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On a theorem of Eilenberg-MacLane

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1. Introduction

Introducing the notion of '*classes of Abelian groups*', J–P. Serre gave in his recent paper [7][1] very useful generalizations of the classical theorems of W. Hurewicz and of J.H.C. Whitehead. Among the same kind of these classical theorems, we find a theorem due to S. Eilenberg and S. MacLane [4]: *If Y is an arcwise connected space with vanishing homotopy groups $\pi_i(Y)$ for $i<n$ and $n<i<q$, then the homotopy group $\pi_n(Y)$ determines the homology groups of Y in dimensions $<q$, and partially the q–dimensional homology group of Y.*

In the present note, we generalize the theorem of Eilenberg-MacLane after the fashion of Serre. The generalizations are stated in §2 and are proved in §3. As applications we have two theorems. We solve in §4 a special kind of homotopy type problem, and prove in §5 a theorem by which we can obtain an information about the homotopy groups from calculations of the Betti numbers.

Throughout this note all spaces will be assumed to be arcwise connected.

2. Statements of theorems

Let $\mathcal{C}$ be a *class* in the sense of [7; Chap. I]. Namely $\mathcal{C}$ is a *non-vacuous* collection of Abelian groups satisfying the condition:

(I) *If, in the exact sequence $L \rightarrow M \rightarrow N$, the groups L and N are in $\mathcal{C}$, then M is also in $\mathcal{C}$.*

We further throughout this and next sections assume that $\mathcal{C}$ satisfies the conditions:

(II_B) *If M is in $\mathcal{C}$, then the tensor product $M \otimes N$ is in $\mathcal{C}$ for any group N.*

(III) *If M is in $\mathcal{C}$, then the i–dimensional homology group of M, $H_i(M, 1) = H_i(M)$, is in $\mathcal{C}$ for any $i>0$.*

We call that a homomorphism $f: M \rightarrow N$ is *$\mathcal{C}$–on* if the cokernel $N/f(M)$ is in $\mathcal{C}$, and that f is *$\mathcal{C}$–isomorphic* if the kernel and the cokernel are both in $\mathcal{C}$. For two given groups M and N, if there exist a group L and two $\mathcal{C}$–isomorphisms (i.e. $\mathcal{C}$–isomorphic homomorphisms) $f: L \rightarrow M$ and $g: L \rightarrow N$, then we call that M and N are *$\mathcal{C}$–isomorphic*, and write $M \underset{\mathcal{C}}{\approx} N$. See [7] for the detailed accounts of classes.

Let us denote by $\mathcal{K}(\pi, n)$ any one of spaces X such that $\pi_n(X) = \pi$ and $\pi_i(X) = 0$ for $i \neq n$. Then it is well known [4] that the singular homology groups

1) Numbers in brackets refer to the references cited at the end of the paper.

$H_i(\mathcal{K}(\pi, n); G)$ with coefficient group G are determined by π, n and G alone, and are denoted by $H_i(\pi, n; G)$.

We can now state the generalized theorem of Eilenberg-MacLane as follows:

THEOREM 1. *Let Y be a simply connected space such that $\pi_i(Y)$ is in $\mathcal{C}$ for $i < n$ and $n < i < q$. Then we have*

$$H_i(Y; G) \underset{\mathcal{C}}{\approx} H_i(\pi_n(Y), n; G) \quad \text{if} \quad i < q,$$
$$H_q(Y; G) / \Sigma_q(Y; G) \underset{\mathcal{C}}{\approx} H_q(\pi_n(Y), n; G),$$

where $\Sigma_q(Y; G)$ denotes the spherical subgroup [3].

