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On $\mathfrak{l}$- relative cohomology groups of an associative algebra

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Introduction

Ordinary cohomology theory for associative algebras was first established by G. Hochschild in his papers [4], [5], [6]. Recently M. Ikeda, T. Nakayama and the writer succeeded, in the joint paper [8], in determing the structure of algebras with vanishing n-dimensional cohomology groups; S. Eilenberg has given an alternative approach to our result ([1]). In our treatment a use was made of a notion of $\mathfrak{l}$-(relative) cohomology groups introduced by T. Nakayama [11]. Nakayama further extended our result to a characterization of algebras with vanishing n-dimensional $\mathfrak{a}$-cohomology groups, with a two-sided ideal $\mathfrak{a}$. His unpublished result reads: Let A be an algebra of finite rank over a ground field, N be its radical, and let $\mathfrak{a}$ be a two-sided ideal of A. All $n(\geqq 2)$-dimensional $\mathfrak{a}$-relative cohomology groups of A vanish if and only if (i) $A/(\mathfrak{a}+N)$ is separable and (ii) for every left ideal $\mathfrak{l}$ containing $\mathfrak{a}$, $Q^{n-1}_{\mathfrak{l}/\mathfrak{a}}$[1] is an (M_0)-module[2] as an $A*$-left module[3].

In the present paper, we introduce the notion of $[\mathfrak{n}]$-cohomology groups of an algebra, which is a generalization of the notion of factor sets to higher dimensional cases, and by considering some exact sequences, extend the result of our joint paper [8] and the above result by Nakayama to $\mathfrak{l}$-relative case.

In section 1, we repeat briefly the notion of $\mathfrak{l}$-(relative) cohomology groups, and introduce the notion of $[\mathfrak{n}]$-cohomology groups. Then we get an exact sequence which clarifies the relation between the ordinary, $\mathfrak{l}$- and $[\mathfrak{l}]$-cohomology groups. In fact, the method of Nakayama essentially depends on the exactness of this sequence. In section 2, we relate the $[\mathfrak{n}]$-cohomology groups to the enlargement of modules, and, in section 3, we state some properties of algebras with vanishing ordinary or

1) $Q^{n-1}_{\mathfrak{l}/\mathfrak{a}} = A\times \cdots \times A\times \mathfrak{l}/\mathfrak{a}$ is the Kronecker product of the vector space of $(n-2)$-fold Kronecker product of A and the underlying vector space of $\mathfrak{l}/\mathfrak{a}$. We define the $*$-operation of A by setting
$x*(x_1\times \cdots \times x_{n-1}) = xx_1\times \cdots \times x_{n-1} - x\times x_1x_2\times \cdots \times x_{n-1} + \cdots + (-1)^n x\times x_1\times \cdots \times x_{n-2}x_{n-1}$,
where $x, x_1, \dots, x_{n-2}\in A$, $x_{n-1}\in \mathfrak{l}/\mathfrak{a}$. This makes $Q^{n-1}_{\mathfrak{l}/\mathfrak{a}}$ an A-left module. We shall speak of $A*$-left module $Q^{n-1}_{\mathfrak{l}/\mathfrak{a}}$ in order to make distinction from $Q^{n-1}_{\mathfrak{l}/\mathfrak{a}}$ considered as A-left module in usual fashion.

2) For the notion of (M_0)-modules, see [10].

3) See footnote 1).

$\mathfrak{l}$-cohomology groups. In section 4, we first prove a theorem on ordinary cohomology groups (Theorem 6), which is a generalization of our main theorem in [8] and seems to the writer to be some interest for itself. By combining this theorem and a theorem in section 3 (Theorem 2), we obtain two main theorems. In the appendix, we consider algebras with vanishing 1-dimensional $\mathfrak{l}$-cohomology groups with respect to the enlargement of modules.

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1. Cohomology groups $H^n_{\mathfrak{l}}(A, \mathfrak{m})$, $H^n_{[\mathfrak{n}]}(A, \mathfrak{m})$

Let A be an associative algebra, of finite or infinite rank, over a field Ω, and let $\mathfrak{l}$ be a left ideal of A. We consider an A-A-module $\mathfrak{m}$ satisfying

$$\mathfrak{m}\mathfrak{l} = 0 . \tag{1}$$

We briefly repeat the notion of $\mathfrak{l}$-(relative) cohomology groups of A in $\mathfrak{m}$ as was introduced in [11]. Let $P^n = A \times \cdots \times A$ be the n-fold Kronecker product of the underlying vector space of A over Ω, and let $C^n_{\mathfrak{l}}(A, \mathfrak{m})$ be the module of all Ω-linear mappings f of P^n into $\mathfrak{m}$ such that $f(x_1, \ldots, x_n) = 0$ whenever $x_n \in \mathfrak{l}$. On the other hand $C^0_{\mathfrak{l}}(A, \mathfrak{m})$ is identified with the Ω-submodule of $\mathfrak{m}$ consisting of all elements u such that $\mathfrak{l}u = 0$. The coboundary operater δ, which maps each $C^n_{\mathfrak{l}}(A, \mathfrak{m})$ linearly into $C^{n+1}_{\mathfrak{l}}(A, \mathfrak{m})$, is defined as usual. Namely, if $f \in C^n_{\mathfrak{l}}(A, \mathfrak{m})$, $x_1, \ldots, x_{n+1} \in A$, then

$$\begin{aligned}\delta f(x_1, \ldots, x_{n+1}) = x_1 f(x_2, \ldots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \ldots, x_i x_{i+1}, \\ \ldots, x_{n+1}) + (-1)^{n+1} f(x_1, \ldots, x_n) x_{n+1} .\end{aligned} \tag{2}$$

Thus, we have a cochain complex $C_{\mathfrak{l}}(A, \mathfrak{m}) = \sum_{n=0}^{\infty} C^n_{\mathfrak{l}}(A, \mathfrak{m})$ which we want to call the *$\mathfrak{l}$-cochain complex* of A in $\mathfrak{m}$; we shall also speak of *$\mathfrak{l}$-cochains*, *$\mathfrak{l}$-cocycles* and *$\mathfrak{l}$-coboundaries.* We denote the n-dimensional cohomology group of $C^n_{\mathfrak{l}}(A, \mathfrak{m})$ by $H^n_{\mathfrak{l}}(A, \mathfrak{m})$, and call it the *$n$-dimensional $\mathfrak{l}$-cohomology group* of A in $\mathfrak{m}$. If we speak of an (ordinary) cochain, cocycle, coboundary or cohomology group, we shall always mean a 0-cochain, -cocycle, -coboundary or -cohomology group, and denote the 0-cochain complex and 0-cohomology group, omitting the suffixes 0, by $C^n(A, \mathfrak{m})$ and $H^n(A, \mathfrak{m})$ respectively.

Now, we consider another cochain complex. Let $\mathfrak{n}$ be an A-left module, and put

$$Q^n_{\mathfrak{n}} = A \times \cdots \times A \times \mathfrak{n} \quad (\text{with } n-1 \ A\text{'s}) . \tag{3}$$

Let $n \geqq 1$, and let $\mathfrak{m}$ be an A-A-module. We denote by $C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$ the module of all Ω-linear mappings of $Q^n_{\mathfrak{n}}$ into $\mathfrak{m}$, and define the coboundary operator δ, which maps each $C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$ linearly into $C^{n+1}_{[\mathfrak{n}]}(A, \mathfrak{m})$, as follows; for $f \in C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$, $x_1, \dots, x_n \in A$, $x_{n+1} \in \mathfrak{n}$, we set

$$(4) \quad \delta f(x_1, \dots, x_{n+1}) = x_1 f(x_2, \dots, x_{n+1}) + \sum_{i=1}^{n} (-1)^i f(x_1, \dots, x_i x_{i+1}, \dots, x_{n+1}).$$

Then, we see, by direct computations, that $\delta\delta f = 0$, and thus we have a cochain complex $C_{[\mathfrak{n}]}(A, \mathfrak{m}) = \sum_{n=1}^{\infty} C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$ which we want to call $[\mathfrak{n}]$-*cochain complex* of A in $\mathfrak{m}$; we shall also speak of $[\mathfrak{n}]$-*cochains*, $[\mathfrak{n}]$-*cocycles* and $[\mathfrak{n}]$-*coboundaries.* We denote the n-dimensional cohomology group of $C_{[\mathfrak{n}]}(A, \mathfrak{m})$ by $H^n_{[\mathfrak{n}]}(A, \mathfrak{m})$, and call it the *$n$-dimensional $[\mathfrak{n}]$-cohomology group* of A in $\mathfrak{m}$. It is readily seen, from the definition, that $H^n_{[\mathfrak{n}]}(A, \mathfrak{m})$ is independent of the A-right module structure of $\mathfrak{m}$.

