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INFINITE DIVISIBILITY OF RANDOM MEASURES ASSOCIATED TO SOME RANDOM SCHRÖDINGER OPERATORS

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Abstract

We study a random measure which describes distribution of eigenvalues and corresponding eigenfunctions of random Schrödinger operators on $L^2(\mathbf{R}^d)$. We show that in the natural scaling every limiting point is infinitely divisible.

1. Introduction

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. We consider a family $\{H_\omega\}_{\omega\in\Omega}$ of Schrödinger operators parametrized by $\omega \in \Omega$:

$$H_\omega := -\triangle + V_\omega(x), \quad \text{on} \quad L^2(\mathbf{R}^d),$$
$$V_\omega(x) := \sum_{k\in\mathbf{Z}^d} \lambda_k(\omega) U(x-k).$$

We assume that V_ω satisfies the following condition.

H1. (1) U ($\neq 0$) is bounded, measurable with $|\partial(\operatorname{supp} U)| = 0$ and satisfies the following "overlapping condition": for some positive constants c, C, r_0 ($r_0 > 1/2$), we have

$$c\chi_{B(0,1/2)}(x) \le U(x) \le C\chi_{B(0,r_0)}(x)$$

where

$$B(a, r) := \{y = (y_1, y_2, \dots, y_d) \in \mathbf{R}^d : d(y, a) < r\},$$
$$d(y, a) := \max_{j=1,2,\dots,d} |y_j - a_j|,$$

is the cube of size $2r$ centered at $a \in \mathbf{R}^d$ and χ_B is the characteristic function of B. For a subset A of $\mathbf{R}^n$, $|A|$ is its n-dimensional Lebesgue measure.
(2) $\{\lambda_k(\omega)\}_{k\in\mathbf{Z}^d}$ are independent, identically distributed real-valued random variables whose common distribution has a bounded density $\rho \in L^\infty$ with $\operatorname{supp}\rho \subset [0, \infty)$ being compact and $0 \in \operatorname{supp}\rho$.

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H1 is assumed so that there exist some intervals where the fractional moment bound (1.1) and Wegner's estimate (Lemma 3.2) are satisfied. It is known that $\sigma(H_\omega) = [0, \infty)$, a.s. [7] and we can find $E_1 > 0$ such that the spectrum of H_ω in $I = [0, E_1]$ are a.s. pure point with exponentially decaying eigenfunctions. This phenomenon is called Anderson localization. See e.g., [1, 2, 3, 15] and references therein. One method for proving this is fractional moment method [1] and another one is multiscale analysis [2, 15]. The purpose of this paper is to describe the distribution of eigenvalues and eigenfunctions in I in the product space of energy and space, and study its properties. In order to do that, we consider the following as is done in [6].

DEFINITION. We define a measure ξ on $\mathbf{R}^{d+1}$ by setting

$$\xi(J \times B) := \mathrm{Tr}(\chi_B(x) P_J(H) \chi_B(x))$$

for $J \in \mathcal{B}(\mathbf{R})$, $B \in \mathcal{B}(\mathbf{R}^d)$, where $P_J(H)$ is the spectral projection of H w.r.t. J.

Since $\chi_B(x) P_J(H)$ is Hilbert-Schmidt for bounded J, B (Lemma 3.1), ξ is locally finite. We set some definitions and notations to state our results.

NOTATION. (1) Let $\mathcal{M}(\mathbf{R}^n)$ (resp. $\mathcal{M}_p(\mathbf{R}^n)$) be the set of locally finite Borel measures (resp. point measures) on $\mathbf{R}^n$ with $\mathcal{B}(\mathcal{M}(\mathbf{R}^n))$ its Borel field generated by the vague topology. A random measure (resp. point process) on $\mathbf{R}^n$ is a measurable mapping from $(\Omega, \mathcal{F}, \mathbf{P})$ to $(\mathcal{M}(\mathbf{R}^n), \mathcal{B}(\mathcal{M}(\mathbf{R}^n)))$ (resp. to $(\mathcal{M}_p(\mathbf{R}^n), \mathcal{B}(\mathcal{M}_p(\mathbf{R}^n)))$). For a random measure ζ, $\mathbf{E}[\zeta(dx)]$ is called its intensity measure. Since $f(H)$ is weakly measurable for bounded Borel function f on $\mathbf{R}$ [3], and since $\mathcal{B}(\mathcal{M}(\mathbf{R}^n))$ is generated by mappings $\{\mu \mapsto \mu(A)\}$ for bounded Borel sets $A \in \mathcal{B}(\mathbf{R}^n)$, ξ a random measure on $\mathbf{R}^{d+1}$.
(2) A point process ζ is called an infinitely divisible point process iff for any $n \in \mathbf{N}$ there exists independent identically distributed sequence of point processes $\{\xi_{n,j}\}_{j=1}^n$ such that $\zeta \stackrel{d}{=} \xi_{n,1} + \xi_{n,2} + \cdots + \xi_{n,n}$.
(3) A sequence $\{\xi_n\}_{n=1}^\infty$ of random measures is said to converge in distribution to a random measure ζ (and we write $\xi_n \stackrel{d}{\to} \zeta$) if the distribution of ξ_n converges weakly to that of ζ. It is equivalent to the following statement: for any $k \in \mathbf{N}$, any interval $J_1 \times B_1, J_2 \times B_2, \ldots, J_k \times B_k$ and any $A_1, \ldots, A_k \in \mathcal{B}(\mathbf{R})$ such that $\mathbf{P}(\zeta(J_j \times B_j) \in \partial A_j) = 0$ for $j = 1, 2, \ldots, k$,

$$\mathbf{P}(\xi_n(J_j \times B_j) \in A_j,\ j = 1, 2, \ldots, k) \xrightarrow{n\to\infty} \mathbf{P}(\zeta(J_j \times B_j) \in A_j,\ j = 1, 2, \ldots, k).$$

(4) Let $H_{\Lambda_L} := H|_{\Lambda_L}$ ($\Lambda_L := [-L/2, L/2]^d$) with the periodic boundary condition[1]. It is known that, with probability 1, the following limit finitely exists for any $E \in \mathbf{R}$ and is independent of $\omega \in \Omega$

$$N(E) := \lim_{L \to \infty} \frac{1}{|\Lambda_L|} \sharp\{\text{eigenvalues of } H_{\Lambda_L} \le E\},$$

which is called the integrated density of states, and the corresponding Borel measure ν on $\mathbf{R}$ is called the density of states measure.