More generally we have

THEOREM 2. *Let Y, Y' be simply connected spaces such that $\pi_i(Y)$ and $\pi_i(Y')$ are in $\mathcal{C}$ for $i < n$ and $n < i < q$. If $\pi_n(Y) \underset{\mathcal{C}}{\approx} \pi_n(Y')$ and $G \underset{\mathcal{C}}{\approx} G'$, we have*

$$H_i(Y; G) \underset{\mathcal{C}}{\approx} H_i(Y'; G') \quad \text{if} \quad i < q,$$
$$H_q(Y; G)/\Sigma_q(Y; G) \underset{\mathcal{C}}{\approx} H_q(Y'; G')/\Sigma_q(Y'; G').$$

It is obvious that Theorem 2 is a direct consequence of Theorem 1 and the following

PROPOSITION 1. *If $M \underset{\mathcal{C}}{\approx} N$ and $G \underset{\mathcal{C}}{\approx} G'$, we have*

$$H_i(M, r; G) \underset{\mathcal{C}}{\approx} H_i(N, r; G')$$

for $i \geqq 0$ and $r \geqq 2$.

REMARK. If $\pi_i(Y)$, $\pi_i(Y')$ and so $H_i(Y)$, $H_i(Y')$ are finitely generated for any $i \geqq 0$, the same arguments as in [7; p. 275] show that Theorem 1 and 2 hold for any class $\mathcal{C}$ which does not necessarily satisfy ($\mathrm{II_B}$) and (III).

3. Proofs of theorems

Let us denote by $(E, F, B; p)$ or (E, F, B) a *fiber space* in the sense of [6; p. 443], where E, F, B and p designate respectively the total space, the fiber, the base space and the projection. Then the following proposition is a direct consequence of the homology exact sequence for (E, F) and the Theorem 1. B in [7; p. 268].

PROPOSITION 2. *Let B be simply connected, and let $H_i(F)$ be in $\mathcal{C}$ for $0 < i < r$. Then $p_*: H_i(E) \to H_i(B)$ $(i < r)$ and $p_*: H_r(E)/i_* H_r(F) \to H_r(B)$ induced by the projection p are $\mathcal{C}$-isomorphisms, where $i_*: H_r(F) \to H_r(E)$ is the injection.*[2]

It was proved by H. Cartan and J-P. Serre [1] (see also [9]) that we can associate with any simply connected space X a sequence of simply connected spaces X_r and continuous maps $f_r: X_{r+1} \to X_r$ $(r = 1, 2, \cdots)$ with the following properties:

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2) We denote briefly by $H_i(Y)$ in place of H_i $(Y; Z)$, where Z is the additive group of integers.

(i) $(X_{r+1}, \mathcal{K}(\pi_r(X), r-1), X_r; f_r)$ constitutes a fiber space.

(ii) There exists a fiber space $(X_r', X_{r+1}, \mathcal{K}(\pi_r(X), r))$ such that X_r' is of the same homotopy type as X_r.

(iii) $\pi_i(X_r)=0$ for $i<r$; $X_1=X_2=X$; and the composition $f_1\circ f_2\circ\cdots\cdots\circ f_{r-1}$ induces an isomorphism of $\pi_i(X_r)$ onto $\pi_i(X)$ for $i\geqq r$.

Proof of Theorem 1. Consider the sequence of Cartan-Serre for the space Y. Then we have by (iii) and the assumption that

$$\begin{aligned} &\pi_i(Y_{n+1})=\pi_i(Y)\in\mathcal{C} && \text{for } n<i<q, \\ &\pi_i(Y_{n+1})=0 && \text{for } i\leqq n, \end{aligned}$$

and hence $\pi_i(Y_{n+1})$ is in $\mathcal{C}$ for any $i<q$. Therefore it follows from the generalized Hurewicz theorem [7; p. 271] that

$$\begin{aligned} &H_i(Y_{n+1})\in\mathcal{C} \qquad \text{for } 0<i<q, \\ &\phi:\pi_q(Y_{n+1})\longrightarrow H_q(Y_{n+1}) \text{ is a } \mathcal{C}\text{-isomorphism,} \end{aligned}$$

where ϕ is the natural homomorshism. Thus the i-dimensional homology group of the fiber in the fiber space (ii) with $X=Y$ and $r=n$ is in $\mathcal{C}$ for $0<i<q$, and hence we have by Proposition 2 that

$$(3.1)\qquad \begin{aligned} &H_i(Y_n')\underset{\mathcal{C}}{\approx}H_i(\pi_n(Y), n) \qquad \text{for } i<q, \\ &H_q(Y_n')/i_*H_q(Y_{n+1})\underset{\mathcal{C}}{\approx}H_q(\pi_n(Y), n). \end{aligned}$$

