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ZETA FUNCTION OF DEGENERATE PLANE CURVE SINGULARITY

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Abstract

We introduce in this paper a new resolution graph for an isolated complex plane curve singularity and then calculate the monodromy zeta function and the Alexander polynomial for the singularity in terms of this graph.

1. Introduction

Let $f\colon (\mathbb{C}^{n+1}, O) \to (\mathbb{C}, 0)$ be the germ of a complex analytic function with an isolated singularity at the origin of $\mathbb{C}^{n+1}$. One important topological invariant of the germ f is its Milnor fibration ([11])

$$f_{\epsilon,\eta}\colon B_\epsilon \cap f^{-1}(S^1_\eta) \to S^1_\eta$$

with Milnor fiber $F = f^{-1}_{\epsilon,\eta}(\eta)$ and geometric monodromy $h\colon F \to F$. We consider the singularity (C, O) which is the germ of the hypersurface $C = f^{-1}(0)$ at O. The zeta function of the monodromy h of the singularity (C, O) is defined to be

$$\zeta_f(t) = \prod_{k\geq 0} \det(1 - th_*|H_k(F))^{(-1)^{k+1}}.$$

The earlier important result on monodromy zeta functions belongs to A'Campo. In his celebrated article [2] he described explicitly the zeta function of the singularity (C, O) in terms of numerical data of an embedded resolution of singularity. Let π be a good embedded resolution of singularity for (C, O), let $\{E_s\}_{s\in S}$, with S finite, be the set of exceptional divisors of π together with the irreducible components of the strict transform $\tilde{C}$ of C. For each s in S, we set $E_s^\circ = E_s \setminus \bigcup_{t\neq s} E_t$. Denote by $\chi(E_s^\circ)$ the Euler–Poincaré characteristic of E_s°. Note that if E_s is an irreducible component of $\tilde{C}$, it is noncompact, and then $\chi(E_s^\circ) = 0$. Let m_s be the multiplicity of E_s, $s \in S$. Then the main theorem of [2] says that

(1) $$\zeta_f(t) = \prod_{s\in S} (1 - t^{m_s})^{-\chi(E_s^\circ)}.$$

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Among the other important contributions to monodromy zeta functions we can refer to A'Campo [1], Guseĭn-Zade [8, 9] and Némethi [13] for $n = 1$. In the general dimension but under the condition of nondegeneracy with respect to Newton polyhedron, Milnor and Orlik [12] calculated ζ_f for quasi-homogeneous isolated singularities, Varchenko [18] and Ehlers [6] calculated it in terms of the Newton diagram.

In this paper we are interested in the case where $n = 1$. We consider the complex reduced isolated plane curve singularity (C, O) defined by germ of a complex analytic function f at the origin O of $\mathbb{C}^2$. We use the concept of extended resolution graph of a resolution of singularity for (C, O) introduced in [10]. Fix a resolution of singularity π for (C, O) with the set $\{E_s\}_{s\in S}$ as above. Then the extended resolution graph $\mathbf{G}(f,\pi)$ (or simply $\mathbf{G}$) of π is defined to be a graph in which the vertices correspond to $\{E_s\}_{s\in S}$ and two vertices E_s and $E_{s'}$ are connected by an edge if the intersection $E_s \cap E_{s'}$ is nonempty. The zeta function is described in the formula (1) via the resolution of singularity by A'Campo [2] so that it only requires the multiplicities $m(Q)$ of the pullback of the function f to the vertices Q of $\mathbf{G}$ which is either of degree $d(Q)$ greater than or equal to 3 in $\mathbf{G}$ or an end vertex (i.e., $d(Q) = 1$), because for the case $Q = E_s$ with $d(Q) = 2$ the Euler–Poincaré characteristics $\chi(E_s^\circ) = \chi(S^2 - 2 \text{ points}) = 0$. If E_s corresponds to an irreducible component of $\tilde{C}$, its degree is 1 and E_s is homeomorphic to a disk. Thus $\chi(E_s^\circ) = 0$ and it does not contribute to the zeta function. Therefore it is useful to define the extended simplified resolution graph $\mathbf{G}_\mathrm{s}$ by cutting off the vertices with degree 2 from the extended resolution graph (the construction of $\mathbf{G}_\mathrm{s}$ is actually more complicated, see Section 2 for detail). It is then clear that $\mathbf{G}_\mathrm{s}$ is independent of the choice of the resolution of singularity π.