As is done in [6], we study the following two scaling limits.
(1) Macroscopic limit: we first consider the following scaling ξ_L^M of ξ:

$$\int_{\mathbf{R}^{d+1}} f(E, x)\, d\xi_L^M := L^{-d} \int_{\mathbf{R}^{d+1}} f\left(E, \frac{x}{L}\right) d\xi,$$

in another words,

$$\xi_L^M(J \times B) = L^{-d}\, \mathrm{Tr}(\chi_{LB} P_J(H) \chi_{LB}), \quad J \in \mathcal{B}(\mathbf{R}),\ B \in \mathcal{B}(\mathbf{R}^d).$$

Theorem 1.1. *Under* H1, *we have* $\xi_L^M \stackrel{v}{\to} \nu \otimes dx$ *as* $L \to \infty$ *almost surely.*

$\stackrel{v}{\to}$ means vague convergence. Since ν is interpreted as the number of states per unit volume and per unit energy, this result is natural implying that eigenfunctions are distributed uniformly in the macroscopic scale. In fact, Theorem 1.1 follows quickly from the ergodic theorem.
(2) Natural scaling limit: Pick a reference energy $E_0 \in \mathbf{R}$ and consider the following scaling ξ_L of ξ:

$$\int_{\mathbf{R}^{d+1}} f(E, x)\, d\xi_L := \int_{\mathbf{R}^{d+1}} f\left(L^d(E - E_0), \frac{x}{L}\right) d\xi,$$

equivalently,

$$\xi_L(J \times B) = \mathrm{Tr}(\chi_{LB}(x) P_{E_0 + J/L^d}(H) \chi_{LB}(x)), \quad J \in \mathcal{B}(\mathbf{R}),\ B \in \mathcal{B}(\mathbf{R}^d).$$

We note that if an eigenfunction ϕ of H localizes in a box of size L, then the corresponding energy E satisfies $|E - E_0| \simeq L^{-d}$ [12].

We wish to study the behavior of ξ_L when E_0 is in the localized regime (the region where Anderson localization holds) of H. In order to do that, we assume the following fractional moment estimate.

[1] We always impose periodic boundary condition for the restriction $H|_\Lambda$ of H.

H2 (Fractional moment estimate). Let $H_{\Lambda_L} := H|_{\Lambda_L}$, $\Lambda_L := [-L/2, L/2]^d$ with the periodic boundary condition. We can find positive constants s, C_s, μ $(0 < s < 1)$ and an open interval $I \subset [0, \infty)$ such that for any $E \in I$, $L > 0$, $k, l \in \Lambda_L \cap \mathbf{Z}^d$ we have

$$\sup_{\epsilon > 0} \mathbf{E}[\| \chi_k G_{\Lambda_L}(E + i\epsilon)\chi_l \|_{\mathrm{op}}^s] \le C_s e^{-\mu|k-l|} \tag{1.1}$$

where $\chi_k := \chi_{B(k,1/2)}$, $G_{\Lambda_L}(z) := (H_{\Lambda_L} - z)^{-1}$, $z \in \mathbf{C} \setminus \mathbf{R}$ is the resolvent of H_{Λ_L} and $\| \cdot \|_{\mathrm{op}}$ is the operator norm.

It is known that H2 is satisfied if I is in a neighborhood of $0 = \inf \sigma(H)$ or if $I \subset (0, \infty)$ is an arbitrary bounded interval and $\|\rho\|_\infty$ is sufficiently small under which Anderson localization is proved [1].

Theorem 1.2. *Assume* H1, H2. *If $E_0 \in I$ is the Lebesgue point of ν, we can find a sequence $\{L_k\}_{k=1}^\infty$ with $L_k \xrightarrow{k\to\infty} \infty$ and an infinitely divisible point process ζ on $\mathbf{R}^{d+1}$ such that $\xi_{L_k} \xrightarrow{d} \zeta$ as $k \to \infty$. Furthermore*

$$\mathbf{E}[\zeta(dE\, dx)] \le \frac{d\nu}{dE}(E_0)\, dE \otimes dx. \tag{1.2}$$

As for the related works, Molchanov [10] studied one-dimensional Schrödinger operator H called the Russian school model. Let $H_L := H|_{\Lambda_L}$, $\Lambda_L = [-L, L]$ under the Dirichlet boundary condition and let $\{E_j(\Lambda_L)\}_j$ be its eigenvalues. He considered the point process

$$\mu_L(dE) = \sum_j \delta_{|\Lambda_L|(E_j(\Lambda_L)-E_0)}(dE) \tag{1.3}$$

on $\mathbf{R}$ and proved that it converges in distribution to a Poisson process. Minami [9] proved the same statement for multi-dimensional Anderson model on $l^2(\mathbf{Z}^d)$. In [6] the same model as [9] is studied and it is shown that ξ_L converges to a Poisson process on $\mathbf{R}^{d+1}$. In view of those known results and their proofs, the conclusion of Theorem 1.2 is not surprising, but this paper aims at clarifying which conditions are sufficient to prove this in the continuum case. We also note that, in physics literature, there is a discussion on examples where ζ is infinitely divisible but not Poissonian [8].

REMARK 1.3. The uniqueness of ζ is not known. If we had Minami's estimate

$$\mathbf{P}(\sharp\{\text{eigenvalues of } H_{\Lambda_L} \text{ in } J\} \ge 2) \le C|J|^2 \cdot (L^d)^2 \tag{1.4}$$

then we would be able to prove that ξ_L converges to the Poisson process on $\mathbf{R}^{d+1}$ whose intensity measure is equal to $(d\nu/dE)(E_0)\, dE \otimes dx$. However, (1.4) has not been proved yet. What we obtain from the infinite divisibility of ζ is that ζ has the following representation:

$$\zeta \stackrel{d}{=} \int_{\mathcal{M}(\mathbf{R}^{d+1})} \mu \, d\eta(\mu)$$

where η is a Poisson process on $\mathcal{M}(\mathbf{R}^{d+1})$ [5, Lemma 6.5].

REMARK 1.4. Let $U := [0, 1]^d$ and define a random measure $\xi_{L,f}$ on $\mathbf{R} \times U$ by setting

$$\xi_{L,f}(J \times B) := \mathrm{Tr}(\chi_{LB}(x) P_{E_0+J/L^d}(H|_{LU}) \chi_{LB}(x))$$

for $J \subset \mathcal{B}(\mathbf{R})$, $B \subset U$. Then we can prove the same results for $\xi_{L,f}$ where ζ is now a point process on $\mathbf{R} \times U$. Furthermore, the point process μ_L for $H|_{\Lambda_L}$ converges to an infinitely divisible point process on $\mathbf{R}$ along some subsequence. For one-dimensional case, this is proved in [4].

REMARK 1.5. We used fractional moment bound to prove Theorem 1.2. We can also use the multi-scale analysis, which is presented in Appendix 2, so that the same conclusion also holds whenever the multiscale analysis is applicable.

REMARK 1.6. We can also study the distribution of localization centers (which is done in [13] for discrete case) and can derive essentially the same results as Theorem 1.2.