We consider $C^n_{\mathfrak{l}}(A, \mathfrak{m})$ and $C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$ as A-A-modules, on defining, for $f \in C^n_{\mathfrak{l}}(A, \mathfrak{m})$ or $\in C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$,

$$(5) \quad \begin{aligned} (xf)(x_1, \dots, x_n) &= xf(x_1, \dots, x_n) \\ (fx)(x_1, \dots, x_n) &= xf(x_1, \dots, x_n) - \delta f(x, x_1, \dots, x_n), \end{aligned}$$

where $x, x_1, \dots, x_{n-1} \in A$ and $x_n \in A$ or $\in \mathfrak{n}$ according as $f \in C^n_{\mathfrak{l}}(A, \mathfrak{m})$ or $\in C^n_{[\mathfrak{n}]}(A, \mathfrak{m})$. Then we have the following reduction theorems;

$$(6) \quad H^{n+r}_{\mathfrak{l}}(A, \mathfrak{m}) \simeq H^r(A, C^n_{\mathfrak{l}}(A, \mathfrak{m})),$$

$$(7) \quad H^{n+r}_{[\mathfrak{n}]}(A, \mathfrak{m}) \simeq H^r(A, C^n_{[\mathfrak{n}]}(A, \mathfrak{m})).$$

On the other hand, we consider $C^n(A, \mathfrak{m})$ as an A-A-module, on defining, for $f \in C^n(A, \mathfrak{m})$,

$$(8) \quad \begin{aligned} (xf)(x_1, \dots, x_n) &= f(x_1, \dots, x_n)x + (-1)^n \delta f(x_1, \dots, x_n, x), \\ (fx)(x_1, \dots, x_n) &= f(x_1, \dots, x_n)x, \end{aligned}$$

where $x, x_1, \dots, x_n \in A$. Then we have another reduction theorems;

$$(9) \quad H^{n+r}_{\mathfrak{l}}(A, \mathfrak{m}) \simeq H^r_{\mathfrak{l}}(A, C^n(A, \mathfrak{m})),$$

$$(10) \quad H^{n+r}_{[\mathfrak{n}]}(A, \mathfrak{m}) \simeq H^r_{[n]}(A, C^n(A, \mathfrak{m})).$$

Proofs of these reduction theorems are exactly the same as in the ordinary case.

Now, again, let $\mathfrak{l}$ be a left ideal of A, and $\mathfrak{m}$ be an A-A-module satisfying (1). For the sake of convenience, we define $C^0_{[\mathfrak{l}]}(A, \mathfrak{m})$ as the Ω-module $\mathfrak{m}/C^0_{\mathfrak{l}}(A, \mathfrak{m})$, and coboundary operator δ, which maps $C^0_{[\mathfrak{l}]}(A, \mathfrak{m})$ linearly into $C^1_{[\mathfrak{l}]}(A, \mathfrak{m})$ as follows: for $x \in \mathfrak{l}$ and $\bar{u} \in C^0_{[\mathfrak{l}]}(A, \mathfrak{m})$ (the residue class of $\mathfrak{m}$ modulo $C^0_{\mathfrak{l}}(A, \mathfrak{m})$ which contains an element u), we set

$$(11) \quad \delta \bar{u}(x) = xu.$$

As is easily seen from the property of $C^0_{\mathfrak{l}}(A, \mathfrak{m})$, $\delta\bar{u}$ is independent of the choice of the representative u of the class $\bar{u}$. Since $\delta\delta\bar{u} = 0$, we have a cochain complex

$C_{[\mathfrak{l}]}(A, \mathfrak{m})=\sum_{n=0}^{\infty} C_{[\mathfrak{l}]}^{n}(A, \mathfrak{m})$. Let η be a linear mapping of $C(A, \mathfrak{m})$ into $C_{[\mathfrak{l}]}(A, \mathfrak{m})$ which maps an element u of $C^{0}(A, \mathfrak{m})\,(=\mathfrak{m})$ to the residue class $\bar{u}$ of $\mathfrak{m}$ modulo $C_{\mathfrak{l}}^{0}(A, \mathfrak{m})$, and an element f of $C^{n}(A, \mathfrak{m})\,(n \geqq 1)$ to the element of $C_{[\mathfrak{l}]}^{n}(A, \mathfrak{m})$ obtained from f by restricting the last argument x_{n} to the elements of $\mathfrak{l}$. Then, the kernel of η is $C_{\mathfrak{l}}(A, \mathfrak{m})$, and, as is readily seen from the assumed property (1) of $\mathfrak{m}$, $\delta\eta=\eta\delta$. By the theorem similar to [3], theorem 3.7, we have an exact sequence

$$\cdots \xrightarrow{\delta^{*}} H_{\mathfrak{l}}^{n}(A, \mathfrak{m}) \xrightarrow{i^{*}} H^{n}(A, \mathfrak{m}) \xrightarrow{\eta^{*}} H_{[\mathfrak{l}]}^{n}(A, \mathfrak{m}) \xrightarrow{\delta^{*}} H_{\mathfrak{l}}^{n+1}(A, \mathfrak{m}) \longrightarrow \cdots . \tag{12}$$

2. Modules $Q_{\mathfrak{n}}^{n}$

Let $\mathfrak{n}$ be an A-left module. $Q_{\mathfrak{n}}^{n}$ is an A-left module under the usual operation defined by setting

$$x(x_{1}\times \cdots \times x_{n}) = (x x_{1})\times \cdots \times x_{n}, \tag{13}$$

$(x, x_{1}, \ldots, x_{n-1} \in A,\ x_{n} \in \mathfrak{n})$. However, we introduce, after Hochschild, a new operation $*$ of A by setting

$$x * (x_{1}\times \cdots \times x_{n}) = (x x_{1})\times \cdots \times x_{n} + \sum_{i=1}^{n-1}(-1)^{i} x \times \cdots \times (x_{i}x_{i+1})\times \cdots \times x_{n}, \tag{14}$$

$(x, x_{1}, \ldots, x_{n-1} \in A,\ x_{n} \in \mathfrak{n})$. Under this operation, too, $Q_{\mathfrak{n}}^{n}$ is a left module of A, and we shall speak of $A*$-left module $Q_{\mathfrak{n}}^{n}$ in order to make destinction trom $Q_{\mathfrak{n}}^{n}$ considered as A-left module in usual fashion.