Let Q be a vertex of $\mathbf{G}_\mathrm{s}$ and $E(Q)$ the corresponding exceptional divisor of the fixed resolution of singularity π. We have evidently that, assuming $E(Q) = E_s$ for some s, the Euler–Poincaré characteristic $\chi(E_s^\circ)$ is equal to $-d(Q) + 2$. It thus follows that to compute $\zeta_f(t)$ it suffices to determine the multiplicities on the vertices and the degree of each vertex of $\mathbf{G}_\mathrm{s}$. This is also the main purpose of this paper.

In [7], the tree of contacts was introduced in terms of the Puiseux expansions. Guibert used this tree to compute the motivic Igusa zeta function defined by Denef and Loeser [4] associated with a family of functions, and then related it with the Alexander invariants of the family. In Section 4, we will reformulate Guibert's formula of Alexander polynomial in many variables for (C, O) in terms of the extended simplified resolution graph $\mathbf{G}_\mathrm{s}$.

2. The extended simplified resolution graph

The main references for this section are [3], [10] and [16].

We divide the construction of the extended simplified resolution graph $\mathbf{G}_\mathrm{s}$ into two processes as follows.

2.1. Step 1: The "primitive" graph $\mathbf{G}_\mathrm{p}$. Vertices of $\mathbf{G}_\mathrm{p}$ correspond bijectively to the total space of each toric modification and the base space (the root of $\mathbf{G}_\mathrm{p}$). Thus the number of vertices is one more than the number of necessary toric modifications. Two vertices are connected by an edge of $\mathbf{G}_\mathrm{p}$ if they correspond to a toric modification. Thus the graph $\mathbf{G}_\mathrm{p}$ presents the hierarchy of the toric modifications.

2.2. Step 2: The inductive construction of $\mathbf{G}_\mathrm{s}$. We view the root as the origin O of the base space. For the first toric modification $\pi_1\colon X_1 \to \mathbb{C}^2$, we take the first vertices of $\mathbf{G}_\mathrm{s}$ corresponding to the faces of the Newton boundary $\Gamma(f)$ (such vertices will be called *regular* vertices), and add two vertices named Q^{left} and Q^{right} to the left end and to the right end (these two will be called *leaves*). They make a bamboo and this is the first floor of the extended simplified resolution graph (the bamboo should lie in a horizontal plane—a floor).

Let us give some explanations for this. Assume that $\Gamma(f)$ has faces corresponding to a sequence of ordered primitive weight vectors $P_1, \dots, P_m$. We add other primitive weight vectors to this sequence to obtain a regular simplicial cone subdivision $Q_1, \dots, Q_d$ admissible for $f(x, y)$ (in the terminology of [3]), i.e., every $P_i = Q_j$ for some j and $\det(Q_j, Q_{j+1}) = 1$ for any $j = 0, \dots, d$ where $Q_0 = E_1$ and $Q_{d+1} = E_2$. Then $Q^{\mathrm{left}} = Q_1$ and $Q^{\mathrm{right}} = Q_d$. This left end Q^{left} appears only for the very first modification. The weight vectors $P_1, \dots, P_m$ are the unique ones satisfying that the exceptional divisor $E(P_i)$ has nonempty intersection with the strict transform of C in X_1, $i = 1, \dots, m$. We ignore the exceptional divisors with degree 2, i.e., which do not intersect with the strict transform of C.