Consider the commutative diagram

$$\begin{tikzcd} \pi_q(Y_{n+1}) \arrow[r, "\phi"] \arrow[d, "i_\#"] & H_q(Y_{n+1}) \arrow[d, "i_*"] \\ \pi_q(Y_n') \arrow[r, "\phi'"] & H_q(Y_n'), \end{tikzcd}$$

then we have, since the induced homomorphism $i_\#$ is onto, that

$$\begin{aligned} \textstyle\sum'_q(Y_n') &= \phi'\pi_q(Y_n')=\phi' i_\#\pi_q(Y_{n+1}) \\ &= i_*\phi\pi_q(Y_{n+1})\subset i_*H_q(Y_{n+1}), \end{aligned}$$

and so

$$i_*H_q(Y_{n+1})/\textstyle\sum'_q(Y_n')=i_*H_q(Y_{n+1})/i_*\phi\pi_q(Y_{n+1}).$$

Since $i_*H_q(Y_{n+1})/i_*\phi\pi_q(Y_{n+1})$ is a factor group of $H_q(Y_{n+1})/\phi\pi_q(Y_{n+1})$ and ϕ is $\mathcal{C}$-on, we have $i_*H_q(Y_{n+1})/\sum'_q(Y_n')\in\mathcal{C}$ and hence

$$(3.2)\qquad H_q(Y_n')/\textstyle\sum'_q(Y_n')\underset{\mathcal{C}}{\approx}H_q(Y_n')/i_*H_q(Y_{n+1}).$$

Since $\pi_r(Y)\in\mathcal{C}$ for $r<n$, it follows from Prop. 8 in [7; p. 271] that $H_i(\pi_r(Y), r-1)$ is in $\mathcal{C}$ for $n>r\geqq 2$ and $i\geqq 1$. Therefore the positive dimensional homology groups of the fiber in the fiber space (i) with $X=Y$ are all in $\mathcal{C}$. Thus it follows from Proposition 2 that $f_{r*}: H_i(Y_{r+1})\to H_i(Y_r)$ is $\mathcal{C}$-isomorphic for $2\leqq r<n$ and $i>0$. This implies that

(3.3) $g_*: H_i(Y_n) \to H_i(Y_2) = H_i(Y)$ is $\mathcal{C}$-isomorphic for any $i \geqq 0$, where $g = f_1 \circ f_2 \circ \cdots\cdots \circ f_{n-1}$.

In general, we have

PROPOSITION 3. *If $f: A \to B$ is a $\mathcal{C}$-isomrophism and A' is a subgroup of A, then $\bar{f}: A/A' \to B/f(A')$ induced by f is also $\mathcal{C}$-isomorphic.*

Because the natural homomorphism $f^{-1}(0) \to f^{-1}(f(A'))/A' = Ker.\, \bar{f}$ is onto, and the $Coker.\, \bar{f} = (B/f(A'))/(f(A)/f(A')) = B/f(A)$.

Since $g_\# : \pi_q(Y_n) \to \pi_q(Y)$ is onto, g_* maps $\sum_q(Y_n)$ onto $\sum_q(Y)$. Therefore it follows from (3.3) in virtue of Proposition 3 that

$$
(3.4) \qquad \begin{aligned} &H_i(Y_n) \underset{\mathcal{C}}{\approx} H_i(Y) \qquad \text{for } i < q, \\ &H_q(Y_n)/\textstyle\sum_q(Y_n) \underset{\mathcal{C}}{\approx} H_q(Y)/\textstyle\sum_q(Y). \end{aligned}
$$

Since Y_n and Y_n' are of the same homotopy type, we obtain from (3.1), (3.2) and (3.4)

$$
(3.5) \qquad \begin{aligned} &H_i(Y) \underset{\mathcal{C}}{\approx} H_i(\pi_n(Y), n) \qquad \text{for } i < q, \\ &H_q(Y)/\textstyle\sum_q(Y) \underset{\mathcal{C}}{\approx} H_q(\pi_n(Y), n), \end{aligned}
$$

which is Theorem 1 for the integer coefficient group.