Let $\mathfrak{m}$ be an A-A-module, and let $L(\mathfrak{n}, \mathfrak{m})$ be the module of all Ω-linear mappings of $\mathfrak{n}$ into $\mathfrak{m}$. We may consider $L(\mathfrak{n}, \mathfrak{m})$ as an A-A-module, on defining, for $f \in L(\mathfrak{n}, \mathfrak{m})$,

$$\begin{aligned} (xf)(u) &= xf(u), \\ (fx)(u) &= f(xu), \end{aligned} \tag{15}$$

$(x \in A,\ u \in \mathfrak{n})$. From the definitions, it is readily seen that $C_{[\mathfrak{n}]}^{n}(A, \mathfrak{m})$ may be identified with $L(Q_{\mathfrak{n}}^{n}, \mathfrak{m})$, and, further, the A-A-module structure of $C_{[\mathfrak{n}]}^{n}(A, \mathfrak{m})$ defined in (5) coincides with that of $L(Q_{\mathfrak{n}}^{n}, \mathfrak{m})$ defined in (15) considering $Q_{\mathfrak{n}}^{n}$ as $A*$-left module. The reduction theorem (7) gives, for $n \geqq 2$,

$$H_{[\mathfrak{n}]}^{n}(A, \mathfrak{m}) \simeq H^{1}(A, L(Q_{\mathfrak{n}}^{n-1}, \mathfrak{m})). \tag{16}$$

LEMMA 1. *Let $\mathfrak{m}$ and $\mathfrak{n}$ be two A-left modules. Then the group of equivalence classes of enlargments of $\mathfrak{m}$ by $\mathfrak{n}$ is isomorphic to $H^{1}(A, L(\mathfrak{n}, \mathfrak{m}))$.*

Proof is exactly the same as in [6], §1.

Combining (16) and Lemma 1, we have readily

THEOREM 1. *Let $\mathfrak{n}$ be an A-left module, and let $n \geqq 2$. Then $H_{[\mathfrak{n}]}^{n}(A, \mathfrak{m}) = 0$. for every A-A-module $\mathfrak{m}$ if and only if $Q_{\mathfrak{n}}^{n-1}$ is an (M_{0})-module as an $A*$-left module.*

From the reduction theorem (10) and Theorem 1, we have readily

LEMMA 2. *Let* $\mathfrak{n}$ *be an* A*-left module, and let* $n \geqq 1$. *If* $Q^n_{\mathfrak{n}}$ *is an* (M_0)*-module as an* $A*$*-left module, then* $Q^m_{\mathfrak{n}}$ *is also an* (M_0)*-module as an* $A*$*-left module for every* $m \geqq n$.

Now, let $\mathfrak{m}$ be an A-A-module, $\mathfrak{M}$ be an A-left module, and let $\mathfrak{n}$ be a submodule of $\mathfrak{M}$. The set of cochains of $C_{[\mathfrak{M}]}(A, \mathfrak{m})$ such that $f=0$ whenever the last argument of f is in $\mathfrak{n}$ forms a subcochain of $C_{[\mathfrak{M}]}(A, \mathfrak{m})$. This is clearly isomorphic to $C_{[\mathfrak{M}/\mathfrak{n}]}(A, \mathfrak{m})$, and further, identifying this subcochain with $C_{[\mathfrak{M}/\mathfrak{n}]}(A, \mathfrak{m})$, we have $C_{[\mathfrak{M}]}(A, \mathfrak{m})/C_{[\mathfrak{M}/\mathfrak{n}]}(A, \mathfrak{m}) \simeq C_{[\mathfrak{n}]}(A, \mathfrak{m})$. Hence, we have an exact sequence

$$(17) \qquad \cdots \to H^n_{[\mathfrak{M}/\mathfrak{n}]}(A, \mathfrak{m}) \to H^n_{[\mathfrak{M}]}(A, \mathfrak{m}) \to H^n_{[\mathfrak{n}]}(A, \mathfrak{m}) \to H^{n+1}_{[\mathfrak{M}/\mathfrak{n}]}(A, \mathfrak{m}) \to \cdots$$

By considering this exact sequence, we have, from the reduction theorem (10) and Theorem 1, readily the the following lemmas.

LEMMA 3. *If* $Q^n_{\mathfrak{M}/\mathfrak{n}}$ *is an* (M_0)*-module as an* $A*$*-left module, then* $Q^n_{\mathfrak{M}}$ *is an* (M_0)*-module as an* $A*$*-left module if and only if* $Q^n_{\mathfrak{n}}$ *is so.*

LEMMA 4. *If* $Q^n_{\mathfrak{M}}$ *is an* (M_0)*-module as an* $A*$*-left module, then* $Q^n_{\mathfrak{n}}$ *is an* (M_0)*-module as an* $A*$*-left module if and only if* $Q^{n+1}_{\mathfrak{M}/\mathfrak{n}}$ *is so.*

LEMMA 5. *If* $Q^n_{\mathfrak{n}}$ *is an* (M_0)*-module as an* $A*$*-left module, then* $Q^{n+1}_{\mathfrak{M}}$ *is an* (M_0)*-module as an* $A*$*-left module if and only if* $Q^{n+1}_{\mathfrak{M}/\mathfrak{n}}$ *is so.*

3. Properties of algebras with vanishing $\mathfrak{l}$-cohomology groups

Let A be an algebra of finite or infinite rank over Ω possessing a unit element. Then, either from Theorem 1 in [5], §1 or from Theorem 1 and Lemma 2, $H^n(A, \mathfrak{m})=0$ for every $n \geqq 1$ and A-A-module $\mathfrak{m}$ satisfying $\mathfrak{m}A=0$. By considering the exact sequence (12), we have readily, for every $n \geqq 1$ and A-A-module $\mathfrak{m}$ satisfying $\mathfrak{m}A=0$,

$$(18) \qquad H^n_{[\mathfrak{l}]}(A, \mathfrak{m}) \simeq H^{n+1}_{\mathfrak{l}}(A, \mathfrak{m}) .$$

LEMMA 6. $H^1_{[\mathfrak{l}]}(A, \mathfrak{m})=0$ *for all* A*-*A*-module* $\mathfrak{m}$ *if and only if* $\mathfrak{l}$ *is a principal left ideal generated by an idempotent element.*

Proof. It is readily seen, from the definition, that a 1-dimensional $[\mathfrak{l}]$-cochain of A in $\mathfrak{m}$ is $[\mathfrak{l}]$-cocycle if and only if it induces an A-operator homomorphism from $\mathfrak{l}$ into $\mathfrak{m}$. Assume first that $H^1_{[\mathfrak{l}]}(A, \mathfrak{l})=0$. Then the identical mapping of $\mathfrak{l}$ is an $[\mathfrak{l}]$-cocycle of A in $\mathfrak{l}$, and hence an $[\mathfrak{l}]$-coboundary. Therefore, there exists an element e of $\mathfrak{l}$ such that $x=xe$ for all $x \in \mathfrak{l}$. Such element e is necessarily an idempotent element, and we have $\mathfrak{l}=Ae$.

Conversely, assume that $\mathfrak{l}=Ae$ with an idempotent element e, and let f be a 1-dimensional $[\mathfrak{l}]$-cocycle of A in $\mathfrak{m}$. Since f is an A-operator homomorphism from $\mathfrak{l}$ into $\mathfrak{m}$, $f(ae)=aef(e)$, and hence f is an $[\mathfrak{l}]$-coboundary. This shows that $H^1_{[\mathfrak{l}]}(A, \mathfrak{m})=0$.

LEMMA 7. *Let A possesses a unit element 1. Then $H^n_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}A=0$ if and only if, in case $n=2$, $\mathfrak{l}$ is a principal left ideal generated by an idempotent element, and, in case $n>2$, $Q^{n-2}_{\mathfrak{l}}$ is an (M_0)-module as an $A*$-left module. On the other hand, in case $n=1$, $H^1_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}A=0$.*

Proof. In case $n\geqq 2$, from (15), Theorem 1 and Lemma 6, we have readily the lemma. In case $n=1$, it is readily seen, from the definition that any $\mathfrak{l}$-cocycle of A in $\mathfrak{m}$ induces an A-operator homomorphism from A into $\mathfrak{m}$, if $\mathfrak{m}A=0$. Hence $f(x)=xf(1)$ for all $x\in A$, and, since $f(x)=0$ for all $x\in\mathfrak{l}$, $\mathfrak{l}f(1)=0$. This shows that f is an $\mathfrak{l}$-coboundary, and hence $H^1_{\mathfrak{l}}(A, \mathfrak{m})=0$.