This paper is organized as follows. In Section 2, we prove Theorem 1.1, 1.2. Basically we follow the argument in [9, 6]: we divide the region in concern into small subsystems and approximate as $H \simeq \bigoplus_k H_k$. Since $\Im(H_{\Lambda_L} - z)^{-1}$ does not belong to trace class in the continuum models, we take smooth functions $f \in C_c^\infty(\mathbf{R})$ instead and estimate $\mathrm{Tr}(f(H) - \sum_k f(H_k))$ by using the almost analytic extension of f. Technically, the proof consists of combination of several known methods. In Section 3, we recall some basic estimates needed in Section 2. In Section 4, we prove Theorem 2.1 by using the multiscale analysis. In what follows, unimportant universal constants are written simply as (const.).

2. Proof of Theorems

Proof of Theorem 1.1. We recall that $B(a, r)$ is the cube of size $2r$ centered at $a \in \mathbf{R}^d$ and $\chi_k := \chi_{B(k,1/2)}$, $k \in \mathbf{Z}^d$. It is known that ν has the following representation [3].

$$\nu(J) = \mathbf{E}[\mathrm{Tr}(\chi_0 P_J(H) \chi_0)], \quad J \in \mathcal{B}(\mathbf{R}). \tag{2.1}$$

Since $\nu \otimes dx$ is absolutely continuous w.r.t. the Lebesgue measure on $\mathbf{R}^{d+1}$, by the density argument it suffices to show $\xi_L^M(A) \to \nu(J)|B|$, a.s. for any interval $A = J \times B$ with rational endpoints. Let A be such an interval and suppose that one of its endpoint coincides with the origin. By Birkhoff's ergodic theorem,

$$\lim_{L\to\infty} \xi_L^M(A) = \lim_{L\to\infty} L^{-d} \sum_{B(k,1/2)\cap(LB)\neq\emptyset} \mathrm{Tr}(\chi_k P_J(H)\chi_k) = \nu(J)|B|, \quad \text{a.s.}$$

A subtraction argument completes the proof. □

Proof of Theorem 1.2 is done based on the argument in [6]: we first consider the eigenfunctions of H which are localized in LB, decompose LB into small sets like $LB = \bigcup_p B_p(L)$, and approximate these eigenvalues and eigenfunctions of H by those of $H|_{B_p(L)}$. For that purpose, pick $0 < \alpha < 1$ and let $l_L := [L^\alpha]$. For $p = (p_1, p_2, \ldots, p_d) \in \mathbf{Z}^d$ we set

$$\begin{aligned} B_p(L) &:= \{x \in \mathbf{R}^d : p_j l_L \le x_j < (p_j+1)l_L,\ j = 1, 2, \ldots, d\}, \\ H_{L,p} &:= H|_{B_p(L)}, \quad \text{with periodic boundary condition.} \end{aligned}$$

To approximate ξ_L we consider the following random measure

$$\eta_{L,p}(J \times B) := \mathrm{Tr}(\chi_{LB}(x) P_{E_0+J/L^d}(H_{L,p})\chi_{LB}(x)).$$

Since periodic boundary condition is imposed, $\{\eta_{L,p}\}$ are statistically independent though V satisfies the overlapping condition (H1 (1)). Wegner's estimate (Lemma 3.2) implies that intensity measures of $\xi_L, \eta_{L,p}$ are absolutely continuous (Lemma 3.4). The following proposition is the key to the proof.

Proposition 2.1. *For any $f \in C_c(\mathbf{R}^{d+1})$, we have*

$$\mathbf{E}\left[\left|\xi_L(f) - \sum_{p\in\mathbf{Z}^d} \eta_{L,p}(f)\right|\right] = o(1), \quad L \to \infty.$$

Proof. By Lemma 3.4 it suffices to show Proposition 2.1 for $f(E,x) = \chi_B(x)g(E)$ with B ($\subset \mathbf{R}^d$) bounded rectangle and $g \in C_c^2(\mathbf{R})$. Let $h_L(\lambda) := g(L^d(\lambda - E_0))$. Then

$$\begin{aligned} &\xi_L(f) - \sum_p \eta_{L,p}(f) \\ &= \sum_p \mathrm{Tr}(\chi_{B_p(L)}\chi_{LB}(h_L(H) - h_L(H_{L,p}))\chi_{LB}\chi_{B_p(L)}) \end{aligned}$$

$$
\begin{aligned}
&= \left(\sum_{p:\ B_p(L)\subset(LB)} + \sum_{p:\ B_p(L)\cap(LB)\neq\emptyset,\, B_p(L)\cap(LB)^c\neq\emptyset} \right) \\
&\quad \times \operatorname{Tr}(\chi_{B_p(L)}\chi_{LB}(h_L(H)-h_L(H_{L,p}))\chi_{LB}\chi_{B_p(L)}) \\
&=: K_1 + K_2.
\end{aligned}
$$

Here we regard $h_L(H_{L,p})$ as an operator on $L^2(\mathbf{R}^d)$: to be precise $h_L(H_{L,p})$ should be replaced by $Ph_L(H_{L,p})P$ where P is the orthogonal projection onto $L^2(B_p(L))$. We first show $\mathbf{E}[|K_2|] = o(1)$. Let J ($\subset \mathbf{R}$) be an interval containing the support of g. Then by the inequality $|h_L(\lambda)| \le \|g\|_\infty \chi_{E_0+J/L^d}(\lambda)$ and by Lemma 3.2 we have

$$
\mathbf{E}[|\operatorname{Tr}(\chi_k\chi_{LB}h_L(H)\chi_{LB}\chi_k)|] \le (\text{const.})\frac{|J|}{L^d}\|g\|_\infty, \tag{2.2}
$$

$$
\mathbf{E}[|\operatorname{Tr}(\chi_k\chi_{LB}h_L(H_{L,p})\chi_{LB}\chi_k)|] \le (\text{const.})\frac{|J|}{L^d}\|g\|_\infty \tag{2.3}
$$

for any $k \in \mathbf{Z}^d$. Since $\sharp\{p\colon B_p(L)\cap(LB)\neq\emptyset,\ B_p(L)\cap(LB)^c\neq\emptyset\} = \mathcal{O}((L/l_L)^{d-1})$ and $\sharp(B_p(L)\cap\mathbf{Z}^d) = \mathcal{O}(l_L^d)$ we have

$$
\begin{aligned}
\mathbf{E}[|K_2|] &\le \sum_{p:\ B_p(L)\cap(LB)\neq\emptyset,\ B_p(L)\cap(LB)^c\neq\emptyset}\ \sum_{k\in B_p(L)\cap\mathbf{Z}^d} (\mathbf{E}[|\operatorname{Tr}(\chi_k\chi_{LB}h_L(H_L)\chi_{LB}\chi_k)|] \\
&\qquad\qquad + \mathbf{E}[|\operatorname{Tr}(\chi_k\chi_{LB}h_L(H_{L,p})\chi_{LB}\chi_k)|]) \\
&\le (\text{const.})\left(\frac{L}{l_L}\right)^{d-1} l_L^d\cdot\frac{1}{L^d} = (\text{const.})\left(\frac{l_L}{L}\right) = o(1).
\end{aligned}
$$

