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QUASI-SECTIONS IN LOG GEOMETRY

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Abstract

We systematically study quasi-sections of morphisms of log schemes. As applications, we prove exact log flat descents. Our main tool is the structure theorem of $\mathbb{Q}$-integral homomorphisms of monoids.

Contents

Introduction

Though exact morphisms were already introduced in K. Kato's fundamental paper [3], it is still not very clear what the exactness is in essence. It is known that some pathologies in log geometry disappear under the exactness assumption. For example, the base change of a surjective map in the category of fs log analytic spaces is not necessarily surjective. But any base change of an exact and surjective map is surjective. We can list more, but still can not explain reasonably enough why the exactness prevents various pathologies.

In this article, we add one more in the list, that is, the exactness allows us to have quasi-section statements in log geometry, which generalize those in usual geometry. For example, consider the fact that a smooth and surjective map of analytic spaces admits a section locally over the base. A naive generalization of this statement in log geometry is not valid, that is, a log smooth and surjective map of fs log analytic spaces does not necessarily admit a section even ket (= kummer log étale) locally, because a log blowing up clearly does not necessarily admit a section even ket locally, though it is log smooth and surjective. On the other hand, our 4.3 (1) says that any log smooth, *exact* and surjective map of fs log analytic spaces admits a section ket locally.

Our basic tool of studying exact morphisms is the structure theorem of $\mathbb{Q}$-integral homomorphisms. We explain briefly the notion of $\mathbb{Q}$-integrality introduced in [2]. Let $h\colon P \to Q$ be a homomorphism of fs monoids. By definition, h is $\mathbb{Q}$-integral if $h \otimes_{\mathbb{N}} \mathbb{Q}_{\geq 0}$

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is integral. For us, the point lies in that h is $\mathbb{Q}$-integral if and only if $(\operatorname{Spec}\mathbb{C}[h])_{\mathrm{an}}$ is an exact morphism. In this sense the $\mathbb{Q}$-integrality gives the right notion in the monoid theory which is corresponding to the exactness in the log geometry. Note that h is exact if h is local and $\mathbb{Q}$-integral, but the converse does not necessarily hold. A rather detail analysis of $\mathbb{Q}$-integral homomorphisms reaches a kind of structure theorem [2] (A.3.2.2) of $\mathbb{Q}$-integral homomorphisms, which we exploit in this article.

This article is organized as follows. After a review of $\mathbb{Q}$-integral homomorphisms in Section 1, we prove quasi-section statements in monoid theory as consequences of the structure theorem in Section 2. In Section 3, we start with some variants of quasi-section statements by using the results in Section 2, and as their corollaries, we prove generalizations of kummer log flat descents in [6], which is the original motivation of the present work. In Section 4, we prove quasi-section statements, as generalizations of non-log cases, which can be applied to the existence of log Albanese map (4.4), for example. Finally, in Section 5, we show that, under the assumption of the verticality, some stronger statements hold.

We hope that the results in this article would make progress in our understanding on what the exactness is.

In Sections 3–5, the results are formulated algebraically except a few. Of course, one can write down the analogous statements for fs log analytic spaces. The same proofs work and sometimes the statements would be simpler than the algebraic ones because the characteristic of the complex number field is zero.

NOTATION AND TERMINOLOGY. All monoids in this article are saturated. We say that a monoid P is *sharp* if its group of the invertible elements $P^\times$ is trivial. For a monoid P, the dimension $\dim P$ of P is the maximal length of an increasing sequence of prime ideals of P ([4] (5.4)). For a monoid P, we denote $P/P^\times$ by $\overline{P}$. For a log structure M on a ringed topos $(X, \mathcal{O}_X)$, we denote $M/\mathcal{O}_X^*$ by $\overline{M}$.

We say that a homomorphism of monoids is *local* if the inverse image by it of the maximal ideal is the maximal ideal. We say that a homomorphism $h\colon P \to Q$ of integral monoids is *exact* if $P = (h^{\mathrm{gp}})^{-1}(Q)$ in P^{gp}. We say that a morphism $f\colon X \to Y$ of fs log schemes or fs log analytic spaces is *strict* (resp. *exact*) if $f^*M_Y \to M_X$ is an isomorphism (resp. is exact at stalks).

