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Primitive Locally Free Groups

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§1. Introduction.

A group G is called to be *locally free*,[1] when any finite number of elements generate a free subgroup in G. No element except the identity in a locally free group G is of finite order. Any subgroup of a locally free group is locally free. Free products of locally free groups are also locally free. When we restrict our consideration to a countable locally free group G, it is easily seen that G is either a free group with a finite number of generators or is represented as the set-theoretical sum of an infinite increasing sequence of free groups, each of which has a finite number of generators.

If any finite subset of G can be embeded in free subgroups, which are generated by r elements, where r is a positive integer, the least such integer r is called the *rank* of G. If there exists no such integer, the rank of G is said to be infinite. The rank of a free group is identical with the number of its free generators.

Locally free groups of rank 1 are clearly abelian and they are nothing but *torsionfree abelian groups of rank 1*, or *locally cyclic* groups. They are called *rational groups*, because they are isomorphic to subgroups of the additive group of all rational numbers. The structure of them are quite well known.

In a previous note[2] we studied some conditions for a countable locally free group to be exactly free, that is, to be a free product of infinite cyclic subgroups, and saw that a locally free group of finite rank can not be free, except the trival case when it is generated by a finite number of elements.

The purpose of this paper is to study the class of countable locally free groups of finite rank, which are decomposed into free products of subgroups of rank 1. We shall call such groups to be *completely reducible.*

§2 is devoted to the studies on the *partitions*[3] of locally free groups and on subgroups of rank 1 in them, in §3 we define *primitivity* for locally free groups of finite rank and state some fundamental properties in primitive locally free groups, and in §4 is proved the main theorem, that the primitivity is necessary

1) Cf. A. Kurosch [6]. Numbers in brackets refer to the bibliography at the end of the paper.
2) Cf. M. Takahasi, [11].
3) Cf. M. Takahasi, [10], also P. G. Kontorovitch, [2].

and sufficient for a countable locally free group of finite rank to be completely reducible.

In §5 we remark on the system of invariants. Finally §6 is devoted to the remarks on *p-primitive* locally free groups.

§2. Partitions of locally free groups and their subgroups of rank 1.

In this section the rank of locally free groups considered need not be finite.

A *partition* of a group G is defined, in general, to be a system $\{S_K\}$ of subgroups of G such that every element except the identity of G is contained in one and only one *component* S_K. Partitions of G consisting at least two components are called to be *proper*. If every component of the partition $\{S_K\}$ of G is a rational group, the partition is called to be *complete*.

THEOREM *Locally free groups have the complete partition.*

This fact was already given by P. Kontorovitch, in [2]. And we shall omitt the details of the proof, but it will be noted here that, for any element x of the group, the maximal subgroup M_x of rank 1, which contains x, is determined uniquely and these maximal subgroups of rank 1 are the components of the partition. The component M_x containing the given element x is defined also to be the subset of the group, which consists of all the elements y such that $y^n=x^m$ holds, where n and m are some rational integers. Hence the components M_x is non-extensible, that is, $y^n \in M_x$ (n is an integer) implies alway $y \in M_x$.

Accordingly, if the rank of a locally free group G is more than 1, G has the proper complete partition and any subgroup of rank 1 is contained entirely in some one and only one of these components. Hence the complete partion of G is uniquely determined.

Next we consider *completely reducible* locally free groups. Let $G=\overset{*}{\Pi}A_\tau$ be one of its free product decompositions, where all A_τ are of rank 1. Such a decomposition will be called a *complete free product decomposition.*

According to the results due to F. Levi and R. Baer[4], which follow from the subgroup-theorem[5] in free products of groups, we see that, in any two such complete free product decompositions of G, any non cyclic factor of one is conjugate to one and only one such factor of the other and the numbers of infinite cyclic factors in both decompositions are the same. Therefore the free factors of rank 1 in complete free product decompositions of G are determined uniquely to within isomorphisms.