Next consider any other toric modification $\pi_\xi\colon X_i \to X_j$ with center ξ in an exceptional divisor $E(Q)$ which appears in $\mathbf{G}_\mathrm{p}$ (Step 1), where Q corresponds to a weight vector of the previous modification $X_j \to X_k$. We assume that the partial extended simplified resolution graph is already constructed and let Q be the corresponding *regular* vertex of the simplified graph. (Note that ξ lies in the intersection I_Q of $E(Q)$ and the strict transform of C in X_j.) Suppose that the Newton boundary of the pullback of f has α faces with respect to the toric coordinates (u, v) at ξ so that $u = 0$ is the divisor $E(Q)$, we prepare $\alpha + 1$ vertices in a horizontal bamboo. *We can assume that the right end weight vector is different from the last face of the Newton boundary* (if the right end weight vector is an *exceptional integral* vector, i.e., having the form $(1, b)$, corresponding to the lowest right end edge of the Newton boundary, we add an additional weight vector

$$R = {}^t(1, b) + {}^t(0, 1)$$

between Q^{right} and E_2, then R is the new right end vertex.) We will call a vertex corresponding to a right end weight vector a *leaf* of $\mathbf{G}_\mathrm{s}$.

By the above each ξ in I_Q gives rise to a toric modification, and hence to a bamboo in the next floor. We connect the left end vertex of such a bamboo with Q by a non-horizontal edge. Observe that there is (are) $|I_Q|$ bamboo(s) in the next floor non-horizontally connected with Q, i.e., the degree of Q is equal to $|I_Q| + 2$. Inductively this describes the extended simplified resolution graph $\mathbf{G}_\mathrm{s}$.

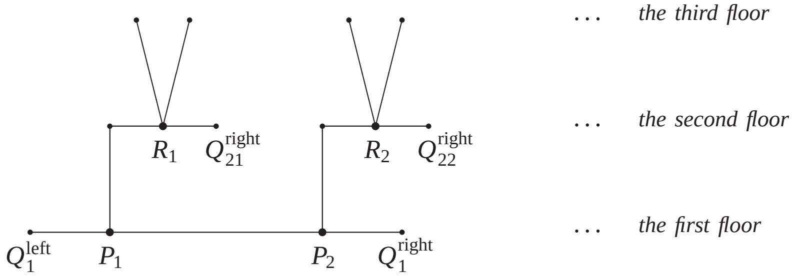
EXAMPLE 2.1. Consider the singularity C: $f(x, y) = (y^2 + x^3)^2(y^3 + x^2)^2 + x^6y^6$. The bamboo in the first floor consists of two regular vertices $P_1 = {}^t(3, 2)$, $P_2 = {}^t(2, 3)$ and two leaves $Q_1^{\mathrm{left}} = {}^t(2, 1)$, $Q_1^{\mathrm{right}} = {}^t(1, 2)$.

The intersection I_{P_1} of $E(P_1)$ and the strict transform of C has only a point ξ. Using the method of [3], there is a standard system of local coordinates (u, v) at ξ such that $\pi_1^* f$ in (u, v) has the form

$$\pi_1^* f(u, v) = Uu^{20}(v^2 + u^{10} + \text{higher terms}),$$

with U a unit. Thus the bamboo of floor 2 corresponding to P_1 has only a regular vertex $R_1 = {}^t(1, 5)$ and two leaves Q_{21}^{left}, Q_{21}^{right}. Similarly, the bamboo of floor 2 corresponding to P_2 has a regular vertex $R_2 = {}^t(5, 1)$ and two leaves Q_{22}^{left}, Q_{22}^{right}.

There are two one-point-bamboos in the third floor corresponding to R_1. Also, there are two one-point-bamboos in the third floor corresponding to R_2. Now we connect each left end vertex to the corresponding previous regular vertex, then we obtain the extended simplified resolution graph $\mathbf{G}_\mathrm{s}$ of $f(x, y)$ as follows



3. The numerical data for $\mathbf{G}_\mathrm{s}$ and the zeta function

In this section we will describe multiplicity and degree of each vertex of $\mathbf{G}_\mathrm{s}$ through the data of resolutions of singularity for the irreducible components of $f(x, y)$ and the relation between them. Based on the main theorem of [2], which is introduced in the first section, we read off the monodromy zeta function of the singularity $f(x, y)$.