We next show $\mathbf{E}[|K_1|] = o(1)$. In what follows, for simplicity, we write $\sum_{k\in B_p(L)}$ instead of $\sum_{k\in B_p(L)\cap\mathbf{Z}^d}$. If $B_p(L)\subset(LB)$, then $\chi_{B_p(L)}\chi_{LB} = \chi_{B_p(L)}$ and hence

$$
\begin{aligned}
|K_1| &= \left| \sum_{p:\ B_p(L)\subset(LB)} \operatorname{Tr}(\chi_{B_p(L)}(h_L(H)-h_L(H_{L,p}))\chi_{B_p(L)}) \right| \\
&\le \sum_{p:\ B_p(L)\subset(LB)}\ \sum_{k\in B_p(L)} |\operatorname{Tr}(\chi_k(h_L(H)-h_L(H_{L,p}))\chi_k)|.
\end{aligned}
$$

Since $\sharp\{p\in\mathbf{Z}^d\colon B_p(L)\subset(LB)\} = \mathcal{O}((L/l_L)^d)$, it suffices to show

$$
\left(\frac{L}{l_L}\right)^d \sum_{k\in B_p(L)} \mathbf{E}[|\operatorname{Tr}(\chi_k(h_L(H)-h_L(H_{L,p}))\chi_k)|] = o(1).
$$

Take $l'_L = o(l_L)$ and let

$$\operatorname{Int} B_p(L) := \{x \in B_p(L) \colon d(x, \partial B_p(L)) \geq l'_L\},$$
$$\operatorname{int} B_p(L) := \{x \in B_p(L) \colon d(x, \partial B_p(L)) \geq 2l'_L\}.$$

We decompose the sum as

$$\begin{aligned}
&\left(\frac{L}{l_L}\right)^d \sum_{k \in B_p(L)} \mathbf{E}[\operatorname{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)] \\
&= \left(\frac{L}{l_L}\right)^d \left(\sum_{k \in \operatorname{int} B_p(L)} + \sum_{k \in B_p(L) \setminus \operatorname{int} B_p(L)}\right) \mathbf{E}[\operatorname{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)] \\
&=: I + II.
\end{aligned} \tag{2.4}$$

We show $I = o(1)$, $II = o(1)$.

Estimate of II: By (2.2), (2.3) and the estimate $\sharp(B_p(L) \setminus \operatorname{int} B_p(L)) = \mathcal{O}(l_L^{d-1} \cdot l'_L)$ we have

$$II \leq (\text{const.}) \left(\frac{L}{l_L}\right)^d \cdot l_L^{d-1} l'_L \cdot 2C_W \|g\|_\infty \frac{|J|}{L^d} \leq (\text{const.}) \frac{l'_L}{l_L} = o(1).$$

Estimate of I: Let $\tilde{h}_L$ be an almost analytic extension of h_L:

$$\tilde{h}_L(x + iy) := (h_L(x) + h'_L(x)(iy))\psi(x + iy)$$

where $\psi \in C^\infty(\mathbf{C})$ and

$$\psi(x + iy) = \begin{cases} 1 & (|y| \leq 1 + |x|), \\ 0 & (|y| \geq 2 + 2|x|). \end{cases}$$

Let

$$G(z) = (H - z)^{-1}, \quad G_{L,p}(z) = (H_{L,p} - z)^{-1}$$

be resolvents of H, $H_{L,p}$. We then have [11]

$$\tilde{h}_L(H) = \frac{-1}{2\pi i} \int_{\mathbf{C}} dz \wedge d\bar{z}\, \partial_{\bar{z}} \tilde{h}_L(z) G(z), \quad \tilde{h}_L(H_{L,p}) = \frac{-1}{2\pi i} \int_{\mathbf{C}} dz \wedge d\bar{z}\, \partial_{\bar{z}} \tilde{h}_L(z) G_{L,p}(z),$$

which gives the following representation.

$$\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k = \frac{-1}{2\pi i} \int_{\mathbf{C}} dz \wedge d\bar{z} (\partial_{\bar{z}} \tilde{h}_L(z)) \chi_k (G(z) - G_{L,p}(z)) \chi_k. \tag{2.5}$$

Let $\Theta \in C_c^\infty(B_p(L))$ such that $\Theta = 1$ on $\mathrm{Int}\, B_p(L)$ and $|\mathrm{supp}\, \nabla\Theta \cup \mathrm{supp}\, \triangle\Theta| \le (\mathrm{const.}) l_L^{d-1}$. As an operator on $L^2(B_p(L))$

$$G_{L,p}(z)[H, \Theta]G(z) = G_{L,p}(z)\{(H-z)\Theta - \Theta(H-z)\}G(z) \\ = \Theta G(z) - G_{L,p}(z)\Theta. \tag{2.6}$$

We multiply χ_k from both sides, use the fact that $\mathrm{supp}\, \nabla\Theta$, $\mathrm{supp}\, \triangle\Theta \subset B_p(L) \setminus \mathrm{Int}\, B_p(L)$, and use Lemma 3.5. It follows that

$$\begin{aligned} \chi_k(G(z) - G_{L,p}(z))\chi_k &= \chi_k G_{L,p}(z)[H, \Theta]G(z)\chi_k \\ &= \sum_l \chi_k G_{L,p}(z)\chi_l[H, \Theta]G(z)\chi_k \\ &= \sum_{l \in B_p(L)\setminus \mathrm{Int}\, B_p(L)} \chi_k G_{L,p}(z)\chi_l[H, \Theta]G(z)\chi_k \\ &= \sum_{l \in B_p(L)\setminus \mathrm{Int}\, B_p(L)} \chi_k G_{L,p}(z)\chi_l T_L(z)G(z)\chi_k. \end{aligned} \tag{2.7}$$

RHS of the above equality now turns out to be in the trace class although $\chi_k G_\Lambda(z)\chi_k$, $\chi_k G_{L,p}(z)\chi_k$ do not for $d \ge 2$. Substituting (2.7) into (2.5) we have

$$\begin{aligned} &\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k \\ &= \frac{-1}{2\pi i}\int_{\mathbf{C}} dz \wedge d\bar{z}(\partial_{\bar{z}}\tilde{h}_L(z)) \sum_{l \in B_p(L)\setminus \mathrm{Int}\, B_p(L)} \chi_k G_{L,p}(z)\chi_l T_L(z)G(z)\chi_k. \end{aligned} \tag{2.8}$$