1. Review of $\mathbb{Q}$-integral homomorphisms

In this section, we review $\mathbb{Q}$-integral homomorphisms briefly. For details, see [2], Appendix (A.3).

DEFINITION 1.1. Let h: $P \to Q$ be a homomorphism of fs monoids. We say that h is *$\mathbb{Q}$-integral* if one of the following equivalent conditions (1) and (2) is satisfied.
(1) $h \otimes_{\mathbb{N}} \mathbb{Q}_{\geq 0}$ is integral.
(2) $\operatorname{Spec}\mathbb{C}[h]$ is an exact morphism of fs log schemes.

For the equivalence of (1) and (2), see [2] (A.3.2) and the remark after it.

An integral homomorphism is $\mathbb{Q}$-integral. A local $\mathbb{Q}$-integral homomorphism $P \to Q$ is exact, and if P is sharp, it is injective. See [2] (A.3.1).

DEFINITION 1.2 (cf. a remark after [2] (A.3.3) for the analytic version). A morphism $f\colon X \to Y$ of fs log schemes is said to be $\mathbb{Q}$-integral if for any $x \in X$, the homomorphism $\overline{M}_{Y,\overline{f(y)}} \to \overline{M}_{X,\overline{x}}$ is $\mathbb{Q}$-integral.

An integral morphism of fs log schemes is $\mathbb{Q}$-integral. A $\mathbb{Q}$-integral morphism of fs log schemes is exact.

Next, we explain that a log flat and exact morphism admits fppf locally a $\mathbb{Q}$-integral chart, and that a log smooth and exact morphism admits étale locally a $\mathbb{Q}$-integral chart. For log flat morphisms, see [6] and [7].

Precise statements are:

Lemma 1.3. *Let $f\colon X \to Y$ be an exact and log flat morphism of fs log schemes. Then there exists fppf locally on X and on Y a chart $\Gamma(Y, \mathcal{O}_Y) \leftarrow P \xrightarrow{h} Q \to \Gamma(X, \mathcal{O}_X)$ of f by fs monoids such that P is sharp, h is local and $\mathbb{Q}$-integral, and such that the induced morphism $X \xrightarrow{i} Y \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$ is strict and flat.*

Lemma 1.4. *Let $f\colon X \to Y$ be an exact and log smooth morphism of fs log schemes. Then there exists étale locally on X and on Y a chart $\Gamma(Y, \mathcal{O}_Y) \leftarrow P \xrightarrow{h} Q \to \Gamma(X, \mathcal{O}_X)$ of f by fs monoids such that P is sharp, h is local and $\mathbb{Q}$-integral, the order of the torsion part of the cokernel of h^{gp} is invertible on X, and such that the induced morphism $X \xrightarrow{i} Y \times_{\operatorname{Spec} \mathbb{Z}[P]} \operatorname{Spec} \mathbb{Z}[Q]$ is strict and smooth.*

Note that [2] (A.3.3) is an analytic version of 1.4.

1.5. Since the proof of 1.4 is similar to that of 1.3, we only explain that for 1.3. The following is parallel to that for [2] (A.3.3).

Let f be as in 1.3 and let $x \in X$. Then, we can take a local chart $h\colon P \to Q$ around x and $f(x)$ which induces the bijection $P \xrightarrow{\cong} \overline{M}_{\overline{f(x)}}$ and which satisfies all the desired conditions except the $\mathbb{Q}$-integrality, and only satisfies the injectivity of h. Localizing X if necessary, we may assume that the chart induces the bijection $\overline{Q} \xrightarrow{\cong} \overline{M}_{\overline{x}}$. Under this assumption, we prove that h is $\mathbb{Q}$-integral, which completes the proof. For this, by [2] (A.3.2) (v) $\Rightarrow$ (ii) and [2] (A.3.2.1), it is enough to show that, for any point $\mathfrak{q}$ in the special fiber S of $\operatorname{Spec} h$, the induced homomorphism $P \to Q_{\mathfrak{q}}$ is exact. Here the special fiber S of $\operatorname{Spec} h\colon \operatorname{Spec} Q \to \operatorname{Spec} P$ means the inverse image of the closed point (= the singleton consisting of the maximal ideal) of $\operatorname{Spec} P$. Then, by the original assumption of the exactness of f, we see that it is enough to show that the image of X in $\operatorname{Spec} Q$ contains S. Since $i\colon X \to X_0 := Y \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q]$ is flat, $\operatorname{Spec}\mathcal{O}_{X,\overline{x}} \to \operatorname{Spec}\mathcal{O}_{X_0,\overline{i(x)}}$ is surjective. Hence the image of $\operatorname{Spec}\mathcal{O}_{X,\overline{x}}$ in $\operatorname{Spec} Q$ contains S. Thus the image of X in $\operatorname{Spec} Q$ contains S.