THEOREM 2. *Let A be an algebra with a unit element 1, and let $\mathfrak{l}$ be a left ideal of A. Then $H^n_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l}=0$ if and only if*

(i) *$H^n(A, \mathfrak{m})=0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l}=0$, and,*

(ii) *in case $n=1$ or 2, $\mathfrak{l}$ is a principal left ideal generated by an idempotent element, and, in case $n>2$, $Q^{n-2}_{\mathfrak{l}}$ is an (M_0)-module as an $A*$-left module.*

Proof. Assume first that $H^n_{\mathfrak{l}}(A, m)=0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l}=0$. Then, from Lemma 7, it is readily seen that the assertion (ii) is valid in case $n\geqq 2$. On the other hand, in case $n=1$, from the reduction theorem (9), $H^2_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all A-A-modules $\mathfrak{m}$ satsfying $\mathfrak{m}\mathfrak{l}=0$, and hence, from lemma 7, $\mathfrak{l}$ is a principal left ideal generated by an idempotent element. From lemma 6, in case $n=1$ or 2, $H^1_{[\mathfrak{l}]}(A, \mathfrak{m})=0$, and hence, from the reduction theorem (10), $H^2_{[\mathfrak{l}]}(A, \mathfrak{m})=0$ for all A-A-modules $\mathfrak{m}$, and, in case $n>2$, from Theorem 1, $H^{n-1}_{[\mathfrak{l}]}(A, \mathfrak{m})=0$. By considering the exact sequence (12), we see now that the assertion (i) is valid for every natural number n.

Conversely, assume that the condition (i), (ii) are satified. In case $n\geqq 2$, from the condition (ii), and Lemma 6 or Theorem 1, we see that $H^{n-1}_{[\mathfrak{l}]}(A, \mathfrak{m})=0$ for all A-A-module $\mathfrak{m}$. Hence, by considering the exact sequence (12), we see that $H^n_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l}=0$. In case $n=1$, we see immediately, from the definition (11), that a 0-dimensional $[\mathfrak{l}]$-cocycle of A in $\mathfrak{m}$ is an element $\bar{u}$ of $\mathfrak{m}/C^0_{\mathfrak{l}}(A, \mathfrak{m})$ such that $\mathfrak{l}\bar{u}=\bar{0}$. Since $\mathfrak{l}=Ae$ with an idempotent element e, $\mathfrak{l}^2=\mathfrak{l}$. Hence $\mathfrak{l}\bar{u}=\bar{0}$ implies $\mathfrak{l}u=0$. and so $\bar{u}=\bar{0}$, because $C^0_{\mathfrak{l}}(A, \mathfrak{m})$ is a submodule of $\mathfrak{m}$ of all element v satisfying $lv=0$. Therefore, $H^0_{[\mathfrak{l}]}(A, \mathfrak{m})=0$ for all $\mathfrak{m}$. By considering the exact sequence (12), we see that $H^1_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l}=0$.

THEOREM 3. *Let A be an algebra with a unit element* 1, *and let $\mathfrak{a}$ be a two-sided ideal of A. If $H^n(A, \mathfrak{m}) = 0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$, then, for every left ideal $\mathfrak{l}_1$ of A containing $\mathfrak{a}$, in case $n = 1$, $\mathfrak{l}_1$ is a principal left ideal generated by an idempotent element, and, in case $n \geqq 2$, $Q^{n-1}_{\mathfrak{l}_1}$ is an (M_0)-module as an $A*$-left module.*

Proof. Let $\mathfrak{l}_1$ be a left ideal of A containing $\mathfrak{a}$, and let $\mathfrak{m}$ be an A-A-module satisfying $\mathfrak{m}\mathfrak{l}_1 = 0$. Then, from the reduction theorem (6), we have

$$H^{n+1}_{\mathfrak{l}_1}(A, \mathfrak{m}) \simeq H^n(A, C^1_{\mathfrak{l}_1}(A, \mathfrak{m})) , \tag{19}$$

where $C^1_{\mathfrak{l}_1}(A, \mathfrak{m})$ is considered as an A-A-module, on defining, for $f \in C^1_{\mathfrak{l}_1}(A, \mathfrak{m})$, $x, y \in A$,

$$\begin{aligned} (xf)\,(y) &= xf(y) , \\ (fx)\,(y) &= f(xy) - f(x)\,y . \end{aligned} \tag{20}$$

If $x \in \mathfrak{a}$, then xy and x belong to $\mathfrak{l}_1$, and hence $f(xy) = f(x) = 0$ for $f \in C^1_{\mathfrak{l}_1}(A, \mathfrak{m})$. This shows that $C^1_{\mathfrak{l}_1}(A, \mathfrak{m})\mathfrak{a} = 0$. Hence, from the assumption and (19), we see that $H^{n+1}_{\mathfrak{l}_1}(A, \mathfrak{m}) = 0$. From theorem 2, we have the theorem immediately.

Combining Theorem 2 and Theorem 3, we have the following theorem.

THEOREM 4. *Let A be an algebra with a unit element* 1, *and let $\mathfrak{l}$ be a left ideal of A. If $H^n_{\mathfrak{l}}(A, \mathfrak{m}) = 0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l} = 0$, then,*

(i) *in case $n = 1$ or 2, $\mathfrak{l}$ is a principal left ideal generated by an idempotent element, and, in case $n > 2$, $Q^{n-2}_{\mathfrak{l}}$ is an (M_0)-module as an $A*$-left module, and,*

(ii) *for any left ideal $\mathfrak{l}_1$ containing $\mathfrak{l}A$, in case $n = 1$, $\mathfrak{l}_1$ is an principal left ideal generated by an idempotent element, and, in case $n \geqq 2$, $Q^{n-1}_{\mathfrak{l}_1}$ is an (M_0)-module as an $A*$-left module.*

Further, we have

THEOREM 5. *Let A be an algebra with a unit element* 1, *and let $\mathfrak{l}$ be a left ideal of A. If $H^n_{\mathfrak{l}}(A, \mathfrak{m}) = 0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l} = 0$, then, for every left ideal $\mathfrak{l}_1$ containing $\mathfrak{l}A$, in case $n = 1$, $\mathfrak{l}_1$ is a principal left ideal generated by an idempotent element, and, in case $n \geqq 2$, $Q^{n-1}_{\mathfrak{l}_1/\mathfrak{l}}$ is an (M_0)-module as an $A*$-left module.*

Proof. In case $n = 1$, the assertion is proved in Theorem 4, and, in case $n > 2$, since $Q^{n-2}_{\mathfrak{l}}$ and $Q^{n-1}_{\mathfrak{l}_1}$ are both (M_0)-modules as $A*$-left modules (Theorem 4), from Lemma 5, we see that $Q^{n-1}_{\mathfrak{l}_1/\mathfrak{l}}$ is an (M_0)-module as an $A*$-left module. In case $n = 2$, by Theorem 4, $\mathfrak{l} = Ae$ with an idempotent element e and $\mathfrak{l}_1$ is an (M_0)-module. As is easily seen, $\mathfrak{l}_1$ is a direct sum of $\mathfrak{l}$ and another submodule which is necessarily isomorphic to $\mathfrak{l}_1/\mathfrak{l}$. Hence, from [10], Lemma 1, we see that $\mathfrak{l}_1/\mathfrak{l}$ is an (M_0)-module.