We know [7; p. 263] that

PROPOSITION 4. *If $A \underset{\mathcal{C}}{\approx} A'$ and $B \underset{\mathcal{C}}{\approx} B'$, we have $A \otimes B \underset{\mathcal{C}}{\approx} A' \otimes B'$ and $A * B \underset{\mathcal{C}}{\approx} A' * B'$, where $*$ denotes the torsion product,*

and it can be easily seen from the definition of $\sum_q(Y; G)$ [3] that

PROPOSITION 5. *$\sum_q(Y; G)$ is the image of the inclusion homomorphism $\sum_q(Y) \otimes G \to H_q(Y) \otimes G$, and so we have $(H_q(Y) \otimes G)/\sum_q(Y; G) = (H_q(Y)/\sum_q(Y)) \otimes G$.*

Therefore Theorem 1 follows from these Propositions and (3.5) in virtue of the universal coefficient theorem [5]

$$
H_i(Y; G) = H_i(Y) \otimes G + H_{i-1}(Y) * G,
$$

and the proof is complete.

Proof of Proposition 1. Since $M \underset{\mathcal{C}}{\approx} N$, there exist a group L and two $\mathcal{C}$-isomorphisms $f; L \to M$, $g; L \to N$. Let K be the kernel of f, and $f(L) = L'$. Then $K \in \mathcal{C}$ and $M/L' \in \mathcal{C}$, and we have two exact sequences:

$$
(3.6)_1 \qquad 0 \longrightarrow K \xrightarrow{\xi} L \xrightarrow{\eta} L/K \longrightarrow 0,
$$

$$
(3.6)_2 \qquad 0 \longrightarrow L/K \xrightarrow{f'} M \xrightarrow{\eta} M/L' \longrightarrow 0,
$$

where ξ is the injection, η is the natural factorization and f' is a homomorphism induced by f. Therefore there exist fiber spaces $(\mathcal{K}(L, r), \mathcal{K}(K, r), \mathcal{K}(L/K, r))$ and $(\mathcal{K}(L/K, r), \mathcal{K}(M/L', r-1), \mathcal{K}(M, r))$ for any $r \geqq 1$, as are shown in (6.1) and (6.2) in [8]. Since $K \in \mathcal{C}$ and $M/L' \in \mathcal{C}$, we have $H_i(K, r) \in \mathcal{C}$, $H_i(M/L', r) \in \mathcal{C}$

for $i \geqq 1$ and $r \geqq 1$. Hence we obtain by Proposition 2 that for $i \geqq 0$ and $r \geqq 2$

$$\begin{aligned} H_i(L, r) &\underset{\mathcal{C}}{\approx} H_i(L/K, r)\,, \\ H_i(L/K, r) &\underset{\mathcal{C}}{\approx} H_i(M, r)\,. \end{aligned} \tag{3.7}$$

Therefore we have

$$H_i(L, r) \underset{\mathcal{C}}{\approx} H_i(M, r)$$

for $i \geqq 0$ and $r \geqq 2$. By the same arguments we have

$$H_i(L, r) \underset{\mathcal{C}}{\approx} H_i(N, r)$$

for $i \geqq 0$ and $r \geqq 2$, and so we obtain Proposition 1 for $G = G' = Z$. From this and Proposition 4, Proposition 1 is obvious in virtue of the universal coefficient theorem. This completes the proof.