2. Quasi-sections of homomorphisms of monoids

Proposition 2.1. *Let $h\colon P \to Q$ be a local homomorphism of integral and sharp $\mathbb{Q}_{\geq 0}$-monoids that are finitely generated as $\mathbb{Q}_{\geq 0}$-monoids. Assume that h is integral. Then the following hold.*
(1) *There is a face G of Q such that $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$ and* $\dim G = \dim Q - \dim P$. *Here and hereafter we identify P with its image $h(P)$ in Q.* (*Note that h is injective.*)
(2) *For any face G of Q satisfying the conditions in* (1), *the induced homomorphism $P \to Q/G$ is an isomorphism.*

Proof. (1) By [2] (A.3.2), h is weak GD ([2] (A.3.1) (4)), i.e., for any $\mathfrak{q} \in \operatorname{Spec} Q$ which lies over the maximal ideal of $\operatorname{Spec} P$, the map $\operatorname{Spec}(Q_{\mathfrak{q}}) \to \operatorname{Spec} P$ is surjective. Then, by [2] (A.3.2.2) (2b), the monoid Q is the union of its submonoids GP, where G ranges over all faces of Q such that $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$. Further, by [2] (A.3.2.2) (2c), each GP as above is isomorphic to the product monoid $G \times P$, which has the dimension $\dim G + \dim P$. Hence there is a G such that $\dim Q = \dim G + \dim P$.

(2) Since $P \to Q/G$ is local and integral, it is exact. On the other hand, $P^{\mathrm{gp}} \to (Q/G)^{\mathrm{gp}} = Q^{\mathrm{gp}}/G^{\mathrm{gp}}$ is bijective because $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$ and $\dim P^{\mathrm{gp}} = \dim(Q^{\mathrm{gp}}/G^{\mathrm{gp}})$. Hence $P \to Q/G$ is an isomorphism. □

Recall that a homomorphism $P \to Q$ of fs monoids is said to be *kummer* if it is injective and for any element a of Q, its some power a^n $(n \geq 1)$ is the image of some element of P.

Proposition 2.2. *Let $h\colon P \to Q$ be a local and $\mathbb{Q}$-integral homomorphism of fs monoids. Assume that P is sharp. Then the following hold.*
(1) *There is a face G of Q such that $G \cap P = \{1\}$* (*or equivalently $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$*) *and* $\dim G = \dim Q - \dim P$. *Here and hereafter we identify P with its image $h(P)$ in Q.*
(2) *For any face G of Q satisfying the conditions in* (1), *the induced homomorphism $P \to Q/G$ is kummer.*

Proof. (1) By [2] (A.3.1.1), the local homomorphism $P_{\mathbb{Q}_{\geq 0}} \to \overline{Q}_{\mathbb{Q}_{\geq 0}}$ induced by h is integral, where $(\ )_{\mathbb{Q}_{\geq 0}}$ denotes $(\ ) \otimes_{\mathbb{N}} \mathbb{Q}_{\geq 0}$. Hence we can apply 2.1 to this homomorphism, and can find a face G_1 of $\overline{Q}_{\mathbb{Q}_{\geq 0}}$ satisfying $G_1^{\mathrm{gp}} \cap P_{\mathbb{Q}_{\geq 0}}^{\mathrm{gp}} = \{1\}$ and $\dim G_1 = \dim \overline{Q}_{\mathbb{Q}_{\geq 0}} - \dim P_{\mathbb{Q}_{\geq 0}}$. On the other hand, since there are the compatible natural bijections $\operatorname{Spec}\overline{Q}_{\mathbb{Q}_{\geq 0}} \xrightarrow{\sim} \operatorname{Spec}\overline{Q} \xrightarrow{\sim} \operatorname{Spec} Q$ and $\operatorname{Spec} P_{\mathbb{Q}_{\geq 0}} \xrightarrow{\sim} \operatorname{Spec} P$, there is a natural bijection between the set of all faces G_1 of $\overline{Q}_{\mathbb{Q}_{\geq 0}}$ satisfying $G_1 \cap P_{\mathbb{Q}_{\geq 0}} = \{1\}$ and