4. Main theorems

LEMMA 8. *Let A be an algebra over Ω, and $\mathfrak{a}$ be a two-sided ideal of A. For any extension field Λ of Ω, n-dimensional (ordinary) cohomology groups of A_Λ in A_Λ-A_Λ-modules $\mathfrak{m}_1$ satisfying $\mathfrak{m}_1\mathfrak{a}_\Lambda = 0$ all vanish if, and only if, n-dimensional (ordinary) cohomology groups of A in A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$ all vanish.*

Proof. Assume first that all n-dimensional cohomology groups of A in A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$ vanish. Let Λ be an extension field of Ω, $\mathfrak{m}_1$ be an A_Λ-A_Λ-module satisfying $\mathfrak{m}_1\mathfrak{a}_\Lambda = 0$, and let f_1 be an n-dimensional cocycle of A_Λ in $\mathfrak{m}_1$. Since a basis $\{x_\alpha\}$ of A over Ω is also a basis of A_Λ over Λ, f_1 is determined by the value $f_1(x_{\alpha_1}, \dots, x_{\alpha_n})$ for $x_{\alpha_1}, \dots, x_{\alpha_n}$. The A_Λ-A_Λ-module $\mathfrak{m}_1$ may be naturally considered as an A-A-module satisfying $\mathfrak{m}_1\mathfrak{a} = 0$, and the cochain $f_1|A$ of A in $\mathfrak{m}_1$ defined by f_1 is cocycle. From the assumption, there exists an $(n-1)$-dimensional cochain g of A in $\mathfrak{m}_1$ such that

$$(f_1|A)(x_{\alpha_1}, \dots, x_{\alpha_n}) = \delta g(x_{\alpha_1}, \dots, x_{\alpha_n}). \tag{21}$$

In case $n \geqq 2$, let g_1 be the cochain of A_Λ in $\mathfrak{m}_1$ obtained from g by linear extension. Then, from (21), we have $f_1 = \delta g_1$ readily. In case $n = 1$, it is obvious from (21) that f_1 is a coboundary of A_Λ, and hence the "if" part of the lemma is proved.

Conversely, let Λ be an extension field of Ω, and assume that all n-dimensional cohomology groups of A_Λ in A_Λ-A_Λ-modules $\mathfrak{m}_1$ satisfying $\mathfrak{m}_1\mathfrak{a}_\Lambda = 0$ vanish. Let $\mathfrak{m}$ be an A-A-module satisfying $\mathfrak{m}\mathfrak{a} = 0$, and let f be an n-dimensional cocycle of A in $\mathfrak{m}$. Since the cochain f_1 of A_Λ in $\mathfrak{m}_\Lambda$ obtained from f by linear extension is also a cocycle, and $\mathfrak{m}_\Lambda\mathfrak{a}_\Lambda = 0$, there exist an $(n-1)$-dimensional cochain g_1 of A_Λ in $\mathfrak{m}_\Lambda$ such that

$$f_1(x_{\alpha_1}, \dots, x_{\alpha_n}) = \delta g_1(x_{\alpha_1}, \dots, x_{\alpha_n}). \tag{22}$$

Let $\{\lambda_0 = 1, \lambda_1, \dots\}$ be a basis of Λ over Ω. Then $\mathfrak{m}_\Lambda$ is the direct sum of submodules $\mathfrak{m}\lambda_i$ which are all isomorphic to $\mathfrak{m}$ as A-A-modules. We denote the $\mathfrak{m}\lambda_0$-component of $g_1(x_{\alpha_1}, \dots, x_{\alpha_{n-1}})$ by $g(x_{\alpha_1}, \dots, x_{\alpha_{n-1}})$ then, since $\delta g_1(x_{\alpha_1}, \dots, x_{\alpha_n}) = f(x_{\alpha_1}, \dots, x_{\alpha_n})$ belongs to $\mathfrak{m}\lambda_0$, we have readify $\delta g(x_{\alpha_1}, \dots, x_{\alpha_n}) = \delta g_1(x_{\alpha_1}, \dots, x_{\alpha_n}) = f(x_{\alpha_1}, \dots, x_{\alpha_n})$. This shows that f is a coboundary of A in $\mathfrak{m}$, and hence the "only if" part of the lemma is proved.

So far, we did not assume that A is finite over Ω. But we assume now that our algebra A over Ω is of finite rank and possesses a unit element.

We shall first prove the following theorem, which gives a generalization of our recently obtained main theorem ([8], Main Theorem).

THEOREM 6. *Let A be an algebra of finite rank over Ω, possessing a unit element, N be its radical, and let $\mathfrak{a}$ be a two-sided ideal of A.*

If $H^n(A, \mathfrak{m}) = 0$ for all A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$, then,

α) $A/(\mathfrak{a}+N)$ is separable, and,

β) for every left ideal $\mathfrak{l}$ of A containing $\mathfrak{a}$, in case $n = 1$, $\mathfrak{l}$ is a principal left ideal generated by an idempotent element, and, in case $n \geqq 2$, $Q_{\mathfrak{l}}^{n-1}$ is an (M_0)-module as an $A*$-left module.

Conversely, if α) is the case, and if,

β_1) in case $n = 1$, $N+\mathfrak{a}$ is a principal left ideal generated by an idempotent element,[4] and, in case $n \geqq 2$, $Q_{N+\mathfrak{a}}^{n-1}$ is an (M_0)-module as an $A*$-left module, then $H^n(A, \mathfrak{m}) = 0$ for all A-A-modules satisfyng $\mathfrak{m}\mathfrak{a} = 0$.

Since we showed, in Theorem 3, the assertion β) in the former half of the theorem, it is sufficient to prove α) in the former half, and the latter half of the theorem. The proof is very similar to that of Main theorem in [8].

Let

$$1 = \sum_{\kappa=1}^{k}\sum_{i=1}^{m_\kappa} e_{\kappa i} \tag{23}$$

be a decomposition of 1 into mutually orthogonal primitive idempotent elements in A such that the left ideals $Ae_{\kappa i}$ and $Ae_{\lambda j}$ are A-operator isomorphic (or, equivalently, the right ideals $e_{\kappa i}A$ and $e_{\lambda j}A$ are A-operator isomorphic) when, and only when $\kappa = \lambda$. Put $e_\kappa = e_{\kappa 1}$ for the sake of simplicity.

We first consider the case where the irreducible representations of A in Ω are all absolutely irreducible. This is equivalent to that $(e_\kappa A e_\kappa / e_\kappa N e_\kappa : \Omega) = 1$ for every κ, and further to that the semi-simple algebra A/N is a direct sum of matric algebras over Ω. Since A/N is separable, by Wedderburn's theorem, there exists a subalgebra $\bar{A}$ of A such that

$$A = \bar{A} \oplus N. \tag{24}$$

This is in fact a consequence of the fact that the 2-dimensional (oridnary) cohomology groups of A/N all vanish. The idempotent elements $e_{\kappa i}$ may, and shall be taken from $\bar{A}$.

We denote $N+\mathfrak{a}$ by N_1. $Q_{N_1}^{n-1}$ and $Q_{\mathfrak{a}}^{n-1}$ may be considered as $A*$-$\bar{A}$-module on defining the right operation of $\bar{A}$ as usual,

Now, assume that n-dimensional (ordinary) cohomology groups of A in $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$ all vanish. We consider first the case $n = 1$. Any $A/\mathfrak{a}$-$A/\mathfrak{a}$-module $\mathfrak{m}$ may be considered as an A-A-module satisfying $\mathfrak{a}\mathfrak{m} = \mathfrak{m}\mathfrak{a} = 0$, and any 1-dimensional (ordinary) cochain, cocycle, coboundary of $A/\mathfrak{a}$ in $\mathfrak{m}$ may be naturally considered as 1-dimensional cochain, cocycle, coboundary of A in $\mathfrak{m}$ respectively. Hence, from the assumption, 1-dimensional cohomology groups of $A/\mathfrak{a}$ all vanish. From [4], Theorem

4) In this case, N is contained in $\mathfrak{a}$, and hence $N+\mathfrak{a} = \mathfrak{a}$.