Conversely, if G and G' are two completely reducible locally free groups whose free factors correspond one-to-one and the corresponding factors are isomorphic to each other, it is obvious that G and G' are isomorphic.

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4) Cf. F. Levi and R. Baer, [7].

5) Cf. F. Levi and R. Baer, [7], and M. Takahasi, [9].

On the other hand, in the complete partition of a completely reducible locally free group $G=\overset{*}{\Pi}A_\tau$, the free factors A_τ and all their conjugate subgroups appear among the components and the other components besides them are all infinite cyclic subgroups.[6)]

Hence *the system of non cyclic free factors in a complete free product decomposition of a completely reducible locally free group G is given, to within isomorphisms, by a complete system of non cyclic maximal rational subgroups of G, any two of which are not conjugate.*

To characterize rational groups we shall use the notions of characteristics and overtypes.[7)]

The set α of natural numbers is called the *characteristic*, if

1) $m|n$ and $n\in\alpha$ imply $m\in\alpha$, and

2) $n, m\in\alpha$ always implies that the least common multiple of n and m belongs also to α.

The characteristics α and β are *equivalent*, if there exist two natural numbers n and m such that, for any $a\in\alpha$ and any $b\in\beta$, we may find numbers $n'|n$ and $m'|m$, for which $n'a=m'b$ holds. This equivalence relation is reflexive, symmetric and transitive, so that all characteristics can be arranged into classes of equivalent. These classes are called *overtypes*.

The natural number n is called a divisor of the element x of the locally free group G, when there exists such an element $y\in G$ that $y^n=x$.

It is evident that such an element y that $y^n=x$ lies always in the component M_x containing x.

The set of all divisors of x is a characteristic, and will be denoted by $\chi(x)$. The overtype containing $\chi(x)$ is called the overtype of x and is denoted by $\sigma(x)$.

It is obvious that any two elements x and y, which are contained in the same component of G, have the same overtypes $\sigma(x)=\sigma(y)$, hence we may define the overtype $\sigma(M_x)$ of the component M_x by the overtype $\sigma(x)$. If M_x is an infinite cyclic subgroup, then $\sigma(M_x)=1$.

Two components M_x and M_y are isomorphic, as rational groups, if and only if $\sigma(M_x)=\sigma(M_y)$ holds.

According to the considerations above, we see that a *completely reducible locally free group G is characterized, to within isomorphisms, by the overtypes of the complete system of non cyclic maximal rational subgroups of G, any two of which are not conjugate, and by the rank of G.*

§ 3. Primitive locally free groups.

Throughout the following, the group G is assumed to be a countable locally

6) Cf. M. Takahasi, [10].

7) Cf. E. Liapin, [8], or R.Baer, [1].

free group of finite rank n.

A free subgroup H of rank n in G is called to be a *basic subgroup* of G, when, for any free subgroup K of rank n which contains H, there exists a suitable free generator system $a_1, \dots, a_n$ of K such that $a_1^{m_1}, \dots, a_n^{m_n}$ are contained in H for some positive integers $m_1, \dots, m_n$.

According to the subgroup-theorem[8], we can conclude then that there exist a free generator system $a_1, \dots, a_n$ of K and positive integers $m_1, \dots, m_n$ such that

$$K = \overset{*}{\Pi}(a_i) \quad \text{and} \quad H = \overset{*}{\Pi}(a_i^{m_i})$$

hold. Here the integer m_i is obviously the least positive integer such that $a_i^{m_i} \in H$.

First, we can prove easily that

any free subgroup H' of rank n which contains a basic subgroup H is also a basic subgroup.

Let K be a free subgroup of rank n, which contains H', then $H \subset H' \subset K$. Since H is a basic subgroup, there exists a free generator system $a_1, \dots, a_n$ of K such that

$$H = \overset{*}{\Pi}(a_i^{m_i}) \subset K = \overset{*}{\Pi}(a_i) .$$

Applying the subgroup-theorem to H', we see that $(a_1^{m_1'}) * \cdots * (a_n^{m_n'})$ is a free fractor of H', where m_i' is a divisor of m_i, $i = 1, \dots, n$. Since the rank of H' is n, H' must be identical with $\overset{*}{\Pi}(a^{m_i'})$. Therefore H' is also a basic subgroup.