4. A homotopy type problem

If π is countable, it is obvious from the theorem in [11] that we can take $\mathcal{K}(\pi, n)$ which is locally finite CW-complex. Let us denote by $\boldsymbol{K}(\pi, n)$ such a space $\mathcal{K}(\pi, n)$. Then it follows from the obstruction theory [2, 12] and a theorem of J.H.C. Whitehead with respect to homotopy type [10] that every complex $\boldsymbol{K}(\pi, n)$ is of the same homotopy type for given π and n

THEOREM 3. *Let Y be a simply connected space with the following properties:*

i) $\pi_{l_1}(Y), \pi_{l_2}(Y), \cdots, \pi_{l_m}(Y)$ *are finite groups such that* $\pi_{l_j}(Y) \otimes \pi_{l_k}(Y) = 0$ *for* $j \neq k$,

ii) $\pi_i(Y) = 0$ *for any i differents from $l_1, l_2, \cdots, l_m$. Then we have*

$$H_i(Y) = \textstyle\sum_{j=1}^m H_i(\pi_{l_j}(Y), l_j)\,.$$

Furthermore, if Y is a CW-complex then Y is of the same homotopy type as the product complex $\boldsymbol{K}(\pi_{l_1}, l_1) \times \boldsymbol{K}(\pi_{l_2}, l_2) \times \cdots\cdots \times \boldsymbol{K}(\pi_{l_m}, l_m)$.

REMARK. Every space $\boldsymbol{K}(\pi_{l_1}, l_l) \times \cdots\cdots \times \boldsymbol{K}(\pi_{l_m}, l_m)$ is on the same homotopy type for given groups $\pi_{l_1}, \cdots\cdots, \pi_{l_m}$ and integers $l_1, \cdots\cdots, l_m$. Moreover this is a CW-complex, because it is proved in [10; p. 227] that the product of locally finite CW-complexes is also a CW-complex[3].

Proof. Let $\boldsymbol{a}_j$ be a set of prime numbers p such that the p-primary component of $\pi_{l_j}(Y)$ is not zero. Then it is clear from $\pi_{l_j}(Y) \otimes \pi_{l_k}(Y) = 0$ $(j \neq k)$ that

(4.1) $\boldsymbol{a}_j, \boldsymbol{a}_k$ *are disjoint if* $j \neq k$.

For $j = 1, 2, \cdots, m$, let $\mathcal{C}_j(\bar{\mathcal{C}}_j)$ be the class of torsion groups which the p-primary

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3) J. H. C. Whitehead say that he does not know if the product of CW-complexes is generally a CW-complex.

component is zero for $p \in \boldsymbol{a}_j (p \notin \boldsymbol{a}_j)$[4]. Then we see from (4.1) and the assumption ii) that $\pi_i(Y) \in \mathcal{C}_j$ if $i \neq l_j$. Therefore, applying Theorem 1 with $n = l_j$, $q = \infty$ and $G = Z$, we have $H_i(Y) \underset{\mathcal{C}_j}{\approx} H_i(\pi_{l_j}(Y), l_j)$ for $i \geqq 0$ and $j = 1, 2, \cdots, m$, and so we have

(4.2) *The p-primary component of $H_i(Y)$ is isomorphic with that of $H_i(\pi_{l_j}(Y), l_j)$ for $p \in \boldsymbol{a}_j$.*

Since $\pi_{l_j}(Y) \in \bar{\mathcal{C}}_j$, we have $H_i(\pi_{l_j}(Y), l_j) \in \bar{\mathcal{C}}_j$ and hence we have

(4.3) *The p-primary component of $H_i(\pi_{l_j}, l_j)$ is not zero if and only if $p \in \boldsymbol{a}_j$.*

Finally let $\bar{\mathcal{C}}$ be the class of torsion groups whose p-primary component is zero for $p \notin \boldsymbol{a}_1 \cup \boldsymbol{a}_2 \cup \cdots\cdots \cup \boldsymbol{a}_m$[4]. Then we have $\pi_i(Y) \in \bar{\mathcal{C}}$ for $i \geqq 0$. Therefore it follows from the generalized Hurewicz theorem that $H_i(Y) \in \bar{\mathcal{C}}$ for $i > 0$, and so we have