$\dim G_1 = \dim \overline{Q}_{\mathbb{Q}_{\geq 0}} - \dim P_{\mathbb{Q}_{\geq 0}}$ and the set of all faces G of Q satisfying $G \cap P = \{1\}$ and $\dim G = \dim Q - \dim P$. Hence there is a face G of Q as in (1). Note that $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$ because $P \to Q/G$ is exact by (1) $\Rightarrow$ (2) in 1.1.

(2) Let the notation be as in the proof of (1) above. In particular, G_1 denotes the face of $\overline{Q}_{\mathbb{Q}_{\geq 0}}$ that corresponds to G. Then, since $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$, we see that $P \to Q/G$ is injective. Since $P_{\mathbb{Q}_{\geq 0}} \to \overline{Q}_{\mathbb{Q}_{\geq 0}}/G_1^{\mathrm{gp}}$ is bijective by 2.1 (2), we conclude that $P \to Q/G$ is kummer. $\square$

Corollary 2.3. *Let $h\colon P \to Q$ be a local and $\mathbb{Q}$-integral homomorphism of sharp fs monoids. Then the following hold.*

(1) *The homomorphism $h \otimes \mathbb{Q}_{\geq 0}$ has a section.*

(2) *For some $n \geq 1$, the n-th power map $P \to P$ factors as $P \xrightarrow{h} Q \to P$.*

Proof. In 2.2 (2), the induced homomorphism $P_{\mathbb{Q}_{\geq 0}} \to (Q/G)_{\mathbb{Q}_{\geq 0}}$ is an isomorphism, which implies (1). (Or (1) is directly deduced from 2.1 (2).) Further, for (2), take an n such that for any $a \in Q/G$, the power a^n belongs to the image of P. Then, the image of Q by the homomorphism $Q \to (Q/G)_{\mathbb{Q}_{\geq 0}} \xleftarrow{\cong} P_{\mathbb{Q}_{\geq 0}}$ is contained in $P^{1/n} := \{a^{1/n} \in P_{\mathbb{Q}_{\geq 0}} \mid a \in P\}$, which implies that the n-th power map factors through Q as desired. $\square$

The following variants of the above results are not used until Section 5.

Proposition 2.4. *Let the notation and the assumption be as in* 2.2. *Let G' be the intersection of all faces G of Q satisfying the conditions in* 2.2 (1). *Then the following hold.*

(1) *The intersection of G' and the minimal face of Q that contains P is $Q^{\times}$.*

(2) *The homomorphism $h \otimes \mathbb{Q}_{\geq 0}$ has a section whose kernel is $G'_{\mathbb{Q}_{\geq 0}}$.*

(2)′ *For some $n \geq 1$, the n-th power map $P \to P$ factors as $P \xrightarrow{h} Q \xrightarrow{s} P$, where* $\mathrm{Ker}(s) = G'$.

Proof. (1) Let a be an element of G'. Assume that a is in the minimal face of Q that contains P. Then, for some $b \in Q$, the product ab is in P. It is enough to show that a is invertible in Q. By applying [2] (A.3.2.2) (2b) to the homomorphism $P_{\mathbb{Q}_{\geq 0}} \to \overline{Q}_{\mathbb{Q}_{\geq 0}}$, we see that the image $\overline{b} \in \overline{Q}_{\mathbb{Q}_{\geq 0}}$ of b belongs to $\overline{G}_{\mathbb{Q}_{\geq 0}} P_{\mathbb{Q}_{\geq 0}}$ for some face G of Q satisfying the conditions in 2.2 (1). Hence, for some $n \geq 1$, the power b^n is in GP. Together with $ab \in P$, we see that $a^n g \in P^{\mathrm{gp}}$ for some $g \in G$. But, since $a \in G' \subset G$ and $G^{\mathrm{gp}} \cap P^{\mathrm{gp}} = \{1\}$, the last implies $a^n g = 1$, that is, a is invertible.