3.1. Irreducible case. Assume that (C, O) is irreducible (this case was already considered in [3]). Let T be a resolution tower for (C, O) by toric modifications

$$T : X_g \to X_{g-1} \to \cdots \to X_0 = \mathbb{C}^2,$$

which corresponds to a sequence of primitive weight vectors R_i, say, ${}^t(a_i, b_i)$, $i = 1, \ldots, g$. Each R_i defines an exceptional divisor $E(R_i)$ which is the unique one containing the center G_i of the next toric modification. We denote $A_i = a_i a_{i+1} \cdots a_g$ for $i = 1, \ldots, g$, $A_{g+1} = 1$. Due to [3] there is a standard way to construct a system of local coordinates (u_i, v_i) at G_i such that $E(R_i)$ is given by $u_i = 0$, and if we denote by Φ_i the composition $X_i \to \cdots \to X_0$, then we have

$$\Phi_i^* f(u_i, v_i) = \begin{cases} u_g^{m(R_g)} v_g, & i = g, \\ u_i^{m(R_i)}((v_i^{a_{i+1}} + \xi_{i+1} u_i^{b_{i+1}})^{A_{i+2}} + (\text{higher terms})), & i < g, \end{cases}$$

where $m(R_i)$ is the multiplicity of $\Phi_i^* f$ on $E(R_i)$, i.e., the multiplicity of the vertex R_i in $\mathbf{G}_{\mathrm{s}}$, which satisfies the following

$$m(R_1) = a_1 b_1 A_2, \quad m(R_i) = a_i m(R_{i-1}) + a_i b_i A_{i+1}, \quad i = 2, \ldots, g.$$

3.2. General case: The first toric modification. Write $f(x, y)$ as a product of irreducible components in $\mathbb{C}\{x\}[y]$,

$$f(x, y) = \prod_{i=1}^{m} \prod_{j=1}^{r_i} \prod_{l=1}^{s_{ij}} g_{i,j,l}(x, y),$$

where

$$g_{i,j,l}(x, y) = (y^{a_i} + \xi_{i,j} x^{b_i})^{A_{i,j,l}} + (\text{higher terms})$$

are irreducible, the $\xi_{i,j}$'s are distinct nonzero complex numbers, $i = 1, \ldots, m$, $j = 1, \ldots, r_i$, $l = 1, \ldots, s_{i,j}$. Then the Newton boundary $\Gamma(f)$ has m faces whose weight vectors are $P_i = {}^t(a_i, b_i)$, $i = 1, \ldots, m$. Consider the first toric modification π_1 for (C, O). By the construction of the extended simplified resolution graph, P_i, $i = 1, \ldots, m$, are regular vertices of $\mathbf{G}_{\mathrm{s}}$ corresponding to π_1 and each P_i has degree $r_i + 2$ in $\mathbf{G}_{\mathrm{s}}$. Let $m(P_i)$ (resp. $m(Q^{\mathrm{left}})$, $m(Q^{\mathrm{right}})$) be the multiplicity of the pullback $\pi_1^* f$ on $E(P_i)$, $i = 1, \ldots, m$, (resp. on $E(Q^{\mathrm{left}})$, on $E(Q^{\mathrm{right}})$). Let the vertices $P_1, \ldots, P_m$ be ordered from the left to the right.

Observe that if we denote by $m_{t,j,l}(Q)$ the multiplicity of $\pi_1^* g_{t,j,l}$ on an exceptional divisor $E(Q)$ then we have

$$m(P_i) = \sum_{t=1}^{m} \sum_{j=1}^{r_t} \sum_{l=1}^{s_{t,j}} m_{t,j,l}(P_i).$$

Due to the irreducible case we have $m_{i,j,l}(P_i) = a_i b_i A_{i,j,l}$. Some similar simple computations also show that

$$m_{t,j,l}(P_i) = a_i b_t A_{t,j,l} \quad \text{for} \quad t < i,$$
$$m_{t,j,l}(P_i) = a_t b_i A_{t,j,l} \quad \text{for} \quad t > i.$$

Denote by A_t the sum $\sum_{j=1}^{r_t} \sum_{l=1}^{s_{t,j}} A_{t,j,l}$. We have just proved

Lemma 3.1. *With the previous notations, the following formulas hold*

$$m(P_i) = a_i \sum_{1 \le t \le i} b_t A_t + b_i \sum_{i+1 \le t \le m} a_t A_t, \quad i = 1, \dots, m,$$
$$m(Q^{\text{left}}) = \sum_{t=1}^{m} a_t A_t, \quad m(Q^{\text{right}}) = \sum_{t=1}^{m} b_t A_t.$$