4.1, $A/\mathfrak{a}$ is semi-simple separable, and hence $N+\mathfrak{a}=\mathfrak{a}$. This proves the assertion α) in case $n=1$.

Next, we consider the case $n \geqq 2$. Associating $x_1 \times x_2 \times \cdots \times x_n \in Q^n_{N_1/\mathfrak{a}}(x_1, \ldots, x_{n-1} \in A,\ x_n \in N_1/\mathfrak{a})$ with the element $x_1 * (x_2 \times \cdots \times x_n)$ of $1 * Q^{n-1}_{N_1/\mathfrak{a}}$, we have an A-operator homomorphic mapping of $Q^n_{N_1/\mathfrak{a}}$, under the ordinary left operation of A, upon $1 * Q^{n-1}_{N_1/\mathfrak{a}}$. The mapping is also $\bar{A}$-operator homomorphism under the ordinary right operation of $\bar{A}$, and its kernel is exactly $1 * Q^n_{N_1/\mathfrak{a}}$. It induces thus an $e_\kappa A e_\kappa - e_\lambda \bar{A} e_\lambda$-homomorphism of $e_\kappa Q^n_{N_1/\mathfrak{a}} e_\lambda$ onto $e_\kappa * Q^{n-1}_{N_1/\mathfrak{a}} e_\lambda$, and the kernel is $e_\kappa * Q^n_{N_1/\mathfrak{a}} e_\lambda$. Hence we have

$$(e_\kappa * Q^n_{N_1/\mathfrak{a}} e_\lambda : \Omega) = (e_\kappa Q^n_{N_1/\mathfrak{a}} e_\lambda : \Omega) - (e_\kappa * Q^{n-1}_{N_1/\mathfrak{a}} e_\lambda : \Omega)\,. \tag{25}$$

Here

$$(e_\kappa * Q^{n-1}_{N_1/\mathfrak{a}} e_\lambda : \Omega) = (e_\kappa * Q^{n-1}_{N_1} e_\lambda : \Omega) - (e_\kappa * Q^{n-1}_{\mathfrak{a}} e_\lambda : \Omega) \tag{26}$$

and, by the same argument as above, we have

$$\begin{aligned} (e_\kappa * Q^{n-1}_{N_1} e_\lambda : \Omega) &= (e_\kappa Q^{n-1}_{N_1} e_\lambda : \Omega) - (e_\kappa * Q^{n-2}_{N_1} e_\lambda : \Omega)\,, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ (e_\kappa * Q^2_{N_1} e_\lambda : \Omega) &= (e_\kappa Q^2_{N_1} e_\lambda : \Omega) - (e_\kappa * Q^1_{N_1} e_\lambda : \Omega)\,, \\ (e_\kappa * Q^1_{N_1} e_\lambda : \Omega) &= (e_\kappa N_1 e_\lambda : \Omega)\,. \end{aligned} \tag{27}$$

Combining (25), (26) and (27), we have

$$\begin{aligned} (e_\kappa * Q^n_{N_1/\mathfrak{a}} e_\lambda : \Omega) - (e_\kappa Q^n_{N_1/\mathfrak{a}} e_\lambda : \Omega) + \sum_{i=1}^{n-2} (-1)^{i-1} (e_\kappa Q^{n-i}_{N_1} e_\lambda : \Omega) \\ - (e_\kappa * Q^{n-1}_{\mathfrak{a}} e_\lambda : \Omega) = (-1)^{n-1} (e_\kappa N_1 e_\lambda : \Omega)\,. \end{aligned} \tag{28}$$

(In case $n=2$, the vacus sum on the left hand is to mean 0.)

Now, we consider generally an A-$\bar{A}$-module $\mathfrak{m}$. Let m be a natural number. If $Q^m_{\mathfrak{m}}$ is an (M_0)-module as an $A*$-left module, then, by [9], Lemma 2.3, it is an (M_0)-module as $A*$-$\bar{A}$-module, where we consider the right operation of $\bar{A}$ as usual. The same is, by [10], Lemma 2, the case with the unitary A $*$-$\bar{A}$-module $1 * Q^m_{\mathfrak{m}}$. Then, by virtue of the structure theorem of (M_0)-modules (see [10], Theorem 1), applied to the Kronecker product algebra of A and an inverse-isomorphic image of $\bar{A}$, $1 * Q^m_{\mathfrak{m}}$ is a direct sum of A-$\bar{A}$-submodules isomorphic to the A-$\bar{A}$-modules of form $Ae_\mu \times e_\nu \bar{A}$. Denoting by $t_{\mu\nu}$ the number of component isomorphic to $Ae_\mu \times e_\nu \bar{A}$, we want to write, symbolically,

$$1 * Q^m_{\mathfrak{m}} \simeq \sum_{\mu, \nu} t_{\mu\nu} (Ae_\mu \times e_\nu \bar{A})\,. \tag{29}$$

Then we have, for each κ, λ, an $e_\kappa A e_\kappa - e_\lambda \Omega$ $(= e_\lambda \bar{A} e_\lambda)$-isomorphism

$$e_\kappa * Q^m_{\mathfrak{m}} e_\lambda \simeq \sum_\mu t_{\mu\lambda} (e_\kappa A e_\mu \times e_\lambda \Omega)\,. \tag{30}$$

Hence

(31) $$(e_\kappa * Q^m_{\mathfrak{m}} e_\lambda : \Omega) = \sum_\mu t_{\mu\lambda} c_{\kappa\mu} ,$$

where

(32) $$c_{\kappa\mu} = (e_\kappa A e_\mu : \Omega)$$

are the Cartan invariants of A.

On the other hand, if $m \geqq 2$, we have, for any A-A-module $\mathfrak{m}$,

(33) $$\begin{aligned}(e_\kappa Q^m_{\mathfrak{m}} e_\lambda : \Omega) &= (e_\kappa A \times A \times \cdots \times A \times \mathfrak{m} e_\lambda : \Omega) \text{ (with } n-2\ A\text{'s)} \\ &= (\sum_\mu c_{\kappa\mu} m_\mu)\,(A : \Omega)^{m-2} (\mathfrak{m} e_\lambda : \Omega) .\end{aligned}$$

Now, by Theorem 3, $Q^{n-1}_{\mathfrak{a}}$ and $Q^{n-1}_{N_1}$ are both (M_0)-modules as $A*$-left modules and hence so as $A*$-$\bar{A}$-modules. By Lemma 4, $Q^n_{N_1/\mathfrak{a}}$ is also an (M_0)-module as an $A*$-$\bar{A}$-module. By (31) and (33), the left hand side of (28) may be described as follows:

(34) $$\sum_\mu c_{\kappa\mu} s_{\mu\lambda} .$$

where $s_{\mu\lambda}$ are certain integers.

On the other hand, since $e_\kappa N e_\kappa$ is a maximal two sided ideal of $e_\kappa A e_\kappa$, and $e_\kappa N e_\kappa \subseteqq e_\kappa N_1 e_\kappa \subseteqq e_\kappa A e_\kappa$, $(e_\kappa N_1 e_\kappa : \Omega) = c_{\kappa\kappa}$ or $= c_{\kappa\kappa} - 1$ according as $e_\kappa \equiv 0$ modulo N_1 or $e_\kappa \not\equiv 0$ modulo N_1, and further $\kappa \neq \lambda$ implies $(e_\kappa N_1 e_\lambda : \Omega) = c_{\kappa\lambda}$. Thus, combining (28) and (34), we have, for each κ such that $e_\kappa \not\equiv 0$ modulo N_1,

(35) $$\sum_\mu c_{\kappa\mu}(s_{\mu\kappa} + (-1)^{n-1}\delta_{\mu\kappa}) = (-1)^n$$

Thus we have

LEMMA 9. *Let A be an algebra over Ω such that the irreducible representations of A in Ω are all absolutely irreducible, and let $\mathfrak{a}$ be a tow-sidd ideal of A. If n-dimensional (ordinary) cohomology groups of A in A-A-modules $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$ all vanish, then the relation (35) holds for each κ such that e_κ is not contained in $\mathfrak{a}$ (or, equivalently, in N_1).*

Further, we have the following lemma; the proof is exactly the same as that of [8], Lemma 5.