(2) By 2.2 (2), for each face G of Q satisfying the conditions in 2.2 (1), there is a section $s_G\colon Q_{\mathbb{Q}_{\geq 0}} \to P_{\mathbb{Q}_{\geq 0}}$ to $h_{\mathbb{Q}_{\geq 0}}$ whose kernel $\{a \in Q_{\mathbb{Q}_{\geq 0}} \mid s_G(a) = 1\}$ coincides with $G_{\mathbb{Q}_{\geq 0}}$. Let $s' = \prod_G s_G^{1/N}\colon Q_{\mathbb{Q}_{\geq 0}} \to P_{\mathbb{Q}_{\geq 0}}$, where N is the number of all such G. Then, s' is another section to $h_{\mathbb{Q}_{\geq 0}}$ and its kernel is the intersection of all $G_{\mathbb{Q}_{\geq 0}}$, that is, $G'_{\mathbb{Q}_{\geq 0}}$.

(2)$'$ In (2), take an n such that for any $a \in Q$, the power $s'(a)^n$ belongs to P. Then, the image of Q by the homomorphism s' is contained in $P^{1/n}$, and the kernel of the induced homomorphism $Q \to P^{1/n}$ coincides with G'. Thus we have (2)$'$. □

3. Exact log flat descents

Proposition 3.1. *Let $f\colon X \to Y$ be an exact and log flat morphism of fs log schemes. Let y be a point in the image $f(X)$ of X. Then there exists an open set U of X such that the composite $U \to X \to Y$ is kummer and its image contains y.*

Proof. Let x be a point of X such that $f(x) = y$. By 1.3, we may assume that there exists a chart $\Gamma(Y, \mathcal{O}_Y) \leftarrow P \xrightarrow{h} Q \to \Gamma(X, \mathcal{O}_X)$ by fs monoids such that P is sharp, h is local and $\mathbb{Q}$-integral, the induced morphism $X \xrightarrow{i} X_0 := Y \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[Q]$ is strict and flat, $\overline{M}_{Y,\overline{y}} = P$ and $\overline{M}_{X_0,\overline{i(x)}} = \overline{Q}$. Apply 2.2 to h. Then there is a face G of Q such that the induced homomorphism $P \to Q/G$ is kummer. Let $U_0 := Y \times_{\operatorname{Spec}\mathbb{Z}[P]} \operatorname{Spec}\mathbb{Z}[G^{-1}Q] \subset X_0$ and $U := U_0 \times_{X_0} X \subset X$. Since $U \to Y$ is kummer, it is enough to show that the image of U contains y. For this, we may assume that Y is $\operatorname{Spec} k[P]$ for some field k, and y is the origin of $\operatorname{Spec} k[P]$. Then U_0 is $\operatorname{Spec} k[G^{-1}Q] \subset \operatorname{Spec} k[Q] = X_0$. On the other hand, since $\overline{M}_{X_0,\overline{i(x)}} = \overline{Q}$, the point $i(x)$ is in the locus $\operatorname{Spec} k[Q]/(Q - Q^{\times})$ of X_0. Hence $i(X)$ contains a point $i(u)$ $(u \in X)$ of the locus $U_0' := \operatorname{Spec} k[G^{-1}Q]/(G^{-1}Q - G^{\times})$, because i is flat so that $i(X)$ is stable under generization. Since this point $i(u)$ belongs to U_0, the point u belongs to U. Since the locus U_0' is lying over y, we have $f(u) = y$. Hence we conclude that the image of U contains y. □

Corollary 3.2. *Let $f\colon X \to Y$ be an exact, log flat and surjective morphism of fs log schemes. Then the following hold.*
(1) *There exists an open set U of X such that the composite $U \to X \to Y$ is kummer and surjective.*
(2) *Further, if the cokernel of $(f^*M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ is torsion free, then for any U in* (1), *the composite $U \to X \to Y$ is strict.*

Proof. (1) is by 3.1. (2) is by the fact that any kummer morphism $f\colon X \to Y$ such that the cokernel of $(f^*M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ is torsion free is strict. □

The above results enable us to generalize some propositions under the kummerness assumption to those under the exactness assumption. For example, we apply the above results to descents whose kummer cases are in [6] Section 7.

