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Note on word-subgroups in free products of groups

By Mutuo TAKAHASI

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In this note we shall apply the subgroup-theorem in free products of groups to a special class of subgroups, called word-subgroups, and study the free product decompositions of such subgroups. The commutator subgroup and related subgroups are especially studied. Elementary properties of word-subgroups are given and the subgroup-theorem is restated, without proof, in sentions **1** and **2**. Results are in section **3**.

§1. Word-subgroups.

The notion of word-subgroups was introduced by F. Levi[1] as follows.

Arbitrary group G is given. Let $x_1, \dots, x_r$ be r variables. By $\rho(x_1, \dots, x_r)$ is denoted a *word* in $x_1, \dots, x_r$, that is, a formal product such as

$$x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \cdots x_{i_s}^{\varepsilon_s},$$

where x_{i_ν} is some one of $x_1, \dots, x_r$ and $\varepsilon = \pm 1$. And let P be a set of such words. If we take, for x_i, an arbitrary element g_i of G, $i=1, \dots, r$, then

$$\rho(g_1, \dots g_r) = g_{i_1}^{\varepsilon_1} g_{i_2}^{\varepsilon_2} \cdots g_{i_s}^{\varepsilon_s}$$

is an element of G. All such elements $\rho(g_1, \dots, g_r)$, for all $\rho \in P$ and for any $g_i \in G$, generate a subgroup in G, which is called a *word-subgroup* of G defined by P in G. We shall denote this subgroup by $P(G)$.

Now if we consider the free group F generated by countably infinite elements $x_1, x_2, \dots$, any subset P of F defines in G a word-subgroup $P(G)$ of G. Obviously the subgroup U of F, which is generated by the subset P in F, defines the same word-subgroup in G as $P(G)$.

In the free group F, for arbitrary elements $w_1, w_2, \dots$, the corespondence $x_i \to w_i$, $i=1, 2, \dots$, gives an endomorphism of F, and $\rho(x_1, \dots, x_r)$ coresponds to $\rho(w_1(x_1, \dots, x_n), \dots, w_r(x_1, \dots, x_n))$ by this endomorphism. And coversely any endomorphism of F is determined, in this way, by the elements $w_1, w_2, \dots$, which are the images of $x_1, x_2, \dots$, respectively by this endomorphism.

Therefore the least fully invariant[2] subgroup V containing U is generated by

1) F.Levi, Über die Untergruppen der freien Gruppen. Math. Zeitschr. Bd 37 (1933), s. 90-97.

2) A subgroup of a group is called *fully invariant* if it admits every endomorphism of the whole group. Such subgroups were first studied by F. Levi. Cf. 1).

the elements of P and those, which are obtained by these substitutions for $x_1, x_2, \ldots$.

The word-subgroup $V(G)$ is clearly identical with $P(G)$.

On the other hand, V itself is a word-subgroup of F defined by the set V in $F: V=V(F)$. Therefore we may consider, without loss of generality, that word-subgroups in G are defined by word-subgroups of the free group F.

A word-subgroup $V(G)$ in an arbitrary group G is necessarily fully invariant. In a free group F, any fully invariant subgroup V is a word subgroup, since the set of all the elements of V can be considered as the defining set of words for V in F.

In arbitrary groups, as is well known, the commutator-groups, the terms of the derived series (iterated commutator-subgroups) and those of the lower central series, for examples, are some of important word-subgroups.

Word-subgroups in a free group and their factor groups were studied by B. H. Neumann.[3]

§2. Subgroup-theorem.

We now consider a group G which is decomposed into a free product of two its subgroups A and B: $G=A*B$.

Elements of A are denoted by $a, a_1, a', \ldots$, elements of B by $b, b_1, b', \ldots$, and elements which belong to either A or B by $c, c_1, c', \ldots$. If g is an element of G, g is represented in the form

$$g=\prod_{i=1}^{\lambda} c_i = c_1 c_2 \ldots c_\lambda ,$$

where the c_i are elements, not equal to 1, alternately out of A and B. Then $\lambda=\lambda(g)$ is called the *length* of g. We define $\lambda(1)$ as 0. Let g, g' be two elements of G; $g=c_1c_2\ldots c_\lambda$, $g'=c_1'c_2'\ldots c'_\mu$. If c_λ and c_1' belong neither to A nor to B at the same time, we call the product gg' to be ***irreducible***, otherwise ***reducible***.

On free product decompositions of subgroups in G, the author formulated and proved[4] the *subgroup-theorem* in free product of groups, which is due first to A. Kurosch[5] and to R. Bare and F. Levi,[6] as follows.

For a subgroup U of G, we consider the coset-decompositions of G

$$G=\sum_i Ur_i=\sum_j Us_jA^{7)}=\sum_k Ut_kB ,$$

3) B. H. Neumann, Identical relations in groups I. Math. Ann. 114 (1937) s. 506-525.