LEMMA 10. *Let A be an algebra over Ω, N be its radical, $\mathfrak{a}$ be a tow-sided ideal of A, and let Λ be the algebraic closure of Ω. If $A/(N+\mathfrak{a})$ is inseparable, then there exists a κ such that the primitive idempotent element e_κ of A_Λ is not contained in $\mathfrak{a}_\Lambda$ and Cartan invariants $c_{\kappa\mu}$ of A_Λ are divisible by the characteristic p of Ω for all μ.*

Combining Lemma 8, 9 and 10, we have easily the assertian α) of the former half of our theorem.

We now prove the latter half of the theorem. If n-dimensional (ordinary) cohomology groups of A in A-A-modules $\mathfrak{m}$ satisfying

(36) $$\mathfrak{m} N_1 = 0$$

all vanish, then n-dimensional cohomology groups of A in $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$ all vanish; this may be easily seen by considering a normal series of a given A-A-module $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{a} = 0$ in which every residue module satisfies (36), and applying a well-known argument by considering residue modules. Therefore, it is sufficient to consider A-A-modules satisfying (36).

We first consider the case $n = 1$. Let $\mathfrak{m}$ be an A-A-module satisfying (36), and f be a 1-dimensional cocycle of A in $\mathfrak{m}$. Put $N_1 = Ae$ with idempotent element e, and $e' = 1 - e$. From the assumed property (36) of $\mathfrak{m}$, it is readily seen that f induces an A-left homomorphism of N_1 into $\mathfrak{m}$. Hence we have $aef(e) = f(ae)$. Since $\delta f(e, e') = ef(e') - f(ee') + f(e)e' = ef(e') + f(e)e' = 0$, and $f(e)e' = f(e)(1-e) = f(e)$, we have $ef(e') = -f(e)$. On the other hand, since Ae' is isomorphic to the semi-simple separable algebra A/N_1, there exists an element v of $\mathfrak{m}$ such that $f(ae') = ae'v - vae'$. Thus, we have $f(e) = -ef(e') = -ef(e'e') = -e(e'v - ve') = eve' = ev$, and hence $\delta v(ae + be') = (ae + be')v - v(ae + be') = aev + be'v - vbe' = aef(e) + f(be') = f(ae + be')$. This shows that f is a coboundary, and hence the latter half of our theorem is proved in case $n = 1$.

The proof in case $n \geqq 2$ is very similar to [9]. We shall state it briefly.

Since A/N_1 is semi-simple and separable, there exists a (separable semi-simple) subalgebra $\bar{A}_1$ such that

$$A = \bar{A}_1 \oplus N_1 . \tag{37}$$

By the similar argument to [9], we have

LEMMA 11. *Let $\mathfrak{m}$ be an A-A-module satisfying $\mathfrak{m}N_1 = 0$, and let $\bar{L}(Q_{N_1}^{n-1}, \mathfrak{m})$ be the module of all $\bar{A}$-right homomorphism of $Q_{N_1}^{n-1}$ into $\mathfrak{m}$, (where we consider $Q_{N_1}^{n-1}$ under the ordinary right operation of $\bar{A}$). We consider $Q_{N_1}^{n-1}$ as $A*$-left module, and define the operation of A on $\bar{L}(Q_{N_1}^{n-1}, \mathfrak{m})$ as in (15). Then, (under the assumption that A/N_1 is separable), we have*

$$H^n(A, \mathfrak{m}) \simeq H^1(A, \bar{L}(Q_{N_1}^{n-1}, \mathfrak{m})) . \tag{38}$$

Now, the right hand side of (38) is 0 for every A-A-module satisfying (36) when, and only when, $Q_{N_1}^{n-1}$ is an (M_0)-module as an $A*$-$\bar{A}_1$-module, the proof is exactly the same as in Hochschild [6], §1. And, further this is equivalent, by [9], Lemma 2.3, to that $Q_{N_1}^{n-1}$ is an (M_0)-module as an $A*$-left module. Thus, if $Q_{N_1}^{n-1}$ is an (M_0)-module as an $A*$-left module, then $H^n(A, \mathfrak{m}) = 0$ for every A-A-module $\mathfrak{m}$ satisfying (36), and hence the latter half of our theorem is proved in case $n \geqq 2$.

Combining Theorem 2 and Theorem 6, we have immediately the following main theorem.

MAIN THEOREM I. *Let A be an algebra of finite rank over a field Ω possessing a unit element 1, N be its radical, and let $\mathfrak{l}$ be a left ideal of A. If n-dimensional $\mathfrak{l}$-cohomology groups of A all vanish, then,*

α) $A/(N+\mathfrak{l}A)$ is seperable,

β) in case $n=1$ or 2, $\mathfrak{l}$ is a principal left ideal of A generated by an idempotent element, and in case $n>2$, $Q_{\mathfrak{l}}^{n-2}$ is an (M_0)-module as an $A*$-left module, and,

γ) for any left ideal $\mathfrak{l}_1$ of A containing $\mathfrak{l}A$, in case $n=1$, $\mathfrak{l}_1$ is a principal left ideal of A generated by an idempotent element, and in case $n\geqq 2$, $Q_{\mathfrak{l}_1}^{n-1}$ is an (M_0)-module as an $A*$-left module.

Conversely, if α) and β) are the cases, and if,

γ_1) in case $n=1$, $N+\mathfrak{l}A$ is a principal left ideal of A generated by an idempotent element, and in case $n\geqq 2$, $Q_{N+\mathfrak{l}A}^{n-1}$ is an (M_0)-module as an $A*$-left module, then all n-dimensional $\mathfrak{l}$-cohomology groups of A vanish.

Further we have

MAIN THEOREM II. *Let A, $\mathfrak{l}$ and N be the same as in Main Theorem I. If n-dimensional $\mathfrak{l}$-cohomology groups of A all vanish, then,*

α) $A/(N+\mathfrak{l}A)$ is separable, and,

δ) for any lef ideal $\mathfrak{l}_1$ of A containing $\mathfrak{l}A$, in case $n=1$, $\mathfrak{l}$ and $\mathfrak{l}_1$ are both principal left ideals of A generated by idempotent elements, and in case $n\geqq 2$, $Q_{\mathfrak{l}_1/\mathfrak{l}}^{n-1}$ is an (M_0)-module as an $A*$-left module.

Conversely, if α) is the case, and if,

δ_1) in case $n=1$, $\mathfrak{l}$ and $N+\mathfrak{l}A$ are both principal left ideals generated by idempotent elements, and, in case $n\geqq 2$ $Q_{A/\mathfrak{l}}^{n-1}$ and $Q_{(N+\mathfrak{l}A)/\mathfrak{l}}^{n-1}$ are (M_0)-modules as $A*$-left modules, then all n-dimensional $\mathfrak{l}$-cohomology groups of A vanish.

Proof. The former half of the theorem is clear from Theorem 5 and Main Theorem I. We prove the latter half. In case $n=1$, it is shown in Main Theorem I. Now, let $\mathfrak{m}$ be an A-A-module such that $\mathfrak{m}A=0$, and let $n\geqq 2$. Then we see readily that $H_{\mathfrak{l}}^{n}(A, \mathfrak{m})$ is isomorphic to $H_{[A/\mathfrak{l}]}^{n}(A, \mathfrak{m})$. By the assumption, $H_{[A/\mathfrak{l}]}^{n}(A, \mathfrak{m})=0$ for all A-A-modules $\mathfrak{m}$ (Theorem 1), and hence $H_{\mathfrak{l}}^{n}(A, \mathfrak{m})=0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}A=0$. By Lemma 7, in case $n=2$, $\mathfrak{l}=Ae$ with an idempotent element e of A, and, in case $n>2$, $Q_{\mathfrak{l}}^{n-2}$ is an (M_0)-module as an $A*$-left module. Hence $Q_{\mathfrak{l}}^{n-1}$ is also an (M_0)-module, and hence, from Lemma 3, we see readily that $Q_{N+\mathfrak{l}A}^{n-1}$ is an (M_0)-module (as an $A*$-left module). Thus, from Main Theorem I, we have our theorem.