Theorem 3.3. *Let $f\colon X \to Y$ be a morphism of fs log schemes, and let $g\colon Y' \to Y$ be an exact, log flat and surjective morphism locally of finite presentation of fs log schemes. Let $X' = X \times_Y Y'$ and let $f'\colon X' \to Y'$ be a morphism induced by f.*

Then f is log étale (resp. log smooth, resp. log flat, resp. kummer) if and only if so is f'.

Proof. It is enough to show the if part. For this, by 3.2 (1), we may assume that g is kummer. That is, the problem is reduced to the kummer case [6] Theorem 7.1. (It is known that the Noetherian assumption and the local finite presentation assumption put in [6] 7.1 (2) are not necessary.) □

The next gives an affirmative answer to a question of L. Illusie.

Theorem 3.4. *Let $X' \xrightarrow{g} X \xrightarrow{f} Y$ be morphisms of fs log schemes, and assume that g is surjective and exact.*

If g and $f \circ g$ is log étale (resp. log smooth, resp. log flat), then f is log étale (resp. log smooth, resp. log flat).

Proof. The case where g is kummer is [6] Theorem 7.2. (It is known that the Noetherian assumption and the local finite presentation assumption put in [6] 7.2 (2) are not necessary.) The log étale case of the theorem is included in it because a morphism of fs log schemes is kummer and log étale if and only if it is exact and log étale. The other cases are reduced to [6] 7.2 by applying 3.2 (1) to g. □

4. Quasi-sections of log flat or log smooth morphisms

The following are generalizations of some results in [1] 17.16.

Proposition 4.1. *Let $f : X \to Y$ be a log flat, exact and surjective morphism locally of finite presentation of fs log schemes. Then the following hold.*

(1) *There exists a log flat, kummer, surjective and locally quasi-finite morphism locally of finite presentation $Y' \to Y$ such that there exists a Y-morphism $Y' \to X$.*

(2) *Further, if the cokernel of $(f^*M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ is torsion free, then $Y' \to Y$ in* (1) *can be taken to be a strict flat, surjective and locally quasi-finite morphism locally of finite presentation.*

Proof. To show (1) (resp. (2)), by 3.2 (1) (resp. (2)), we may assume that f is kummer (resp. strict). Then, we see that (2) is nothing but [1] 17.16.2 by forgetting log structures. Further, (1) is also reduced to [1] 17.16.2 by taking a kummer chart étale locally on Y because, for a kummer homomorphism h of fs monoids, the morphism $\operatorname{Spec} \mathbb{Z}[h]$ is log flat, kummer, surjective, finite, and is of finite presentation. □

Proposition 4.2. *Let $f : X \to Y$ be a log smooth, exact and surjective morphism of fs log schemes. Assume that the cokernel of $(f^*M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ is torsion free. Then there exists a strict étale and surjective morphism $Y' \to Y$ such that there exists a Y-morphism $Y' \to X$.*

Proof. Similarly as 4.1 (2), we may assume that f is strict by 3.2 (2), and use [1] 17.16.3. □

Note that an analogue of 4.1 (1) in the log smooth case fails, that is, even if f is log smooth, exact and surjective in 4.1, there may not be a section ket (= kummer log étale) locally. A counter example is as follows. Let Y be a standard log point over the spectrum of a field k of characteristic $p > 0$. Assume that we are given a chart $\mathbb{N} \to M_Y$. Let $P = \mathbb{N}^2 = \langle x, y \rangle$, where x, y is the canonical basis. Let $h\colon \mathbb{N} \to P$ be the homomorphism sending 1 to $x^p y \in P$. Let $X_0 = Y \times_{\operatorname{Spec} \mathbb{Z}[\mathbb{N}]} \operatorname{Spec} \mathbb{Z}[P]$, which is log smooth and exact over Y. Let X be the open fs log subscheme of X_0 defined by $y \neq 0$. Then $f\colon X \to Y$ has no section even after ket localization of the base because for any $x \in X$, the order of the cokernel of the homomorphism $\mathbb{Z} \cong \overline{M}^{\mathrm{gp}}_{Y,\overline{f(x)}} \to \overline{M}^{\mathrm{gp}}_{X,\bar{x}} \cong \mathbb{Z}$ is p.