4) M. Takahasi, Bemerkungen über den Untergruppensatz in freien Produkte, Proc. Imp. Acad. Tokyo, 20 (1945) pp. 539-594.

5) A. Kurosch, Die Untergruppen der freien Produkte von beliebigen Gruppen, Math. Ann. 109 (1934), s. 647.

6) R. Baer und F. Levi, Freie Prdukte und ihre Untergruppen, Compositio Math. 3 (1946) p. 391.

7) Double coset decompositition modulo (U, A). Us_jA consists of all the elements of the form us_ja, $u\in U$, $a\in A$.

and take the set of representatives $\{r_i\}$, $\{s_j\}$ and $\{t_k\}$ respectively. We put moreover,

$$A=\sum_{\tau}(s_j^{-1}Us_j\cap A)a_{j\tau}$$

and

$$B=\sum_{\sigma}(t_k^{-1}Ut_k\cap B)b_{k\sigma}\,.$$

Then

$$G=\sum_j Us_jA=\sum_j(Us_j\sum_{\tau}(s_j^{-1}Us_j\cap A)a_{j\tau})=\sum_j\sum_{\tau}Us_ja_{j\tau}\ ,$$

and

$$G=\sum_k\sum_{\sigma}Ut_kb_{k\sigma}.$$

If we take any one coset Ur_i, s_j, $a_{j\tau}$, t_k, b_σ are determined *uniquely* respectively, such that

$$Ur_i=Us_ja_{j\tau}=Ut_kb_{k\sigma} \qquad \text{hold.}$$

Now, if we put here

$$u(r_i,\,A)=r_ia_{j\tau}^{-1}s_j^{-1} \qquad \text{and} \qquad u(r_i,\,B)=r_ib_{k\sigma}^{-1}t_k^{-1}\,,$$

those elements u belong to U.

It is easily proved that the subgroup U is generated by all of these elements u and together by all the elements out of $U\cap s_jAs_j^{-1}$ $(=s_j(s_j^{-1}Us_j\cap A)s_j^{-1})$ and of $U\cap t_kBt_k^{-1}$.

Now we can choose the sets of regresentatives $\{r_i\}$, $\{s_j\}$ and $\{t_k\}$ subject to the following conditions.

(i) r_i is one of the elements of the shortest length in the coset Ur_i, and if $r_i=c_1c_2\dots c_\lambda$ is taken as the representative of the coset Ur_i, then any initial segment $r_i'=c_1\dots c_{\lambda'}$ $(\lambda'<\lambda)$ must also be taken as the representative of the coset Ur_i'.

(ii) s_j is one r_i, which has the shortest length among the representatives chosen as in (i), which belong to Us_jA.

(iii) t_k is one r_i, which has the shortest length, among the representatives chosen as in (i), which belong to Ut_kB.

If we have chosen the three sets of representatives in this way respectively, we can prove that *irreducible $u(r_i, A)$ and irreducible $u(r_i,B)$ generate a free subgroup H of U* and that

$$U=H*\prod_j^{*}(s_jAs_j^{-1}\cap U)*\prod_k^{*}(t_kBt_k^{-1}\cap U)$$

holds.

Even when the number of free factors of G is more than 2, the theorem can be proved analogously by the same way.

§3. Decomposition of word-subgroups.

We shall apply the results above to a word-subgroup $P(G)$ of G. First we prove

Lemma 1. *If $U=P(G)$ is a word-subgroup generated by $P=\{\rho_\nu\}$ in* G, *then $U\cap A=P(A)$ and $U\cap B=P(B)$.*

Proof. It is obvious that $P(A)\subseteqq U\cap A$.
Take an element a from $U\cap A$, a is represented as a product of the form

$$a=\Pi\rho_{\nu}(g_1,\dots,g_r),\quad g_i\in G.$$

We consider the mapping α, which leaves any element a' of A invariant and mapps every element b of B to 1. This mapping is clearly an endomorphism of G onto A. Hence

$$a=a^{\alpha}=\Pi\rho_{\nu}(g_1^{\alpha},\dots,g_1^{\alpha})^{8)}\ \text{and}\quad g_i^{\alpha}\in A.$$

a must belong to $P(A)$.
Therefore we have

Theorem 2. *If* $U=P(G)$ *is a word subgroup of* $G=A*B$,

$$P(G)=H*\overset{*}{\Pi}s_jP(A)s_j^{-1}*\overset{*}{\Pi}t_kP(B)t_k^{-1},$$

where H is a free subgroup.