Let H be a basic subgroup of G and K be a free subgroup of rank n which contains H. If we take a free generator system $a_1, a_2, \dots, a_n$ of K such that

$$K = \overset{*}{\Pi}(a_i) \quad \text{and} \quad H = \overset{*}{\Pi}(a_i^{m_i})$$

hold, the system of n positive integers $m_1, m_2, \dots, m_n$ is determined uniquely only by H and K and is independent on the choice of free generator system $a_1, \dots, a_n$ of K.

For, if we denote by $\bar{H}$ the normal subgroup of K generated by H in K, $K/\bar{H}$ is a free product of the form

$$(\bar{a}_1) * (\bar{a}_2) * \cdots * (\bar{a}_n) ,$$

where $(\bar{a}_i)$ is a cyclic group of order m_i. Therefore, the system of $m_1, \dots, m_n$ is dependent only on the structure of $K/\bar{H}$, which is determined only by the groups H and K.

The system of n positive integers $m_1, \dots, m_n$ will be called the *index set* of H with respect to K.

DEFINITION. If a locally free group G of rank n contains in it at least one basic subgroup H, G will be called to be *primitive*.

Let G be a primitive locally free group of rank n and H be one of its

8) Cf. [7] and [9].

basic subgroups. For any free subgroup K of rank n containing H, we shall denote by $s(H,K)$ the number of m_i not equal to 1 in the index set of H with respect to K.

Obviously $s(H,K)$ is either 0 or is a positive integer equal to or less than n. Hence there exists the maximum value of $s(H,K)$, which we shall denote by $s(H)$. Again,

$$0 \leqq s(H) \leqq n .$$

If $H \subseteqq K \subseteqq L$, where K and L are both free subgroups of rank n, K is also a basic subgroup and we shall see that

$$s(H,K) \leqq s(H,L) \quad \text{and} \quad s(K,L) \leqq s(H,L) .$$

For, as before,

$$L = \overset{*}{\Pi}(a_i) \supseteqq K \supseteqq H = \overset{*}{\Pi}(a_i^{m_i}) \text{ implies } K = \overset{*}{\Pi}(a_i^{m_i'}) ,$$

where m_i' is a divisor of m_i: $m_i = m_i' m_i''$, $i = 1, 2, \dots, n$.

The index set of H with respect to L, is $m_1, \dots, m_n$ while that of H with respect to K and that of K with respect to L are $m_1', \dots, m_n'$ and $m_1'', \dots, m_n''$ respectively.

Here $m_i' \neq 1$ or $m_i'' \neq 1$ implies $m_i \neq 1$, hence the inequalities hold.

Particularly, if $s(H,K) = s(H)$, then $s(H,L) = s(H)$.

Among all the basic subgroups of G, we can find one, say H, such that $s(H)$ is minimum.

Such a basic subgroup we shall call a *principal subgroup*, and this minimum value r of $s(H)$, which is determined only by the group G itself, will be denoted by $r(G)$ and called *the reduced rank* of G.

Now, for any two basic subgroups $H \subset K$, we take free subgroups L and M of rank n respectively, such that

$$H \subseteqq L,\ K \subseteqq M \quad \text{and} \quad s(H,L) = s(H),\ s(K,M) = s(K) .$$

Then there exists a free subgroup N of rank n which contains both L and M, because G is a locally free group of rank n. As seen just above, since $H \subseteqq L \subseteqq N$ and $H \subseteqq K \subseteqq M \subseteqq N$, the inequalities

$$s(K,M) \leqq s(K,N) \leqq s(H,N),$$

hold and $\quad s(H,L) = s(H),\ L \subseteqq N$ imply $s(H,N) = s(H)$.

Therefore $\quad s(K) \leqq s(H)$.