But, if we work over the base of characteristic zero, such an analogue exists. For example, we have the following.

Proposition 4.3. *Let $f\colon X \to Y$ be a log smooth, exact and surjective morphism of fs log analytic spaces. Then the following hold.*
(1) *There exists a kummer log étale and surjective morphism $Y' \to Y$ such that there exists a Y-morphism $Y' \to X$.*
(2) *Further, if the cokernel of $(f^*M_Y)^{\mathrm{gp}} \to M_X^{\mathrm{gp}}$ is torsion free, then f has a section locally on Y.*

Proof. The proof is similar to that for 4.1. To show (1) (resp. (2)), by an analytic version of 3.2 for log smooth morphisms, we may assume that f is kummer (resp. strict). Then, (2) is a non-log statement. Further, by [2] (5.3), (1) is also reduced to the strict case. □

As an application, we have

Corollary 4.4. *Let $f\colon P \to S$ be a projective, log smooth, vertical and exact morphism with connected fibers of log smooth fs log analytic spaces. Assume that the cokernel of $(f^*M_S)^{\mathrm{gp}} \to M_P^{\mathrm{gp}}$ is torsion free. Then locally on S, there is a log Albanese map for f.*

Proof. By [5] 10.5, there is a log Albanese map for such an f whenever f has a section. By 4.3 (2), f has a section locally on S. Hence there is a log Albanese map locally on S. □

5. Vertical case

In the non-log situation, there can often exist a quasi-section whose image contains a prescribed point (cf. [1] 17.16.1, 17.16.3). In this section, we prove some variants of this type. For this, we impose the verticality assumption (cf. 5.2). This is because, without the verticality, we cannot expect that this type of results hold in general. For example, let us consider the non-vertical morphism $\operatorname{Spec}\mathbb{C}[\mathbb{N}] \to \operatorname{Spec}\mathbb{C}$, where the log structure on the base is trivial. Then, clearly, there is no section whose image is the origin of $\operatorname{Spec}\mathbb{C}[\mathbb{N}]$.

Recall that a homomorphism $P \to Q$ of fs monoids is said to be *dominating* if any element of Q divides the image of some element of P.

Proposition 5.1. *Let $h\colon P \to Q$ be a local, dominating and $\mathbb{Q}$-integral homomorphism of fs monoids. Assume that P is sharp. Then there is a local homomorphism $s\colon Q \to P$ such that the composite $P \xrightarrow{h} Q \xrightarrow{s} P$ is the n-th power map $P \to P$ for some $n \geq 1$.*

Proof. Apply 2.4. Since h is dominating, the minimal face of Q generated by P is Q itself. Hence G' in 2.4 is nothing but $Q^{\times}$ by 2.4 (1), and s in 2.4 (2)$'$ is local. $\square$

Recall that a morphism $f\colon X \to Y$ of fs log schemes is said to be *vertical* at $x \in X$ if the induced homomorphism $M_{Y,\overline{f(x)}} \to M_{X,\bar{x}}$ is dominating. We say that f is *vertical* if f is vertical at any $x \in X$.

The following three propositions are proved in 5.5.

Proposition 5.2. *Let $f\colon X \to Y$ be a log flat and exact morphism of fs log schemes. Let x be a point of X. Then the following hold.*

(1) *Assume that f is vertical at x. Then, there exist morphisms $U \xrightarrow{a} V \xrightarrow{b} X$ of fs log schemes, where a is log étale and b is strict, flat and locally of finite presentation, such that the image of U in X contains x and such that the composite $U \to X \to Y$ is kummer.*

(2) *Assume that f is vertical. Then, there exists a log flat and surjective morphism $X' \to X$ locally of finite presentation of fs log schemes such that the composite $X' \to X \to Y$ is kummer.*

Proposition 5.3. *Let $f\colon X \to Y$ be a log smooth and exact morphism of fs log schemes. Let x be a point of X. Then the following hold.*

(1) *Assume that f is vertical at x. Then, there exists a log étale morphism $U \to X$ of fs log schemes whose image contains x such that the composite $U \to X \to Y$ is kummer.*

