subcomplex $M(Y_i)$ of $S(Y_i)$ where $i=1, 2$. Thus, there are one-to-one correspondences between $M(Y_1)$ and $M(Y_2)$ through the complex $K(\pi_n, n, \pi_q, q, k)$. From this identification we may conclude that

(6.3) $P_M(Y_1)$ and $P_M(Y_2)$ are of the same homotopy type.

The theorem III is an immediate consequence of (6.1), (6.2) and (6.3).

7. Generalizations

In the following, let Y be a simpy connected topological space. We shall construct complexes by induction on r

$$K_r = K(1, \pi_2, \dots, \pi_r, 0, \boldsymbol{k}_2, \dots, \boldsymbol{k}_{r-1})$$

which is an approximation for the minimal subcomplex M of $S(Y)$ in the sence of the theorem I.

Assume that the complexes $\{K_1, K_2, \dots, K_{r-1}\}$ are already constructed, and we shall determine the invariant $\boldsymbol{k}_{r-1}$ as follows:

For any r-cell Φ of K_{r-1}, there is at least one simplex T of the minimal complex M such that $\kappa^{(r-1)}(T) = \Phi$ as in §4. Although this correspondence is not determined uniquely since π_r is not assumed to vanish, we shall select one of these simplexes and denoted by $\bar{\kappa}^{(r-1)}(\Phi)$.

For any $(r+1)$-simplex Φ of K_{r-1}, we obtain a characteristic map

$$f_\Phi : \Delta_{r+1 \cdot r} \longrightarrow Y$$

and also an element $c(f_\Phi)$ of π_r as in §5. Consequently, we obtain a cochain $k_{r-1} = k_{r-1}^{r+1}$ of $C^{r+1}(K_{r-1}; \pi_r)$ writing

$$k_{r-1}(\Phi) = c(f_\Phi).$$

This cochain determine a unique cohomology class $\boldsymbol{k}_{r-1}$ of $H^{r+1}(K_{r-1}; \pi_r)$ as in §5.

We shall next define a complex K_r as follows:

A p-cell of K_r is a pair (Φ, ψ) where Φ is a p-cell of K_{r-1}, and ψ is an element of $F_p(\pi_r, r)$ subject to the condition:

$$\textstyle\sum_{i=0}^{r+1}(-1)^i\psi(\gamma\varepsilon_{r+1}^i) + k_{r-1}(\Phi_\gamma) = 0 \quad \text{for any element } \gamma \text{ of } K_{r+1}(p).$$

Then it is obvious that $(\iota_{p\cdot 1}, \iota_{p\cdot 2}, \dots, \iota_{p\cdot r})$ is the p-identity of K_r. Further K_r becomes an FD-complex if we define for any element β of $K_{p'}(p)$ a homomorphism

$$\beta^* : K_{r\cdot p} \longrightarrow K_{r\cdot p'}$$

by the formula

$$\beta^*(\Phi, \psi) = (\Phi, \psi)_\beta = (\Phi_\beta, \psi_\beta).$$

The uniquencess of K_r and the existence of *FD*-maps $\kappa^{(r)}$, $\bar{\kappa}^{(r)}$ are proved by the arguments similar to those used in the preceding sections.

Thus we have the following;

THEOREM IV. *The r-th homology and cohomology groups of a simply connected topological space Y are quite determined by the complex* K_{r+1}.

THEOREM V. *If a simply connected topological space Y has vanishing homotopy groups* π_i *for* $i > r$ *and is dominated by a CW-complex, then the homotopy type of Y is quite determined by the system*

$$\{1, \pi_2, \dots, \pi_r, 0, \boldsymbol{k}_2, \dots, \boldsymbol{k}_{r-1}\}.$$

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