From this inequality we have:

If H is a principal subgroup of G, then any free subgroup K of rank n, which contains H, is also a principal subgroup.

Now we shall prove a following lemma which will be used in the proof of the main theorem in the next section.

LEMMA. *Let F be a free group of rank n and H be a subgroup of F such that $H=\overset{*}{\Pi}(a_i^{m_i})$, for a suitably chosen free generator system $a_1, \dots, a_n$ of F. Then an element x of F, which does not belong to H but whose power x^s is contained in H, is necessarily of the form $x=ha_j^t h^{-1}$, where $h\in H$ and t is a rational integer which is not a multiple of m_j.*

Proof. Let

$$x=a_{i_1}^{\varepsilon_1}a_{i_2}^{\varepsilon_2}\dots a_{i_\lambda}^{\varepsilon_\lambda}, \quad \varepsilon=\pm 1,$$

be the normal form of x with respect to the free generator system $a_1, \dots, a_n$ of F, and assume that

$$a_{i_1}=a_{i_\lambda},\ a_{i_2}=a_{i_{\lambda-1}}, \dots, a_{i_{\rho-1}}=a_{i_{\lambda-\rho+2}},$$

$$\varepsilon_1+\varepsilon_\lambda=\varepsilon_2+\varepsilon_{\lambda-1}=\cdots=\varepsilon_{\rho-1}+\varepsilon_{\lambda-\rho+2}=0,$$

but

$$a_{i_\rho}\neq a_{i_{\lambda-\rho+1}},$$

or

$$a_{i_\rho}=a_{i_{\lambda-\rho+1}},\ \varepsilon_\rho+\varepsilon_{\lambda-\rho+1}\neq 0.$$

Obviously $\rho-1<\left[\frac{\lambda+1}{2}\right]$, since otherwise x must be equal to the identity.

Putting

$$u=a_{i_1}^{\varepsilon_1}a_{i_2}^{\varepsilon_2}\cdots a_{i_{\rho-1}}^{\varepsilon_{\rho-1}}$$

and

$$v=a_{i_\rho}^{\varepsilon_\rho}\cdots a_{i_{\lambda-\rho+1}}^{\varepsilon_{\lambda-\rho+1}},$$

we can write

$$x=uvu^{-1}.$$

Then

$$x^s=uv^s u^{-1}.$$

If x^s belongs to $H=\overset{*}{\Pi}(a_i^{m_i})$, x^s is represented in an irreducible word with respect to $a_1^{m_1}, \dots, a_n^{m_n}$. And this representation of x^s must be, at the same time, the normal form in F with respect to $a_1, \dots, a_n$. Therefore it can be concluded that $u\in H$ and $v^s\in H$.

According to the assumption that x does not belong to H, v does not belong to H also, and if v contains more than two different generators in $a_1, \dots, a_n$, then the power v^s can not belong to H. Hence v must be of the form a_j^t and t is not a multiple of m_j, but st is divisible by m_j.

§4. Main Theorem.

Return to the countable primitive locally free group G of rank n and of reduced rank r, and represent it as the set-theoretical sum of an increasing sequence of free subgroups H_j of rank n:

$$G=\cup H_j,$$

where

$$H_1\subseteqq H_2\subseteqq\cdots\subseteqq H_j\subseteqq\cdots.$$

According to the primitivity of G, we may assume here that every H_j is a principal subgroup,[9] and that

9) If we take a principal subgroup for H_1, then every H_j is also a principal subgroup.

$$s(H_1, H_2)=s(H_1)=r(G)$$

holds. Then there can be chosen the free generator system $a_{j1}, \dots, a_{jn}$ of H_j respectively such that

$$H_1=\overset{*}{\Pi}(a_{ji}^{m_{ji}})$$

and $m_{j1}, \dots, m_{jr}$ are not equal to 1, but the other $(n-r)$ members $m_{jr+1}, \dots, m_{jn}$, if exist, are all 1.