(2) *Assume that f is vertical. Then, there exists a log étale and surjective morphism $X' \to X$ of fs log schemes such that the composite $X' \to X \to Y$ is kummer.*

EXAMPLE. Let h be the diagonal map $\mathbb{N} \to \mathbb{N}^2 = \langle e_1, e_2\rangle$; $a \mapsto (a, a)$. Let f be the morphism $\operatorname{Spec}\mathbb{C}[h]$. Let x be the origin of $\operatorname{Spec}\mathbb{C}[\mathbb{N}^2]$. Then we can take U in

5.3 (1) as the log product $\operatorname{Spec}\mathbb{C}[\mathbb{N}] \times^{\log}_{\operatorname{Spec}\mathbb{C},[\mathbb{N}]} \operatorname{Spec}\mathbb{C}[\mathbb{N}] = \operatorname{Spec}\mathbb{C}[e_1, e_2, (e_1/e_2)^{\pm 1}]$ in the sense of [7].

Proposition 5.4. *Let $f\colon X \to Y$ be a log smooth and exact morphism of fs log analytic spaces. Let x be a point of X. Assume that f is vertical at x. Then there is a kummer log étale morphism $Y' \to Y$ such that there exists a Y-morphism $Y' \to X$ whose image contains x.*

5.5. We prove 5.2–5.4. First we prove 5.2 (resp. 5.3) (1). This implies (2). By 1.3 (resp. 1.4), we may assume that there is a local and $\mathbb{Q}$-integral chart $P \to Q$ with P sharp which induces a bijection $\overline{Q} \xrightarrow{\cong} \overline{M}_{\bar{x}}$. By the verticality, $P \to Q$ is dominating. Under this assumption, we prove the existence of a log étale $U \to X$ satisfying the conditions, that is, we can take $V = X$ in 5.2. Apply 5.1, and let Q' be the inverse image of P by $s^{\mathrm{gp}}\colon Q^{\mathrm{gp}} \to P^{\mathrm{gp}}$ (notation is the same as there). Let $U = X \otimes_{\operatorname{Spec}\mathbb{Z}[Q]} \operatorname{Spec}\mathbb{Z}[Q']$, and we show that this U satisfies the desired conditions. First, $U \to X$ is clearly log étale. Next we show that $U \to X \to Y$ is kummer. For this, since the face $\operatorname{Ker}(s^{\mathrm{gp}})$ of Q' is the minimal face, the induced homomorphism $\overline{Q'} \to P$ is injective. Hence $P \to \overline{Q'}$ is kummer and $U \to X \to Y$ is kummer. The rest is to show that the image of U contains x. To see this, we may assume that $X = \operatorname{Spec}\mathbb{Z}[Q]$ so that $U = \operatorname{Spec}\mathbb{Z}[Q']$. It is enough to show that the image of U contains the locus $\operatorname{Spec}\mathbb{Z}[Q]/(Q - Q^{\times}) \subset X$. But, since s is local, $Q \hookrightarrow Q'$ is also local, and the locus $\operatorname{Spec}\mathbb{Z}[Q']/(Q' - (Q')^{\times}) \subset U$ is mapped onto $\operatorname{Spec}\mathbb{Z}[Q]/(Q - Q^{\times})$.

We prove 5.4. By the analytic version of 5.3 (1), we may assume that f is kummer. By [2] (5.3), the problem is reduced to the case where f is strict, which is essentially a non-log statement.

As an application, for example, we have

Proposition 5.6. *A log flat, vertical and exact morphism locally of finite presentation of fs log schemes is an open map.*

Proof. By 5.2 (2), we may assume that the morphism is kummer. Then, it is an open map by [6] Proposition 2.5. □

REMARK 5.7. In fact, in 5.6, the verticality is not necessary. A proof for it goes as follows (we do not use 5.2). By 1.3, we may assume that there is a $\mathbb{Q}$-integral chart. Then, by [2] (A.3.4), the morphism is integral after the base change by a kummer flat covering. Hence, by [6] 2.5, we may further assume that the morphism is integral. Since the underlying morphism of schemes of a log flat and integral morphism is flat (cf. [3] (4.5)), the problem is reduced to the non-log case.

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