We take trace and use the inequalities $\|\chi_k G_{L,p}(z)\chi_l\|_{\mathrm{op}} \le \|\chi_k G_{L,p}(z)\chi_l\|_{\mathrm{op}}^s \cdot \|G_{L,p}(z)\|_{\mathrm{op}}^{1-s}$ $(0 < s < 1)$, $\|G(z)\|_{\mathrm{op}} \le |\Im z|^{-1}$, $\|G_{L,p}(z)\|_{\mathrm{op}} \le |\Im z|^{-1}$ and $\|T_L(z)\|_1 \le C_{d,z}(l_L)^{(d-1)(d+1)}$. Here we write $z = x + iy$ and note that $\mathrm{supp}|\partial_{\bar{z}}\tilde{h}_L(x+iy)|$ is compact in $\mathbf{R}^2$.

$$\begin{aligned} &|\mathrm{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)| \\ &\le (\mathrm{const.}) \int_{\mathbf{R}^2} dx\, dy |\partial_{\bar{z}}\tilde{h}_L(x+iy)| \\ &\quad \times \sum_{l \in B_p(L)\setminus \mathrm{Int}\, B_p(L)} \|\chi_k G_{L,p}(x+iy)\chi_l\|_{\mathrm{op}}^s |y|^{-(1-s)-1}(l_L)^{(d-1)(d+1)}. \end{aligned} \tag{2.9}$$

We use H2 (1.1) here. Since $\mathrm{supp}\, h_L \subset I$ for L sufficiently large, and since $|k - l| \ge l'_L$ for $k \in \mathrm{int}\, B_p(L)$, $l \in B_p(L) \setminus \mathrm{Int}\, B_p(L)$, (1.1) implies

$$\begin{aligned} &\mathbf{E}[|\mathrm{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)|] \\ &\le (\mathrm{const.}) \int_{\mathbf{R}^2} dx\, dy |\partial_{\bar{z}}\tilde{h}_L(x+iy)|(l_L^{d-1} \cdot l'_L)e^{-\mu l'_L}|y|^{-(1-s)-1}(l_L)^{(d-1)(d+1)}. \end{aligned} \tag{2.10}$$

By the definition of almost analytic extension and h_L,

$$|\partial_{\bar{z}}\tilde{h}_L(x+iy)| \le (\text{const.})|y| \sum_{j=0}^{2} |h_L^{(j)}(x)|,$$

$$\int |h_L^{(j)}(x)|\,dx \le \begin{cases} (\text{const.})L^d & (j=2), \\ (\text{const.}) & (j=1), \\ (\text{const.})L^{-d} & (j=0), \end{cases}$$

which shows that

$$\int_{\mathbf{R}^2} dx\,dy |\partial_{\bar{z}}\tilde{h}_L(x+iy)| \cdot |y|^{-(1-s)-1} \le (\text{const.})L^d.$$

With this estimate (2.10) yields

$$\mathbf{E}[|\mathrm{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)|] \le (\text{const.})L^{\gamma}e^{-\mu l'_L}$$

for some $\gamma > 0$. Substituting it into (2.4) and taking $l'_L = \beta \log L$ $(= o(l_L))$ with $\beta \gg 1$ proves $I = o(1)$. □

REMARK 2.2. The argument of showing $I = o(1)$ in the proof of Proposition 2.1 also proves

$$\mathbf{E}[|\mathrm{Tr}(\chi_0(f(H) - f(H_{\Lambda_L}))\chi_0)|] = o(1), \quad L \to \infty, \tag{2.11}$$

for $f \in C_0^\infty(I)$. We note Lemma 3.2 is not used in the estimate of I.

The rest of our argument is similar to that in [6]. To prove the infinite divisibility of ζ as a point process, we approximate $\eta_{L,p}$ by point processes. For that purpose let $\{E_{j,p}\}_j$ be the eigenvalues of $H_{L,p}$ and define point processes $\tilde{\eta}_{L,p}$

$$\int_{\mathbf{R}^{d+1}} f(E,x)\,d\tilde{\eta}_{L,p} := \sum_j f\left(L^d(E_{j,p} - E_0), \frac{pl_L}{L}\right), \quad f \in C_c(\mathbf{R}^{d+1}).$$

Proposition 2.3. *For* $f \in C_c(\mathbf{R}^{d+1})$

$$\mathbf{E}\left[\sum_p \left|\int_{\mathbf{R}^{d+1}} f(E,x)\,d\eta_{L,p} - \int_{\mathbf{R}^{d+1}} f(E,x)\,d\tilde{\eta}_{L,p}\right|\right] = o(1).$$

Proof. For $\delta > 0$ let

$$w(\delta) := \sup\{|f(E,x) - f(E,x')|;\ |x - x'| < \delta,\ E \in \mathbf{R}\}.$$

Since f is uniformly continuous, $\lim_{\delta \to 0} w(\delta) = 0$. Let $\psi_{j,p}$ be the normalized eigenfunctions corresponding to $E_{j,p}$. We then have

$$\int_{\mathbf{R}^{d+1}} f(E, x)\, d\eta_{L,p} = \sum_j \int_{B_p(L)} f\left(L^d(E_{j,p} - E_0), \frac{x}{L}\right) |\psi_{j,p}(x)|^2\, dx,$$

$$\int_{\mathbf{R}^{d+1}} f(E, x)\, d\tilde{\eta}_{L,p} = \sum_j \int_{B_p(L)} f\left(L^d(E_{j,p} - E_0), \frac{pl_L}{L}\right) |\psi_{j,p}(x)|^2\, dx.$$

Take an interval $J \subset \mathbf{R}$ such that $\operatorname{supp} f \subset J \times \mathbf{R}^d$. Then

$$\left| \int_{\mathbf{R}^{d+1}} f\, d\eta_{L,p} - \int_{\mathbf{R}^{d+1}} f\, d\tilde{\eta}_{L,p} \right| \le \| f \|_\infty w\left(\frac{l_L}{L}\right) \sharp \left\{ \text{eigenvalues of } H_{L,p} \in E_0 + \frac{J}{L^d} \right\}.$$

Therefore, by Lemma 3.2

$$\begin{aligned} &\mathbf{E}\left[\sum_p \left| \int_{\mathbf{R}^{d+1}} f(E, x)\, d\eta_{L,p} - \int_{\mathbf{R}^{d+1}} f(E, x)\, d\tilde{\eta}_{L,p} \right| \right] \\ &\le (\text{const.}) \left(\frac{L}{l_L}\right)^d \| f \|_\infty w\left(\frac{l_L}{L}\right) \cdot l_L^d \cdot \frac{|J|}{L^d} = o(1). \end{aligned}$$

□

For the estimate on the intensity measure, we have

Proposition 2.4. *Let $E_0 \in I$ be the Lebesgue point of ν. For intervals J $(\subset \mathbf{R})$, A $(\subset \mathbf{R}^d)$ we have*

$$\mathbf{E}[\xi_L(J \times A)] \to \frac{d\nu}{dE}(E_0)|J| \cdot |A|.$$

As in [6] Proposition 2.4 is proved by using (2.1) and the Lebesgue differentiation theorem.