REMARK 3.2. It is easily checked that $m(Q^{\text{left}})$ is equal to the degree n of $f(x, y)$ in $\mathbb{C}\{x\}[y]$. Using the Weierstrass preparation theorem, we can write $f(x, y)$ in the form $f(x, y) = ug(x, y)$, with $u = u(x, y)$ a unit in $\mathbb{C}\{x, y\}$, $g(x, y)$ being monic in $\mathbb{C}\{y\}[x]$. Then $m(Q^{\text{right}})$ is equal to the degree of $g(x, y)$ in the variable x.

3.3. General case: Vertices on a bamboo of floor ≥ 2. For such a bamboo $\mathcal{B}$, let $P_{\mathcal{B},i}$, $i = 1, \dots, m_{\mathcal{B}}$, be the regular vertices (i.e., the left end vertex $Q_{\mathcal{B}}^{\text{left}}$ and the right end vertex $Q_{\mathcal{B}}^{\text{right}}$ not included) of $\mathbf{G}_{\text{s}}$ lying on $\mathcal{B}$, and P the vertex of $\mathbf{G}_{\text{s}}$ non-horizontally connected to the left end vertex $Q_{\mathcal{B}}^{\text{left}}$ of $\mathcal{B}$, i.e., the bamboo $\mathcal{B}$ is arisen by a toric modification π_k centered at a point in the exceptional divisor $E(P)$. As above, we regard P as the predecessor of the $P_{\mathcal{B},i}$'s in $\mathbf{G}_{\text{s}}$. We assume that the multiplicity $m(P)$ is already described.

We give an explicit description for the relation between $m(P_i^{\mathcal{B}})$ and $m(P)$ as follows. Let Φ be the composition of the sequence of toric modifications starting from π_1 in Subsection 3.2 to the previous toric modification of π_k just mentioned. Suppose that, in the standard system of local coordinates (u, v) (at the center of π_k) constructed as in [3], the pullback $\Phi^* f(u, v)$ has the form

$$\Phi^* f(u, v) = U(u, v) \prod_{i=1}^{m_{\mathcal{B}}} \prod_{j=1}^{r_{\mathcal{B},i}} \prod_{l=1}^{s_{\mathcal{B},ij}} g_{\mathcal{B},i,j,l}(u, v),$$

where

$$g_{\mathcal{B},i,j,l}(u, v) = (v^{a_{\mathcal{B},i}} + \xi_{\mathcal{B},i,j} u^{b_{\mathcal{B},i}})^{A_{\mathcal{B},i,j,l}} + (\text{higher terms})$$

are irreducible in $\mathbb{C}\{u\}[v]$, the $\xi_{\mathcal{B},i,j}$'s are distinct and nonzero, $i = 1, \dots, m_{\mathcal{B}}$, $j = 1, \dots, r_{\mathcal{B},i}$, $l = 1, \dots, s_{\mathcal{B},i,j}$, and $U(u, v)$ is a unit in $\mathbb{C}\{u, v\}$. The $P_{\mathcal{B},i} = {}^t(a_{\mathcal{B},i}, b_{\mathcal{B},i})$,

$i = 1, \dots, m_{\mathcal{B}}$, are different faces of the Newton boundary $\Gamma(\Phi^* f, u, v)$. As a vertex of $\mathbf{G}_s$, $P_{\mathcal{B},i}$ has degree $r_{\mathcal{B},i} + 2$. As usual we assume that the faces $P_{\mathcal{B},i}$'s are ordered from the left to the right. We denote by $m_{\mathcal{B},t,j,l}(Q)$ the multiplicity of the pullback of the irreducible component of $f(x, y)$ corresponding to $g_{\mathcal{B},t,j,l}$ on an exceptional divisor $E(Q)$. Then due to the irreducible case, we have

$$m_{\mathcal{B},i,j,l}(P_{\mathcal{B},i}) = a_{\mathcal{B},i} m_{\mathcal{B},i,j,l}(P) + a_{\mathcal{B},i} b_{\mathcal{B},i} A_{\mathcal{B},i,j,l},$$