(4.4) *The p-primary component of $H_i(Y)$ is not zero if and only if $p \in \boldsymbol{a}_1 \cup \boldsymbol{a}_2 \cup \cdots\cdots \cup \boldsymbol{a}_m$.*

We see from (4.1)-(4.4) that $H_i(Y)$ is isomorphic with $\sum_{j=1}^{m} H_i(\pi_{l_j}(Y), l_j)$. Thus we have the first part of Theorem 3.

To proceed the proof of the second part, we assume that Y is a CW-complex, and we consider all homotopy classes of maps of Y into $\boldsymbol{K}_j = \boldsymbol{K}(\pi_{l_j}(Y), l_j)$. Let u be the basic cohomology class of $\boldsymbol{K}_j$, and $f^*: H^{l_j}(\boldsymbol{K}_j; \pi_{l_j}(\boldsymbol{K}_j)) \to H^{l_j}(Y; \pi_{l_j}(\boldsymbol{K}_j))$ be the homomorphism between cohomology groups induced by a map $f: Y \to \boldsymbol{K}_j$. Then we have from the obstruction theory [2, 12]

(4.5) *All homotopy classes of Y into $\boldsymbol{K}_j$ are in one-to-one correspondence with elements of $H^{l_j}(Y; \pi_{l_j}(\boldsymbol{K}_j))$ by means of the map $\{f\} \to f^*u$, where $\{f\}$ denotes a homotopy class containing f.*

Moreover we have by the universal coefficient theorem

$$H^{l_j}(Y; \pi_{l_j}(\boldsymbol{K}_j)) \approx \mathrm{Hom}\,(H_{l_j}(Y), \pi_{l_j}(\boldsymbol{K}_j)) + \mathrm{Ext}\,(H_{l_j-1}(Y), \pi_{l_j}(\boldsymbol{K}_j)).$$

On the other hand we have $\pi_i(Y) \in \mathcal{C}_j$ for $i < l_j$, and hence the generalized Hurewicz theorem implies that $H_{l_j-1}(Y) \in \mathcal{C}_j$ and $\phi: \pi_{l_j}(Y) \to H_{l_j}(Y)$ is a $\mathcal{C}_j$-isomorphism. Since $\pi_{l_j-1}(\boldsymbol{K}_j) \in \bar{\mathcal{C}}_j$, it follows from (3.1) by the well known properties of Hom and Ext [5] that Ext $(H_{l_j-1}(Y), \pi_{l_j}(\boldsymbol{K}_j)) = 0$ and ϕ^*: Hom $(H_{l_j}(Y), \pi_{l_j}(\boldsymbol{K}_j)) \to$ Hom $(\pi_{l_j}(Y), \pi_{l_j}(\boldsymbol{K}_j))$ induced by ϕ is isomorphic. We have now the following commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}\,(\pi_{l_j}(\boldsymbol{K}_j), \pi_{l_j}(\boldsymbol{K}_j)) \arrow[r, "f^{\#}"] & \mathrm{Hom}\,(\pi_{l_j}(Y), \pi_{l_j}(\boldsymbol{K}_j)) \\
\mathrm{Hom}\, H_{l_j}(\boldsymbol{K}_j), \pi_{l_j}(\boldsymbol{K}_j)) \arrow[u, "\phi^*", "\wr"'] \arrow[r, "f^*"] & \mathrm{Hom}\,(H_{l_j}(Y), \pi_{l_j}(\boldsymbol{K}_j)) \arrow[u, "\phi^*", "\wr"'] \\
H^{l_j}(\boldsymbol{K}_j, \pi_{l_j}(\boldsymbol{K})) \arrow[u, "\theta", "\wr"'] \arrow[r, "f^*"] & H^{l_j}(Y; \pi_{l_j}(\boldsymbol{K}_j)) \arrow[u, "\theta", "\wr"']
\end{tikzcd},$$