As immediate corollaries we have

Corollary 3. *The commutator-subgroup of a free product of abelian groups is a free group.*

Corollary 4. *In a free product of soluble groups with at most n-th derived group=1, the n-th derived group is a free group.*

Corollary 5. *In a free product of nilpotent groups of class at most c, the c-th subgroup in the lower central series is a free group.*

We now consider the *commutator-subgroup* G' of $G=A*B$. Taking the results in section 2 into consideration, we can take the set $\{a_\sigma b_\tau\}$ for the set $\{r_i\}$ of representatives in $G=\sum G'r_i$, where a_σ and b_τ are representatives of A mod A' and of B mod B' respectively.

Since $G'(ab)A=G'bA$ and $G'(ab)B=G'aB$, we can take the sets $\{b_\tau\}$ and $\{a_\sigma\}$ for $\{s_j\}$ and $\{t_k\}$ respectively.

These sets of representatives satisfy the conditisns (i), (ii) and (iii) in section 2. If $a\neq 1$, $b\neq 1$, $u(ab,A)=aba^{-1}b^{-1}$ is irreducible, and the other $u(r_i,A)$, $u(r_i,B)$ are all reducible and can be ommitted from the generators. Hence we have

Theorem 6. *The commutator-subgroup G' of $G=A*B$ is decomposed into a free product of the form*

$$G=H*\overset{*}{\Pi}b_\tau A'b_\tau^{-1}*\overset{*}{\Pi}a_\sigma B'a_\sigma^{-1},$$

where A' (B') is the commutator-subgroup of A (B), $A=\sum_\sigma A'a_\sigma$, $B=\sum_\tau B'b_\tau$ and the totality of $a_\sigma b_\tau a_\sigma^{-1}b_\tau^{-1}$ is a free generator system of H.

Corollary 7. *If $G=A*B$, where $A=(a)$, $B=(b)$ are cyclic groups, then G is generated freely by all the elements $a^nb^ma^{-n}b^{-m}$ such that $a^n\neq 1$ and $b^m\neq 1$.*

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8) Because $G=A*B$ implies $A\frown\bar{B}=1$, where $\bar{B}$ is the normal subgroup generated by B.

We next consider the subgroup M of $G=A*B$, which is generated by all the elements $aba^{-1}b^{-1}$, $(a\in A,\ b\in B)$. Of course, M is a normal subgroup of G and $G/M\cong A\times B$. Since $G=\sum M(ab)$, like as in the case of G' above, we see that M is generated freely by all the elements $aba^{-1}b^{-1}$, $a\in A$, $b\in B$.

On the other hand, if we denote by $\bar{\bar{A}}$ and by $\bar{\bar{B}}$ the least normal subgroups of G containing $\bar{A}$ and $\bar{B}$ respectively, then $M=\bar{\bar{A}}\cap\bar{\bar{B}}$.

$M\subseteqq\bar{\bar{A}}\cap\bar{\bar{B}}$ is eveident.

If we consider the endomorphism β just like as α above,

$$(\bar{\bar{A}})^{\beta}=1 \quad\text{and}\quad (\bar{\bar{B}})^{\alpha}=1,$$

hence
$$(\bar{\bar{A}}\cap\bar{\bar{B}})^{\alpha}=(\bar{\bar{A}}\cap\bar{\bar{B}})^{\beta}=1.$$

Conversely an element g is contained in $\bar{\bar{A}}\cap\bar{\bar{B}}$, when $g^{\alpha}=g^{\beta}=1$ hold.

If we take an element x of $\bar{\bar{A}}\cap\bar{\bar{B}}$, $x=mab$, where $m\in M$. $x=mab\in\bar{\bar{A}}\cap\bar{\bar{B}}$ implies $m^{\alpha}a=m^{\beta}b=1$. But $M\subseteqq\bar{\bar{A}}\cap\bar{\bar{B}}$ implies $m^{\alpha}=m^{\beta}=1$, therefore $a=b=1$ and $x\in M$.

Theorem 8. *The subgroup $M=\bar{\bar{A}}\cap\bar{\bar{B}}$ of $G=A*B$ is generated freely by all the elements $aba^{-1}b^{-1}(a\in A,\ b\in B)$, and* $G/M\cong A\times B$.

We next prove the following

Lemma 9. *Let $G=H*A$, where H is a free group generated freely by $\{h_\nu\}$, and $h_\nu\to a_\nu\in A$ be an one valued mapping, then $\{h_\nu a_\nu\}$ generate freely a free subgroup K and $G=K*A$.*

Proof.

If an element in K, of the form

$$(h_{\nu_1}a_{\nu_1})^{\varepsilon_1}(h_{\nu_2}a_{\nu_2})^{\varepsilon_2}\dots(h_{\nu_t}a_{\nu_t})^{\varepsilon_t}\ ,$$

is equal to 1 in $H*A$, there must exists some one s such that

$$\varepsilon_s=-\varepsilon_{s+1},\quad a_{\nu_s}=a_{\nu_{s+1}}\ \text{and}\ h_{\nu_s}=h_{\nu_{s+1}}.$$

Hence there exists no non-trivial relation between elements of $\{h_\nu a_\nu\}$.