and similarly,

$$m_{\mathcal{B},t,j,l}(P_{\mathcal{B},i}) = a_{\mathcal{B},i} m_{\mathcal{B},t,j,l}(P) + a_{\mathcal{B},i} b_{\mathcal{B},t} A_{\mathcal{B},t,j,l} \quad \text{for} \quad t < i,$$
$$m_{\mathcal{B},t,j,l}(P_{\mathcal{B},i}) = a_{\mathcal{B},i} m_{\mathcal{B},t,j,l}(P) + a_{\mathcal{B},t} b_{\mathcal{B},i} A_{\mathcal{B},t,j,l} \quad \text{for} \quad t > i.$$

Thus we have

Lemma 3.3. *For* $i = 1, \dots, m_{\mathcal{B}}$,

$$m(P_{\mathcal{B},i}) = a_{\mathcal{B},i} m(P) + a_{\mathcal{B},i} \sum_{1 \le t \le i} b_{\mathcal{B},t} A_{\mathcal{B},t} + b_{\mathcal{B},i} \sum_{i+1 \le t \le m_{\mathcal{B}}} a_{\mathcal{B},t} A_{\mathcal{B},t},$$

moreover,

$$m(Q_{\mathcal{B}}^{\text{right}}) = m(P) + \sum_{t=1}^{m_{\mathcal{B}}} b_{\mathcal{B},t} A_{\mathcal{B},t},$$

where $A_{\mathcal{B},t} = \sum_{j=1}^{r_{\mathcal{B},t}} \sum_{l=1}^{s_{\mathcal{B},t,j}} A_{\mathcal{B},t,j,l}$.

EXAMPLE 3.4. Continue Example 2.1. Due to Lemma 3.1 we have

$$m(P_1) = 3 \cdot 2 \cdot 2 + 2 \cdot 2 \cdot 2 = 20, \qquad m(P_2) = 2(2 \cdot 2 + 3 \cdot 2) = 20,$$
$$m(Q_1^{\text{left}}) = 3 \cdot 2 + 2 \cdot 2 = 10, \qquad m(Q_1^{\text{right}}) = 2 \cdot 2 + 3 \cdot 2 = 10.$$

Similarly, applying Lemma 3.3 we get

$$m(R_1) = m(Q_{21}^{\text{right}}) = 30,$$
$$m(R_2) = m(Q_{22}^{\text{right}}) = 30.$$

3.4. The monodromy zeta function of the singularity $f(x,y)$. As in the introduction part, by a theorem of A'Campo [2], each exceptional divisor E_s of a resolution of singularity π contributes a factor $(1 - t^{m_s})^{-\chi(E_s^\circ)}$ to the zeta function $\zeta_f(t)$. Thus the bamboo corresponding to π_1 described in Subsection 3.2 contributes the following factor to $\zeta_f(t)$

$$\zeta_{\pi_1}(t) := (1 - t^{m(Q^{\text{left}})})^{-1} (1 - t^{m(Q^{\text{right}})})^{-1} \prod_{i=1}^{m} (1 - t^{m(P_i)})^{r_i}.$$

Each bamboo $\mathcal{B}$ of floor ≥ 2 contributes a factor to $\zeta_f(t)$ as follows

$$\zeta_{\mathcal{B}}(t) := (1 - t^{m(Q_{\mathcal{B}}^{\text{right}})})^{-1} \prod_{i=1}^{m_{\mathcal{B}}} (1 - t^{m(P_{\mathcal{B},i})})^{r_{\mathcal{B},i}}.$$

Let $\mathbf{B}$ be the set of bamboos of $\mathbf{G}_{\mathrm{s}}$, which coincides with the set of necessary toric modifications of resolution of singularity π. Note that $m(Q^{\text{left}})$ is equal to the degree n of $f(x, y)$ as a polynomial in $\mathbb{C}\{x\}[y]$. Then we have

Theorem 3.5. *The monodromy zeta function $\zeta_f(t)$ of the singularity $f(x, y)$ is described via $\mathbf{G}_{\mathrm{s}}$ as follows*

$$\zeta_f(t) = (1 - t^n)^{-1} \prod_{\mathcal{B} \in \mathbf{B}} (1 - t^{m(Q_{\mathcal{B}}^{\text{right}})})^{-1} \prod_{i=1}^{m_{\mathcal{B}}} (1 - t^{m(P_{\mathcal{B},i})})^{r_{\mathcal{B},i}}.$$