4) $\mathcal{C}_j$, $\bar{\mathcal{C}}_j$ and $\bar{\mathcal{C}}$ are classes satisfying (I), ($\mathrm{II_B}$) and (III). (see [7, p. 265])

where ϕ^* and θ are isomorphisms. Thus if we associate each homotopy class $\{f\}$ of Y into $\boldsymbol{K}_j$ with the induced homomorphism $f_{\#}: \pi_{l_j}(Y) \to \pi_{l_j}(\boldsymbol{K}_j)$, we see from (4.5) that it gives a one-to-one correspondence between all homotopy classes of Y into $\boldsymbol{K}_j$ and $\mathrm{Hom}_{\Lambda}(\pi_{l_j}(Y), \pi_{l_j}(\boldsymbol{K}_j))$. Therefore there exists a map $h_j: Y \to \boldsymbol{K}_j$ such that $h_{j_{\#}}: \pi_{l_j}(Y) \to \pi_{l_j}(\boldsymbol{K}_j)$ is isomorphic. Let $h: Y \to \boldsymbol{K}_1 \times \boldsymbol{K}_2 \times \cdots \times \boldsymbol{K}_m$ be a map defined by

$$h(y) = h_1(y) \times h_2(y) \times \cdots \times h_m(y), \ y \in Y,$$

and consider the induced homomorphism $h_{\#}: \pi_i(Y) \to \pi_i(\boldsymbol{K}_1 \times \boldsymbol{K}_2 \times \cdots \times \boldsymbol{K}_m)$. When i is not some l_j, $h_{\#}$ is isomorphic[5] since $\pi_i(Y)$ and $\pi_i(\boldsymbol{K}_1 \times \boldsymbol{K}_2 \times \cdots \times \boldsymbol{K}_m)$ are both zero. For $i = l_j$, consider the commutative diagram

$$\begin{tikzcd} \pi_{l_j}(Y) \arrow[r, "h_{\#}"] \arrow[dr, "h_{j\#}"'] & \pi_{l_j}(\boldsymbol{K}_1 \times \boldsymbol{K}_2 \times \cdots \times \boldsymbol{K}_m) \arrow[d, "p_{j\#}"] \\ & \pi_{l_j}(\boldsymbol{K}_j) \end{tikzcd},$$

where $p_j(z_1 \times z_2 \times \cdots \times z_m) = z_j (z_i \in \boldsymbol{K}_i)$ is the projection. Then $h_{j_{\#}}$ and $p_{j_{\#}}$ are isomorphic onto, and so $h_{\#}$ is also an isomorphism. Thus $h_{\#}: \pi_i(Y) \to \pi_i(\boldsymbol{K}_1 \times \boldsymbol{K}_2 \times \cdots \times \boldsymbol{K}_m)$ is isomorphic[5] for every i. Since Y and $\boldsymbol{K}_1 \times \boldsymbol{K}_2 \times \cdots \times \boldsymbol{K}_m$ are CW-complexes as is noted in the above remark, it follows from a theorem of J.H.C. Whitehead [10; p. 215] that h is a homotopy equivalence. Thus we have the second part of Theorem 3.

REMARK i). In the above proof of the second part of Theorem 3, we did not use Theorem 1 and 2, but we used only the generalized Hurewicz theorem. On the other hand, we used Theorem 1 in the proof of the first part. However, as is easily seen, we can reduce the first part from the second by making use of the result due to J.B. Giever (*On the equivalence of two singular homology theory*, Ann. of Math., 51 (1950), 178–191).

REMARK ii).[6] If we allow to use the results of M.M. Postnikov[7], Theorem 3 is obvious.

5. Betti numbers and homotopy groups

THEOREM 4. *Let Y be a simply connected space such that every dimensional homology group is finitely generated and $\rho_n > 0$, $\rho_i = 0 (0 < i < n)$, where ρ_i denotes the i-dimensional Betti number. Then the homotopy groups of Y are infinite for at least two number of dimensions, if the following relation is not satisfied for some i:*

5) An isomorphism, without quolification, will always mean an isomorphism *onto*.