Then

$$H_j=(a_{j1} \quad)*\cdots*(a_{jr} \quad)*(a_{jr+1})*\cdots*(a_{jn})$$

$$H_2=(a_{21} \quad)*\cdots*(a_{2r} \quad)*(a_{2r+1})*\cdots*(a_{2n})$$

and

$$H_1=(a_{j1}^{m_{j1}})*\cdots*(a_{jr}^{m_{jr}})*(a_{jr+1})*\cdots*(a_{jn})$$

$$=(a_{21}^{m_{21}})*\cdots*(a_{2r}^{m_{2r}})*(a_{2r+1})*\cdots*(a_{2n})\,.$$

According to the subgroup-theorem, we see that

$$H_2=(a_{j1}^{m'_{j1}})*\cdots*(a_{jr}^{m'_{jr}})*(a_{jr+1})*\cdots*(a_{jn})\,,$$

where m'_{jk} is a divisor of m_{jk} for $k=1, 2, \dots, r$.

If $m'_{jk}=m_{jk}$ holds, then $s(H_1, H_2)\leqq r-1$. This contradicts to the assumption that $s(H_1, H_2)=r(G)=r$.

Now

$$m'_{jk}\neq m_{jk}, \; k=1, 2, \dots, r\,.$$

Therefore the element $\bar{a}_{jk}=a_{jk}^{m'_{jk}}$ of H_2 does not belong to H_1 but $\bar{a}_{jk}^{m''_{jk}}\in H$, where $m_{jk}=m'_{jk}m''_{jk}$. According to the lemma in the previous section, we can conclude that, for some l, $1\leqq l\leqq r$,

$$a_{jk}^{m'_{jk}}=\bar{a}_{jk}=h_k\, a_{2l}^{t_{jk}}h_k^{-1}\,,$$

where $h_k\in H_1$ and t_{jk} is an integer not divisible by m_{2l}.

Then the elements a_{jk} and $h_k a_{2l} h_k^{-1}$ must belong to the same component of the complete partion of H_j. Accordingly there must exist an element b_{jk} in H_j such that

$$b_{jk}^x=a_{jk} \quad \text{and} \quad b_{jk}^y=h_k a_{2l} h_k^{-1}\,,$$

where x and y are rational integers and given by

$$t_{jk}=x(t_{jk}, m'_{jk}), \quad m'_{jk}=y(t_{jk}, m'_{jk})$$

for the greatest common divisor (t_{jk}, m'_{jk}) of t_{jk} and m'_{jk}.

But since any proper power of any element in the free group H_j can not be equal to the a_{jk}, one of the members of a free generator system of H_j, x must be equal to 1, hence $m'_{jk}=y\cdot t_{jk}$ and $h_k a_{2l} h_k^{-1}=a_{jk}^y$.

Still more, we can see that $y=m'_{jk}$, because m'_{jk} is the least positive integer such that $a_{jk}^{m'_{jk}}\in H_2$ and $a_{jk}^y\in H_2$.

After all, we have that, to every a_{jk}, $k=1, \dots, r$, there exists one a_{2l}, $1\leqq l\leqq r$ such

that
$$a_{jk}^{m'_{jk}} = h_k a_{2l} h_k^{-1}$$
holds.

Two different a_{jk} and a_{jh} can not correspond to the same a_{2l} in this manner. Otherwise, $a_{jk}^{m'_{jk}}$ and $a_{jh}^{m'_{jh}}$ must be conjugate, and this can not occur, because a_{jk} and a_{jh} are two elements from one and the same free generator system of H_j. Similarly two different a_{2l} and a_{2m} can not be corresponded to the same a_{jk} in the same manner, because otherwise a_{2l} and a_{2m} must be conjugate with respect to $H_1(\subseteqq H_2)$ and this can not be the case by the same reason as above.