REMARK 2.5. Remark 2.2, Lemma 3.4 imply that $\xi_{L,f}$, μ_L (defined in Remark 1.4) satisfy

$$\mathbf{E}[\xi_{L,f}(J \times A)] \to \frac{d\nu}{dE}(E_0)|J| \cdot |A|, \quad \mathbf{E}[\mu_L(J)] \to \frac{d\nu}{dE}(E_0)|J|$$

for intervals $J \subset \mathbf{R}$, $A \subset U$.

Theorem 1.2 is proved by combining these propositions.

Proof of Theorem 1.2. Let $J \subset \mathbf{R}$, $B \subset \mathbf{R}^d$ be bounded intervals. By Lemma 3.4 we have $\mathbf{E}[\xi_L(J \times B)] \le C_W |B| \cdot |J|$. Thus Chebyshev's inequality gives

$$\lim_{t \to \infty} \sup_{L > 0} \mathbf{P}(\xi_L(J \times B) > t) = 0$$

so that $\{\xi_L\}$ is relatively compact [5, Lemma 4.5]: we can find a sequence $\{L_k\}_{k=1}^{\infty}$ and a random measure ζ with $\xi_{L_k} \stackrel{d}{\to} \zeta$. By Proposition 2.1, 2.3, $\xi_L - \sum_p \tilde{\eta}_{L,p} \stackrel{d}{\to} 0$. Since $\mathcal{M}_p(\mathbf{R}^{d+1})$ is closed in $\mathcal{M}(\mathbf{R}^{d+1})$ under the vague topology, ζ is a point process. By Lemma 3.2

$$\begin{aligned}\mathbf{E}[\tilde{\eta}_{L,p}(J \times B)] &\leq \mathbf{E}[\mathrm{Tr}(P_{E_0+J/L^d}(H_{L,p}))] \\ &\leq \sum_{B(k,1/2)\cap B_p(L)\neq\emptyset} \mathbf{E}[\mathrm{Tr}(\chi_k P_{E_0+J/L^d}(H_{L,p})\chi_k)] \\ &\leq (\mathrm{const.}) l_L^d \cdot \frac{|J|}{L^d}.\end{aligned}$$

Hence $\{\tilde{\eta}_{L,p}\}$ is a null-array:

$$\lim_{L\to\infty} \sup_{p\in\mathbf{Z}^d} \mathbf{P}(\tilde{\eta}_{L,p}(J \times B) \geq 1) = 0$$

for any bounded interval $J \subset \mathbf{R}$, $B \subset \mathbf{R}^d$. Therefore ζ is infinitely divisible [5, Theorem 6.1]. The estimate on the intensity measure (1.2) follows from Proposition 2.4 and the inequality $\mathbf{E}[\xi(A)] \leq \liminf_{n\to\infty} \mathbf{E}[\xi_n(A)]$ if $\xi_n \stackrel{d}{\to} \xi$. □

3. Appendix 1: Some basic estimates

We recall some basic estimates used in Section 2. Let $\mathcal{T}_p$ $(1 \leq p \leq \infty)$ be the Schatten class: the ideal of compact operators on $L^2(\mathbf{R}^d)$ with $\|T\|_p := (\mathrm{Tr}(|T|^p))^{1/p} < \infty$.

Lemma 3.1. *Let $g \in L^2(\mathbf{R}^d)$ and let I $(\subset \mathbf{R})$ be an bounded interval. Then $P_I(H)g,\ gP_I(H) \in \mathcal{T}_2$.*

Sketch of proof. Since e^{-tH} is bounded as an operator from L^2 into L^∞, its integral kernel $K(x,y)$ satisfies $\text{ess-sup}_{x\in\mathbf{R}^d} \int_{\mathbf{R}^d} |K(x,y)|^2\,dy < \infty$ implying $ge^{-tH} \in \mathcal{T}_2$ and thus $e^{-tH}g \in \mathcal{T}_2$. By the spectral theorem, $\|P_I(H)g\phi\|^2 \leq (\mathrm{const.})\|e^{-tH}g\phi\|^2$ for $\phi \in L^2(\mathbf{R}^d)$ which proves $P_I(H)g \in \mathcal{T}_2$. □

The following lemma is fundamental to study ξ_L, $\eta_{L,p}$.

Lemma 3.2. *We can find a positive constant C_W such that for any interval $J \subset I$ and any $k \in B_p(L)$,*

(1) $\mathbf{E}[\mathrm{Tr}(\chi_k P_J(H_{L,p})\chi_k)] \leq C_W|J|$,

(2) $\mathbf{E}[\mathrm{Tr}(\chi_k P_J(H)\chi_k)] \leq C_W|J|$.

REMARK 3.3. In the statement of Lemma 3.2 (1), I can be any bounded interval and H2 is not necessary to prove that.

Proof. (1) is proved by the spectral averaging method [2, (4.19)]. The periodic boundary condition on $H_{L,p}$ is used here. For (2), we take $L>0$, $\epsilon>0$ and let $H_{\Lambda_L} := H|_{[-L/2,L/2]^d}$ and $f \in C_c^\infty(\mathbf{R},[0,1])$ with $\operatorname{supp} f \subset \{x \in \mathbf{R}\colon d(x,J) < \epsilon|J|\} \cap I$ and $f=1$ on J. By Remark 2.2,

$$\begin{aligned}\mathbf{E}[\mathrm{Tr}(\chi_0 P_J(H)\chi_0)] &\le \mathbf{E}[\mathrm{Tr}(\chi_0 f(H)\chi_0)]\\ &= \mathbf{E}[\mathrm{Tr}(\chi_0 f(H_{\Lambda_L})\chi_0)] + \mathbf{E}[\mathrm{Tr}(\chi_0(f(H)-f(H_{\Lambda_L}))\chi_0)]\\ &\le (1+\epsilon)C_W|J| + o(1), \quad L\to\infty. \end{aligned}$$ □

Lemma 3.4 given below easily follows from Lemma 3.2.

Lemma 3.4. *Let $f \in L^1(\mathbf{R}^{d+1})$ with compact support. Then for sufficiently large L*
(1) $\mathbf{E}\left[\left|\int_{\mathbf{R}^{d+1}} f(E,x)\,d\xi_L\right|\right] \le C_W\|f\|_1$,
(2) $\mathbf{E}\left[\left|\sum_p \int_{\mathbf{R}^{d+1}} f(E,x)\,d\eta_{L,p}\right|\right] \le C_W\|f\|_1$.