EXAMPLE 3.6. We continue Examples 2.1 and 3.4. With the data of $\mathbf{G}_{\mathrm{s}}$ described in these examples one deduces that

$$\begin{aligned}\zeta_f(t) &= (1 - t^{10})^{-1}(1 - t^{10})^{-1}(1 - t^{20})^2[(1 - t^{30})^{-1}(1 - t^{30})^2]^2 \\ &= (1 + t^{10})^2(1 - t^{30})^2.\end{aligned}$$

4. A formula for the Alexander polynomial

As before, we consider the reduced plane curve singularity $C = \{f(x, y) = 0\}$ at the origin O of $\mathbb{C}^2$. To recall the concept of Alexander polynomial, we write $f(x,y)$ as a product $\prod_{i=1}^{p} f_i(x,y)$ of irreducible components $f_i(x,y)$, $i = 1, \ldots, p$. The Alexander polynomial of this singularity $\Delta^C(T)$, where $T = (T_1, \ldots, T_p)$, is defined to be the Alexander polynomial of the link $C \cap \mathbb{S}^3_\epsilon \subset \mathbb{S}^3_\epsilon$ for sufficiently small $\epsilon > 0$ (see [5]) such that $\Delta^C(0, \ldots, 0) = 1$. Extending this notion to the relative version for regular functions f_i on a complex algebraic variety X, Sabbah [17] gives the Alexander complex viewed as an object of the category $D^b_c(X_0, \mathbb{C}[\mathbb{Z}^p])$ of bounded constructible complexes of $\mathbb{C}[\mathbb{Z}^p]$-modules on X_0, where $X_0 = \bigcap_{i=1}^{p} f_i^{-1}(0)$. Guibert [7] defines an Alexander zeta function associated with $(f_1, \ldots, f_p)$ at neighborhood of a compact set K. In fact, when K is a singular point $\{x\}$ of X_0 this notion reduces to the Alexander polynomial of the singularity (X_0, x). In [17] Sabbah gives an expression of this function in terms of a resolution of singularity for $(f_1, \ldots, f_p)$, which generalizes the formula of A'Campo [2] on the monodromy zeta function of a singularity. Let E_s, $s \in S$, again denote exceptional divisors and strict transforms of a resolution of singularity π for (C, O). Let $\lambda^{(s)}$ be the p-tuple of multiplicities of $(\pi^* f_1, \ldots, \pi^* f_p)$ on the divisor E_s.

Theorem 4.1 (Sabbah [17]). $\Delta^C(T_1, \ldots, T_p) = \prod_{s \in S} (T^{\lambda^{(s)}} - 1)^{-\chi(E_s^\circ)}$.

Now to describe the Alexander polynomial $\Delta^C(T)$ via the extended simplified resolution graph $\mathbf{G}_s$ of (C, O), we use the decompositions and the notations as in Section 3. We firstly consider the ordered vertices $Q^{\text{left}}, P_1, \ldots, P_m, Q^{\text{right}}$ of $\mathbf{G}_s$ on the unique bamboo of the first floor. With the notations as in Subsection 3.2, we have

$$m_{t,j,l}(P_i) = \begin{cases} a_i b_t A_{t,j,l} & \text{for} \quad 1 \leq t \leq i, \\ a_t b_i A_{t,j,l} & \text{for} \quad i < t \leq m, \end{cases}$$

and

$$m_{t,j,l}(Q^{\text{left}}) = a_t A_{t,j,l},$$
$$m_{t,j,l}(Q^{\text{right}}) = b_t A_{t,j,l}.$$

Thus the first bamboo contributes the following factor to the Alexander polynomial of (C, O):

$$(T^{\mathbf{m}(Q^{\text{left}})} - 1)^{-1}(T^{\mathbf{m}(Q^{\text{right}})} - 1)^{-1} \prod_{i=1}^{m}(T^{\mathbf{m}(P_i)} - 1)^{r_i},$$

where $\mathbf{m}(Q^{\text{left}}) = (m_{t,j,l}(Q^{\text{left}}))_{t,j,l}$, $\mathbf{m}(Q^{\text{right}}) = (m_{t,j,l}(Q^{\text{right}}))_{t,j,l}$ and $\mathbf{m}(P_i) = (m_{t,j,l}(P_i))_{t,j,l}$.