Denote by H'_j the subgroup generated by all the elements
$$h_k^{-1} a_{jk} h_k \ (k=1, 2, \dots, r) \quad \text{and} \quad a_{2r+1}, \dots, a_{2n},$$
then obviously $H'_j \subseteqq H_j$. Conversely holds
$$H'_j \supseteqq H_j .$$

Because H'_j contains $h_k^{-1} a_{jk} h_k$, $k=1, 2, \dots, r$, hence it contains $(h_k^{-1} a_{jk} h_k)^{m'_{jk}} = a_{2l}$, $l=1, 2, \dots, r$, and $a_{2r+1}, \dots, a_{2n}$ also, therefore $H'_j \supseteqq H_2 \supset H_1$.

Since $h_1, \dots, h_r \in H_1$,
$$a_{jk} \in H'_j \quad \text{for} \quad k=1, 2, \dots, r.$$
The other $a_{jr+1}, \dots, a_{jn}$ are, of course, contained in H_2 hence also in H'_j.

Thus we have, rearranging the suffices,
$$H_j = (b_{j1}) * \dots * (b_{jr}) * (b_{jr+1}) * \dots * (b_{jn}),$$
where
$$b_{jk} = h_k^{-1} a_{jk} h_k, \ b_{jk}^{m'_{jk}} = a_{2k}, \ k=1, 2, \dots, r,$$
and
$$b_{jh} = a_{2h}, \ h=r+1, \dots, n .$$

Starting from the given free generator system $a_{j1}, \dots, a_{jn}$ of H_j, we have succeeded in constructing a new free generator system $b_{j1}, \dots, b_{jn}$ of H_j, the powers of whose members are equal to $a_{21}, \dots, a_{2n}$ respectively.

Now we consider the subgroup R_i generated by all the elements b_{ji}, $j=3, 4, 5, \dots$.

This subgroup R_i is obviously of rank 1, because every b_{ji} is contained in the same component of the complete partition of G, which contains a_{2i}. And the group G is generated by all these R_i, $i=1, \dots, n$.

Moreover if we take a finite number of elements from *one* of those R_i, these elements belong at least to some one of the free factors $(b_{j1}), \dots, (b_{jn})$ of some one H_j. Hence there exists no non-trivial relation between elements of these R_i. Therefore G is the free product of all these R_i. Naturally among these R_i,
$$R_1, R_2, \dots, R_r$$
are non cyclic subgroups, but the other
$$R_{r+1}, R_{r+2}, \dots, R_n$$
are all infinite cyclic subgroups.

Thus we have:

Any countable primitive locally free group of rank n and of reduced rank r is completely reducible and is decomposed into a free product of r non cyclic subgroups of rank 1 and n-r infinite cyclic subgroups.

Conversely let

$$G=R_1*\cdots*R_r*(c_{r+1})*\cdots*(c_n),$$

where

$$R_k,\ 1\leqq k\leqq r,$$

is a non cyclic locally free group of rank 1, and

$$(c_h),\quad r+1\leqq h\leqq n,$$

is an infinite cyclic group respectively.

Then it is easy to prove that G is countable and is a primitive locally free group of rank n and of reduced rank r.

Any finite number of elements in R_k are contained obviously in a cyclic subgroup (b_k) of R_k, since R_k is of rank 1. Hence any finite number of elements in G can be embeded in a free subgroup H of the form

$$H=(b_1)*\cdots*(b_r)*(c_{r+1})*\cdots*(c_n),$$

which is of rank n. And it is almost evident that n is the least such integer. Therefore G is a locally free group of rank n. Each (c_h) is a maximal cyclic subgroup, hence any subgroup H of the form

$$H=(b_1)*\cdots*(b_r)*(c_{r+1})*\cdots*(c_n),$$

where

$$b_k\in R_k,\ k=1,\dots,r,$$

is a principal subgroup and $r(G)=s(H)=r$.

Thus we have the following main

THEOREM. *A countable locally free group G of finite rank is completely reducible if, and only if, it is primitive.*

The reduced rank r of G *is the number of non cyclic free factors of its complete free product decomposition.*

§5. System of invariants.