Sketch of proof. (1) We first consider the case of $f(E,x) = \chi_J(E)\cdot\chi_A(x)$ for $J\subset\mathbf{R}$, $A\subset\mathbf{R}^d$ intervals. We then have

$$\begin{aligned}\left|\int f(E,x)\,d\xi_L\right| &= \left|\mathrm{Tr}\Big(\chi_A\Big(\frac{x}{L}\Big)\chi_J(L^d(H-E_0))\Big)\right|\\ &\le \sum_{B(k,1/2)\cap(LA)\neq\emptyset} \mathrm{Tr}(\chi_k P_{E_0+L^{-d}J}(H)\chi_k).\end{aligned}$$

Let $r = d(E_0, I^c) > 0$. Since $E_0 + L^{-d}J \subset E_0 + [-r,r] \subset I$ if $J \subset [-rL^d, rL^d]$, we use Lemma 3.2 and conclude

$$\mathbf{E}\left[\left|\int f(E,x)\,d\xi_L\right|\right] \le \sum_{B_k\cap(LA)\neq\emptyset} C_W L^{-d}|J| \le C_W|A|\,|J|.$$

A density argument proves

$$\mathbf{E}\left[\left|\int f(E,x)d\xi_L\right|\right] \le C_W\|f\|_1$$

for general f with $\operatorname{supp} f \subset L^d[-r,r]\times\mathbf{R}^d$. (2) is proved similarly. □

The following lemma is a variant of [1, Lemma 3.3].

Lemma 3.5. *Let $\Theta \in C_c^\infty(B_p(L))$ be as defined in the proof of* Proposition 2.1 *and let $k \in \operatorname{int} B_p(L)$. Then we can find a trace class operator $T = T_L(z)$ and a positive constant $C_{d,z}$ such that*

$$[H,\Theta]G(z)\chi_k = T_L(z)G(z)\chi_k, \quad \|T_L(z)\|_1 \le C_{d,z}\, l_L^{(d-1)(d+1)},$$

where $C_{d,z}$ is locally bounded w.r.t. $z \in \rho(H)$ and $\|\cdot\|_1$ is the trace norm.

Proof. Take $\Phi_0 \in C_c^\infty(B_p(L))$ with $\Phi_0 = 1$ on $\operatorname{supp} \nabla\Theta \cup \operatorname{supp} \triangle\Theta$ and let $H_0 = -\triangle$, $G_0 = (H_0+1)^{-1}$. We then have

$$\begin{aligned}
&[H, \Theta]\Phi_0^2 G(z)\chi_k \\
&= [H, \Theta]\Phi_0 G_0 (H_0+1)\Phi_0 G(z)\chi_k \\
&= [H, \Theta]\Phi_0 G_0(\Phi_0(z+1-V) + [H, \Phi_0])\Phi_1^2 G(z)\chi_k \\
&= T_0 \Phi_1^2 G(z)\chi_k
\end{aligned}$$

where

$$T_0 := [H, \Theta]\Phi_0 G_0(\Phi_0(z+1-V) + [H, \Phi_0])$$

is a bounded operator and $\Phi_1 \in C_c^\infty(B_p(L))$ is a function which satisfies $\Phi_1 = 1$ on $\operatorname{supp} \nabla\Phi_0 \cup \operatorname{supp} \triangle\Phi_0$. It is possible to let $|\operatorname{supp} \Phi_1| \leq (\text{const.}) l_L^{d-1}$. Furthermore, by the same argument,

$$\begin{aligned}
\Phi_1^2 G(z)\chi_k &= \Phi_1 G_0(\Phi_1(z+1-V) + [H, \Phi_1])\Phi_2^2 G(z)\chi_k \\
&=: T_1 \Phi_2^2 G(z)\chi_k
\end{aligned}$$

with some $\Phi_2 \in C_c^\infty(B_p(L))$ satisfying $\Phi_2 = 1$ on $\operatorname{supp} \nabla\Phi_1 \cup \operatorname{supp} \triangle\Phi_1$ and $|\operatorname{supp} \Phi_2| \leq (\text{const.}) l_L^{d-1}$. We repeat this procedure: for $j = 1, 2, \ldots, n+1$ we can find $\Phi_j \in C_c^\infty(B_p(L))$ with $\Phi_j = 1$ on $\operatorname{supp} \nabla\Phi_{j-1} \cup \operatorname{supp} \triangle\Phi_{j-1}$ and $|\operatorname{supp} \Phi_j| \leq (\text{const.}) l_L^{d-1}$ such that

$$\begin{aligned}
&[H, \Theta]\Phi_0^2 G(z)\chi_k = T_0 T_1 \cdots T_n \Phi_{n+1}^2 G(z)\chi_k, \\
&T_j := \Phi_j G_0(\Phi_j(z+1-V) + [H, \Phi_j]), \quad j = 1, 2, \ldots, n.
\end{aligned}$$

By the fact that

$$T_j = \Phi_j (H_0+1)^{-1/2} C_j$$

for some bounded operator C_j and by Lemma 3.6, $T_j \in \mathcal{I}_p$ for $p > d$ with $\|T_j\|_p \leq (\text{const.}) l_L^{d-1}$. We note that the (const.) appearing in this inequality is locally bounded w.r.t. $z \in \rho(H)$. Taking $n = d+1$, we have $T = T_0 T_1 \cdots T_n \in \mathcal{I}_1$. The estimate for $\|T\|_1$ follows from the inequality $\|T\|_1 \leq \|T_0\|_{\mathrm{op}} \prod_{j=1}^n \|T_j\|_n$. $\square$

Lemma 3.6 ([14]). *Let $g \in L^p(\mathbf{R}^d)$, $2 \leq p \leq \infty$ and f is bounded measurable on $\mathbf{R}$ with $|f(\lambda)| \leq C_f \langle\lambda\rangle^{-\alpha}$. Then $g(x) f(H) \in \mathcal{I}_p$ for $\alpha p > d/2$ and for some positive constant $C_{f,p}$ which depends only on p, d and C_f we have*

$$\|g(x) f(H)\|_p \leq C_{f,p} \|g\|_p.$$

4. Appendix 2: Proof of Theorem 1.2 by the multiscale analysis

We first set some notations. Let

$$\Lambda_L(x) := \left\{ y \in \mathbf{R}^d : |y_j - x_j| \le \frac{L}{2},\ j = 1, 2, \ldots, d \right\}$$

be a finite box of size L centered at $x \in \mathbf{R}^d$ and let

$$\Lambda_L^{\text{out}}(x) := \{ y \in \Lambda_L(x) : d(y, \partial \Lambda_L(x)) \le 1 \}, \quad \chi^{\text{out}}_{\Lambda_L(x)} := 1_{\Lambda_L^{\text{out}}(x)}$$

be a strip of width 1 on the boundary of $\Lambda_L(x)$ and its characteristic function. For $\gamma > 0$ and $E \in \mathbf{R}$ we say $\Lambda_L(x)$ is (γ, E)-regular iff $E \notin \sigma(H_{\Lambda_L})$ and the following estimate hold.

$$\sup_{\epsilon > 0} \| \chi_x G_{\Lambda_L(x)}(E + i\epsilon) \chi^{\text{out}}_{\Lambda_L(x)} \|_{\text{op}} \le e^{-\gamma L/2}$$

where $G_{\Lambda_L(x)}(z) = (H_{\Lambda_L(x)} - z)^{-1}$ is the resolvent of $H_{\Lambda_L(x)} := H|_{\Lambda_L(x)}$ and $\chi_x = \chi_{B(x,1/2)}$. We assume

H3 (Initial length scale estimate). We can find a bounded open interval $I \subset [0, \infty)$ and $\gamma > 0$ such that for each $E \in I$

$$\mathbf{P}(\Lambda_{L_0}(0) \text{ is } (\gamma, E)\text{-regular}) \ge 1 - L_0^{-p}, \quad p > 2d^2 + 8d + 2$$

for sufficiently large $L_0 = L_0(E)$.