Consider a bamboo $\mathcal{B}$ of floor ≥ 2 with the ordered vertices $P_{\mathcal{B},1}, \ldots, P_{\mathcal{B},m_{\mathcal{B}}}, Q_{\mathcal{B}}^{\text{right}}$ as in Subsection 3.3. If $\Phi^* g_{t,j,l} = g_{\mathcal{B},t',j',l'}$ for some (t', j', l'), then we put

$$\begin{aligned} m_{t,j,l}(P_{\mathcal{B},i}) &:= m_{\mathcal{B},t',j',l'}(P_{\mathcal{B},i}) \\ &= \begin{cases} a_{\mathcal{B},i} m_{\mathcal{B},t',j',l'}(P) + a_{\mathcal{B},i} b_{\mathcal{B},t'} A_{\mathcal{B},t',j',l'} & \text{for} \quad 1 \leq t' \leq i, \\ a_{\mathcal{B},i} m_{\mathcal{B},t',j',l'}(P) + a_{\mathcal{B},t'} b_{\mathcal{B},i} A_{\mathcal{B},t',j',l'} & \text{for} \quad i < t' \leq m_{\mathcal{B}}. \end{cases} \end{aligned}$$

and

$$m_{t,j,l}(Q_{\mathcal{B}}^{\text{right}}) := m_{\mathcal{B},t',j',l'}(Q_{\mathcal{B}}^{\text{right}}) = m_{\mathcal{B},t',j',l'}(P) + b_{\mathcal{B},t'} A_{\mathcal{B},t',j',l'}.$$

Otherwise for a triple (t, j, l) such that $\Phi^* g_{t,j,l} = g_{\mathcal{B},t'',j'',l''}$ with $\mathcal{B} \neq \mathcal{B}$ (actually in the same floor), let $\overline{P}$ be the closest common "ancestor" of vertices on $\mathcal{B}$ and $\mathcal{B}$. Then we put

$$m_{t,j,l}(P_{\mathcal{B},i}) = m_{t,j,l}(Q_{\mathcal{B}}^{\text{right}}) := m_{\mathcal{B},t'',j'',l''}(\overline{P}).$$

Now we set $\mathbf{m}(Q_{\mathcal{B}}^{\text{right}}) = (m_{t,j,l}(Q_{\mathcal{B}}^{\text{right}}))_{t,j,l}$, $\mathbf{m}(P_{\mathcal{B},i}) = (m_{t,j,l}(P_{\mathcal{B},i}))_{t,j,l}$. Then the bamboo contributes the following factor to $\Delta^C(T)$:

$$(T^{\mathbf{m}(Q_{\mathcal{B}}^{\text{right}})} - 1)^{-1} \prod_{i=1}^{m_{\mathcal{B}}}(T^{\mathbf{m}(P_{\mathcal{B},i})} - 1)^{r_{\mathcal{B},i}}.$$

Denote $\mathbf{n} = (\deg_y g_{t,j,l}(x, y))_{t,j,l}$. Thus $\mathbf{n} = \mathbf{m}(Q^{\text{left}})$.

Proposition 4.2. *The Alexander polynomial* $\Delta^C(T)$ *is described via* $\mathbf{G}_s$ *as follows*

$$\Delta^C(T) = (T^{\mathbf{n}} - 1)^{-1} \prod_{\mathcal{B}\in\mathbf{B}} (T^{\mathbf{m}(Q_{\mathcal{B}}^{\text{right}})} - 1)^{-1} \prod_{i=1}^{m_{\mathcal{B}}} (T^{\mathbf{m}(P_{\mathcal{B},i})} - 1)^{r_{\mathcal{B},i}}.$$

In the irreducible case this formula reduces to that of Eisenbud–Neumann (cf. [5]) and that of A'Campo and Oka (cf. [3]).

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