Let G be a countable primitive locally free group of rank n and of reduced rank r. The integers n and r are, of course, invariants of G. The group G is decomposed into a free product of the form

$$G=R_1*\cdots*R_r*(c_{r+1})*\cdots*(c_n).$$

According to the investigations in §2, the group G is characterized, to within isomorphisms, by the overtypes of the factors

$$R_1,\cdots,R_r,\ \text{and}\ (c_{r+1}),\dots,(c_n).$$

Among these overtypes, the last $n-r$ are, of course, equal to 1 but the first r are not equal to 1, since $R_1,\dots,R_r$ are non cyclic.

Here

$$\sigma(R_1),\dots,\sigma(R_r)$$

can be obtained directly from the group G, independently on its free product decomposition, as follows.

In the (unique) complete partition $G=\sum S_K$ of G, we can take just r components

$$S_1, S_2, \dots, S_r,$$

which are non cyclic and not conjugate to each other. Then there exists always a free product decomposition of G with r factors isomorphic to these r components and $n-r$ infinite cyclic factors. Thus the group G is determined completely by the numbers n and r and the overtypes $\sigma_1, \dots, \sigma_r$ of such components, that is, by the system of overtypes:

$$\sigma_1, \sigma_2, \dots, \sigma_r, \underbrace{1, \dots, 1}_{n-r}.$$

§6. *p*-primitive locally free groups of finite rank.

Finally we shall consider the special class of *p-primitive* groups.

DEFINITION. For a prime number p, we shall call a free subgroup H of rank n in a locally free group G of rank n to be a *p-basic subgroup*, when any free subgroup K of rank n, containing H, has a free generator system $a_1, \dots, a_n$ such that there exists n non-negative integers $e_1, \dots, e_n$, and powers of p: $p^{e_1}, \dots, p^{e_n}$, for which

$$a_i^{p^{e_i}} \in H, \ i=1,2,\dots,n,$$

hold.

Analogously as in §3, we can prove:

Any free subgroup of rank n containing a p-basic subgroup H is also a p-basic subgroup.

If H is a p-basic subgroup and K is a free subgroup of rank n, which contains H, the index set of H with respect to K consists of n powers of p, some of which may be equal to 1.

DEFINITION. A locally free group G of rank n will be called *p-primitive*, when G has at least one p-basic subgroup.

Now we have

THEOREM. *If p and q are two different prime numbers and if G is p-primitive and, at the same time, q-primitive, then G is a free group of rank n.*

Proof. Let H and K be a p-basic subgroup and a q-basic subgroup of G respectively. We can take a free subgroup L of rank n, which contains both H and K, because $H \cup K$ is of finite rank and G is of rank n. Then L is also a p-basic subgroup and, at the same time, a q-basic subgroup.

Now take any element x of G. There exists a free subgroup M of rank n such that $x \in M$ and $L \subseteq M$. The index set of L with respect to M is

$$p^{e_1}, p^{e_2}, \dots, p^{e_n}$$

and, at the same time,

$$q^{f_1}, q^{f_2}, \dots, q^{f_n}.$$

Since $p \neq q$, all those powers must be equal to 1. That is, $M = L$.

Hence any element $x (\in G)$ must belong to L, and

$$G = L.$$

According to this theorem,

if G is p-primitive, the prime number p is uniquely determined.

Hence this prime number p will be called the *characteristic* of the group G.

p-principal subgroups and *p-reduced rank* of a p-primitive locally free group of rank n will be defined in the quite same manner as in § **3**.

p-primitive locally free groups of rank 1 are either infinite cyclic or are *all* isomorphic to the additive group generated by the rational numbers

$$\frac{1}{p^i}, \quad i = 1, 2, \dots.$$

This group will be denoted by R_p.

Then we have, quite analogously as in §§ **3-4**,

THEOREM. *A countable locally free group G of rank n is decomposed into a free product of r subgroups isomorphic to R_p and $n-r$ infinite cyclic subgroups, if,* and *only if, G is p-primitive and of reduced rank r.*

THEOREM. *A p-primitive locally free group of finite rank is completely characterized by three integers, namely, by the characteristic p, the rank n, and the reduced rank r.*

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