As an immediate consequence of our Main theorems, we mention the following corollary.

COROLLARY. *Let A be a quasi-Frobenius algebra over a field Ω, and $\mathfrak{l}$ be its left ideal. For every natural number n, n-dimensional $\mathfrak{l}$-cohomology groups of A all vanish (if and) only if 1-dimensional $\mathfrak{l}$-cohomology groups of A all vanish.*

Proof. Quasi-Frobenius algebras are characterized as algebras (with unit element) whose (M_0)-left modules are always (M_u)-left modules[5] and conversely ([10]). By the same argument as in the proof of Corollary of main theorem in [8], we see that $Q^n_{\mathfrak{m}}(n \geqq 1)$ is an (M_0) module as A-left module (if and) only if $\mathfrak{m}$ is an (M_0)-left module, or, equivalently an (M_u)-left module. Therefore, if $H^n_{\mathfrak{l}}(A, \mathfrak{m}) = 0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l} = 0$, then, by Main Theorem I, $\mathfrak{l}$ and $N + \mathfrak{l}A$ are (M_u)-modules and hence generated by idempotent elements. This shows that $H^1_{\mathfrak{l}}(A, \mathfrak{m}) = 0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l} = 0$.

Appendix: Significance of 1-dimensional $\mathfrak{l}$-cohomology groups

The 1-, 2- and 3-dimensional ordinary cohomology groups of algebras were interpreted, by Hochschild, with reference to classical notions of structure, and a significance of 3-dimensional $\mathfrak{l}$-cohomology groups has been given by Nakayama in his paper [11].

For the significance of 1-dimensional cohomology groups, we shall prove the following theorem.

THEOREM 7. *All 1-dimensional $\mathfrak{l}$-cohomology groups of A vanish if and only if either of the following conditions is satisfied.*

(i) *For any A-A-modules $\mathfrak{n}$ and $\mathfrak{m}$ satisfying $\mathfrak{l}\mathfrak{m} = 0$, every right inessential enlargement of $\mathfrak{n}$ by $\mathfrak{m}$ splits.*

(ii) *For any A-A-modules $\mathfrak{m}$ and $\mathfrak{n}$ satisfying $\mathfrak{n}\mathfrak{l} = 0$, every left inessential enlargement of $\mathfrak{n}$ by $\mathfrak{m}$ splits.*

Proof. Assume first that 1-dimensional cohomology groups of A all vanish. Let $\mathfrak{m}$ and $\mathfrak{n}$ be two A-A-modules, and assume that $\mathfrak{l}\mathfrak{m} = 0$. We denote by $\mathfrak{R}(\mathfrak{m}, \mathfrak{n})$ the module of all A-right operator homomeorphism of $\mathfrak{m}$ into $\mathfrak{n}$, and consider it as an A-A-module on defining the operation of A as in (15). Clearly $\mathfrak{R}(\mathfrak{m}, \mathfrak{n})\mathfrak{l} = 0$. Hence, by Theorem 2, we have $H^1(A, \mathfrak{R}(\mathfrak{m}, \mathfrak{n})) = 0$. By [6], Theorem 1.3, this proves (i). In order to prove (ii), let $\mathfrak{m}$ and $\mathfrak{n}$ be two A-A-modules, and assume that $\mathfrak{n}\mathfrak{l} = 0$. We denote by $\mathfrak{L}(\mathfrak{m}, \mathfrak{n})$ the modules of all A-left operator homomorphism of $\mathfrak{m}$ into $\mathfrak{n}$, and consider it as an A-A-module on defining the operation of A as follows; for $f \in \mathfrak{L}(\mathfrak{m}, \mathfrak{n})$, we set

$$(39)\qquad \begin{aligned}(xf)(u) &= f(xu)\,,\\ (fx)(u) &= f(u)x\,,\end{aligned}$$

$(x \in A,\ u \in \mathfrak{m})$. Then, clearly $\mathfrak{L}(\mathfrak{m}, \mathfrak{n})\ \mathfrak{l} = 0$, and it is proved, by a similar way to [6], Theorem 1, 3, that the group of equivalent classes of left inessential enlargement of $\mathfrak{n}$ by $\mathfrak{m}$ is isomorphic to $H^1(A, \mathfrak{L}(\mathfrak{m}, \mathfrak{n}))$. But, by Theorem 2, $H^1(A, \mathfrak{L}(\mathfrak{m}, \mathfrak{n})) = 0$, hence we have (ii).

Conversely, assume that (i) is satisfied. Let $(1, A)$ be the algebra obtained from

5) For the notion of (M_u)-modules, see [10].

A by adjoining a new identity element 1, and let $\mathfrak{m}$ be an A-A-module satisfying $\mathfrak{m}\mathfrak{l}=0$. Then $\mathfrak{m}$ may be naturally considered as a unitary $(1, A)$-$(1, A)$-module. Associate every 1-dimensional $\mathfrak{l}$-cochain f of A in $\mathfrak{m}$ with a 1-dimensional cochain $f^{\blacktriangle}$ of A in $\mathfrak{R}((1, A)/\mathfrak{l}A, \mathfrak{m})$ defined by

$$f^{\blacktriangle}(x)(\bar{y}) = f(x)y, \tag{40}$$

where $x \in A$, $y \in (1, A)$ and $\bar{y}$ is the residue class of $(1, A)$ modulo $\mathfrak{l}A$ which contains y. Then f is an $\mathfrak{l}$-cocycle or $\mathfrak{l}$-coboundary when, and only when, $f^{\blacktriangle}$ is so. From the assumption, $H^1(A, \mathfrak{R}((A)/\mathfrak{l}A, \mathfrak{m}))=0$, hence we have $H^1_{\mathfrak{l}}(A, \mathfrak{m})=0$. By the same argument, we can conclude from (ii) that $H^1_{\mathfrak{l}}(A, \mathfrak{m})=0$ for all $\mathfrak{m}$ satisfying $\mathfrak{m}\mathfrak{l}=0$.

References

[1] S. Eilenberg, Algebras of cohomologically finite dimension, to appear in Commentarii Math. Helvetici.

[2] S. Eilenberg-M. Ikeda-T. Nakayama, On the dimension of modules and algebras, to appear in Nagoya Math. J.

[3] S. Eilenberg and N. Steenrod, Foundations of algebraic topology, (1952).

[4] G. Hochschild, On the cohomology groups of an associative algebra, Ann. Math. 46 (1945), 58-67.

[5] ——, On the cohomology theory for associative algebras, Ann. Math. 47 (1946), 568-579.

[6] ——, Cohomology and representations of associative algebras, Duke Math. J. 14 (1947), 921-948.

[7] M. Ikeda, On absolutely segregated algebras, Nagoya Math. J. 6 (1953), 63-75.

[8] M. Ikeda-H. Nagao-T. Nakayama, Algebras with vanishing n-cohomology groups, to appear in Nagoya Math. J.

[9] H. Nagao, Note on the cohomology groups of associative algebras, Nagoya Math. J. 6 (1953), 85-92.

[10] H. Nagao-T. Nakayama, On the structure of (M_0)- and (M_u)- modules, Math. Zeitshr. 59 (1953), 164-170.

[11] T. Nakayama, On absolutely segregated algebras and relative 3-cohomology groups of an algebra, Nagoya Math. J. 6 (1953), 177-185.