$G=H\cup A$ implies immediately $G=K\cup A$. If there exists a non-trivial relation between elements of K and elements of A, we take one of them such that it has the shortest length with respect to K and A. Let it be

$$R=(h_{\nu_1}a_{\nu_1})^{\varepsilon_1}a_1(h_{\nu_2}a_{\nu_2})^{\varepsilon_2}a_2\dots a_{t-1}(h_{\nu_t}a_{\nu_t})^{\varepsilon_t}a_t\,.$$

and $\varepsilon_1=1$ (without loss of generality). a_t may be equal to 1.

For R to be equal to 1 in $H*A$, considering it as a product in $H*A$, there must exists an initial part T of R such that

$$T=h_{\nu_1}\dots h_{\nu_1}^{-1}$$

and the part $(\dots)$ between h_{ν_1} and $h_{\nu_1}^{-1}$ is again equal to 1 in $H*A$.

Since $h_\nu\to a_\nu$ is an one valued mapping, T must be of the form $T=(h_{\nu_1}a_{\nu_1})\dots\dots(h_{\nu_1}a_{\nu_1})^{-1}$. Hence, according to the assumption on the length of R, T must be identical with R.

But if $\quad R=(h_{\nu_1}a_{\nu_1})a_1\ldots\ldots a_{r-1}(h_{\nu_1}a_{\nu_1})^{-1}$,

and accordingly $\quad R'=a_{\nu_1}a_1(h_{\nu_2}a_{\nu_2})^{\varepsilon_2}\ldots\ldots(h_{\nu_{t-1}}a_{\nu_{t-1}})^{\varepsilon_{t-1}}a_{t-1}a_{\nu_1}^{-1}$

also, is equal to 1 in $H*A$, R' is a non-trivial relation of shorter lengtht han R'. This is a contradiction. Hence there exists no non-trivial relation between K and A, and $G=K*A$ holds.

Return to Theorem 2, and consider the decomposition of a word-subgroup $U=P(G)$ of $G=A*B$:

$$U=P(G)=H*\overset{*}{\prod}s_jP(A)s_j^{-1}*\overset{*}{\prod}t_kP(B)t_k^{-1}.$$

We replace each h_ν, the free generator of H, by

$$k_\nu=h_\nu(h_\nu^\alpha)^{-1}(h_\nu^\beta)^{-1},$$

where α and β are the endomorphisms of G considerd above.

Since $\quad h_\nu^\alpha\in P(G)\cap A=P(A),\quad h_\nu^\beta\in P(G)\cap B=P(B),$

$$k_\nu^\alpha=h_\nu^\alpha(h_\nu^\alpha)^{-1}=1 \quad\text{and}\quad k_\nu^\beta=1.$$

Hence $\quad k_\nu\in M=\bar{A}\cap\bar{B}.$

Applying lemma 9, we have

Theorem 10. *If $U=P(G)$ is a word-subgroup of $G=A*B$, then*

$$U=P(G)=K*\overset{*}{\prod}s_jP(A)s_j^{-1}*\overset{*}{\prod}t_kP(B)t_k^{-1},$$

where K is part of the free subgroup

$$M=\overset{*}{\prod_{a\in A,\ b\in B}}(aba^{-1}b^{-1}).$$

Corollary 11. *If V is a word subgroup of a free group F, than V is defined by some power x^m (m is an integer) and, besides it, by some words of commutator from.*

Remark. Let $G=G_1\supset\ldots\supset G_n\supset\ldots$ be the lower central series of $G=A*B$, and G_ω be the meet of all G_n. Then holds also

$$G_\omega=H*\overset{*}{\prod}sA_\omega s^{-1}*\overset{*}{\prod}tB_\omega t^{-1}.$$

When G is a free group, it is known that $G_\omega=1$ holds. But in the case of $G=A*B$, in general, the factor H of G_ω need not be equal to 1.

This is shown by the following example.

Let A and B be finite cyclic groups (a) and (b) of order p and q respectively, where p and q are two different prime numbers. The commutator-subgroup $G'=G_2$ of G is a free group and $G_2=\overset{*}{\prod\limits_{n\not\equiv0(p),\ m\not\equiv0(q)}}(a^nb^ma^{-n}b^{-m})$. Modulo $G_3=G_2\circ G$, (a,b) is commutative with any element of G, and $(a^n,b^m)\equiv(a,b)^{nm}$ mod G_3. Hence $(a,b)^p\equiv(a^p,b)=1$ and $(a,b)^q\equiv(a,b^q)=1$. Therefore $G_2=G_3$, which implies $G_\omega=G_2$. Obviously $A_\omega=B_\omega=1$, hence $H=G_\omega=G_2\neq1$.