6) This is a remark by Mr. Mizuno.

7) M. M. Postnikov: *Determination of the homology groups of a space by means of the homotopy invariants*, Doklady Akad. Nauk SSSR. 76 (1951), 359–362; *On the homotopy type of polyhedra*. ibid, 789–791. See also the paper of K. Mizuno in this journal and the mimeographed note due to P. J. Hilton.

$$(5.1)\qquad \rho_i = \binom{q+\rho_n-1}{q} \quad \textit{if} \quad i=qn \textit{ and } n \textit{ is even,}$$
$$= \binom{\rho_n}{q} \quad \textit{if} \quad i=qn \textit{ and } n \textit{ is odd,}$$
$$= 0 \quad \textit{if} \quad i \textit{ is not divided by } n,$$

where $\binom{k}{j}$ *is the binomial coefficient with the convention*: $\binom{k}{j}=0$ *if* $j>k$.

Proof. Assume that all dimensional homotopy groups $\pi_i(Y)$ are finite. Then all dimensional homology groups are also finite, by the generalized Hurewicz theorem and the assumption i). This contradicts the assumption i), and so there exists the smallest integer m such that $\pi_m(Y)$ is infinite. Let $\mathcal{C}_f$ be the class of finite groups. Then, since $\pi_i(Y)\in\mathcal{C}_f$ for $i<m$, it follows from the generalized Hurewicz theorem that[8]

$$H_i(Y)\in\mathcal{C}_f \quad \text{for} \quad i<m,$$
$$\pi_m(Y) \underset{\mathcal{C}_f}{\approx} H_m(Y).$$

Therefore we have $m=n$.

Let us now assume that $\pi_i(Y)$ is finite except $i=m=n$, and let $\Gamma=Z+Z+\cdots+Z$, where the number of Z is ρ_n. We have then $\pi_i(Y)\in\mathcal{C}_f$ and $\pi_i(\mathcal{K}(\Gamma,n))\in\mathcal{C}_f$ for $i\neq n$. Further $\pi_n(Y)\underset{\mathcal{C}_f}{\approx}\Gamma=\pi_n(\mathcal{K}(\Gamma,n))$. Therefore it follows from Theorem 2 that[6]

$$H_i(Y)\underset{\mathcal{C}_f}{\approx} H_i(\Gamma,n) \quad \text{for} \quad i\geqq 0,$$

and hence

$$(5.2)\qquad H_i(Y,k)=H_i(\Gamma,n;k) \quad \text{for} \quad i\geqq 0,$$

where k is a field of characteristic zero.

J-P. Serre proved in [6; p. 501] that the cohomology algebra $H^*(Z,n;k)=\sum H^i(Z,n;k)$ is a commutative polynomial algebra or an exterior algebra generated by one element of $H^n(Z,n;k)$ according as n is even or odd. Therefore if n is even, $H_i(Z,n;k)$ is one or zero dimensional vector space according as n is divided or not divided by n. If n is odd, $H_0(Z,n;k)$ and $H_n(Z,n;k)$ is one dimensional, and the other is zero. On the other hand, we have

$$(5.3)\qquad H_i(\Gamma,n;k)=\sum_{j_1+\cdots+j_{\rho_n}=i} H_{j_1}(Z,n;k)\underset{k}{\otimes}\cdots\cdots\underset{k}{\otimes}H_{j_{\rho_n}}(Z,n;k)$$

by the Künneth relation. From (5.2) and (5.3), we have (5.1). Therefore if the relation (5.1) is not satisfied for some i, the homotopy groups are infinite for at least two number of dimensions. This completes the proof.

8) $\mathcal{C}_f$ is a class which does not satisfy (II$_B$). However Theorem 1 and the generalized Hurewicz theorem hold for this class, since we have the assumption i). See Remark in § 2).

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