This condition for $p > d$ together with Lemma 3.2 (1) are sufficient condition to prove Anderson localization [15]. For a technical reason p must be larger here. However we can still find an interval $I \subset [0, \infty)$ such that H3 holds, in those situations described after H2. By H3 we can deduce the following facts: let $\alpha = 2p/(p + 2d)$ and define a set of growing scales $\{L_k\}_{k=1}^{\infty}$ as

$$L_{k+1} := L_k^{\alpha}, \quad k = 0, 1, 2, \ldots,$$

then for any $x \in \mathbf{Z}^d$ we have

$$\mathbf{P}(\Lambda_{L_k}(x) \text{ is } (\gamma, E)\text{-regular}) \ge 1 - L_k^{-p}, \quad k = 1, 2, \ldots. \tag{4.1}$$

Furthermore for $\gamma' = \gamma/8$, $\Lambda_L := [-L/2, L/2]^d$ and for $k, m \in \Lambda_L \cap \mathbf{Z}^d$,

$$\begin{aligned} &\mathbf{P}\left(\left\{ \omega \in \Omega : \sup_{\epsilon > 0} \| \chi_k G_{\Lambda_L}(E + i\epsilon) \chi_m \|_{\text{op}} \le e^{-\gamma' |k-m|} \right\} \right) \\ &\quad \ge 1 - C |\Lambda_L| \, |k - m|^{-(p/2+d)}, \quad |k - m| \ge L_0 \end{aligned} \tag{4.2}$$

for some positive constant C. These estimates (4.1), (4.2) are proved as in the discrete case (see e.g., [3, 15]) by Lemma 3.2 (1) and the following geometric resolvent estimate: for $\Lambda_L(x) \subset \Lambda'$ and $B \subset \Lambda' \setminus \Lambda_L(x)$ we have

$$\|\chi_x G_{\Lambda'}(z)\chi_B\|_{\mathrm{op}} \leq (\mathrm{const.})\|\chi_x G_{\Lambda_L(x)}(z)\chi^{\mathrm{out}}_{\Lambda_L(x)}\|_{\mathrm{op}} \cdot \|\chi^{\mathrm{out}}_{\Lambda_L(x)} G_{\Lambda'}(z)\chi_B\|_{\mathrm{op}}. \tag{4.3}$$

We note that (4.3) follows from (2.6) and the argument in the proof of Lemma 3.5.

Theorem 4.1. *Assume* H1, H3. *Then the same conclusion as in* Theorem 1.2 *holds*.

REMARK 4.2. It is known that H1, H3 with $p > 2(d-1)$ implies H2 [1, Theorem 5.1]. However the argument in this section also applies to various models (divergence type Hamiltonian for instance) even when H1 is not satisfied, provided Lemma 3.2 (1) and H3 hold.

Proof. Statements in Proposition 2.3, 2.4 and the equation $II = o(1)$ in the proof of Proposition 2.1 follow from Lemma 3.2, Remark 3.3 and Remark 4.3 below. Hence all we need to prove is $I = o(1)$. We start from (2.8) with $k \in \mathrm{int}\, B_p(L)$. By the argument to deduce (2.9), we have

(4.4)

$$\begin{aligned}
&\mathbf{E}[|\mathrm{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)|] \\
&\leq \sum_{m \in B_p(L)\setminus \mathrm{Int}\, B_p(L)} \int_{\mathbf{R}^2} dx\, dy |\partial_{\bar z}\tilde{h}_L(x+iy)|\, \mathbf{E}[|\mathrm{Tr}(\chi_k G_p(x+iy)\chi_m T_L(z) G(x+iy)\chi_k)|] \\
&\leq (\mathrm{const.}) \sum_{m \in B_p(L)\setminus \mathrm{Int}\, B_p(L)} \int_{\mathbf{R}^2} dx\, dy |\partial_{\bar z}\tilde{h}_L(x+iy)| C(L)|y|^{-1}\mathbf{E}[\|\chi_k G_p(x+iy)\chi_m\|_{\mathrm{op}}]
\end{aligned}$$

where $C(L) = l_L^{(d-1)(d+1)}$. We define an event $G_{km}(E)$ by

$$G_{km}(E) := \left\{\omega \in \Omega\colon \sup_{\epsilon \neq 0}\|\chi_k G_{L,p}(E+i\epsilon)\chi_m\|_{\mathrm{op}} \leq e^{-\gamma'|k-m|}\right\}$$

for $E \in I$ and $k, m \in \mathbf{Z}^d$ whose probability is estimated by (4.2)

$$\mathbf{P}(G_{km}(E)) \geq 1 - C|B_p(L)|\,|k-m|^{-(p/2+d)}, \quad |k-m| \geq L_0.$$

Therefore for $k \in \mathrm{int}\, B_p(L)$ and $m \in B_p(L) \setminus \mathrm{Int}\, B_p(L)$, and for sufficiently large L with $l'_L > L_0$, we have

$$\mathbf{E}[\|\chi_k G_p(x+iy)\chi_m\|_{\mathrm{op}};\ G_{km}(x)] \leq e^{-\gamma'|k-m|} \leq e^{-\gamma' l'_L}, \tag{4.5}$$

$$\begin{aligned}
\mathbf{E}[\|\chi_k G_p(x+iy)\chi_m\|_{\mathrm{op}};\ G_{km}(x)^c] &\leq (\mathrm{const.})|B_p(L)|\,|k-m|^{-(p/2+d)}|y|^{-1} \\
&\leq (\mathrm{const.})|B_p(L)| l_L^{\prime -(p/2+d)}|y|^{-1}.
\end{aligned} \tag{4.6}$$

The second one (4.6) is dominant. Since $|y|^{-2}$ factor appears when (4.6) is substituted into (4.4), we take higher order term in the definition of the almost analytic extension of h_L: we take $g \in C_c^3(\mathbf{R})$, $h_L(\lambda) = g(L^d(\lambda - E_0))$ and

$$\tilde{h}_L(x+iy) := \left(h_L(x) + h'_L(x)(iy) + \frac{h''_L(x)}{2}(iy)^2 \right) \psi(x+iy).$$

Then we have

$$|\partial_{\bar{z}} \tilde{h}_L(x+iy)| \le (\text{const.})|y|^2 \sum_{j=0}^{3} |h_L^{(j)}(x)|,$$

$$\sum_{j=0}^{3} \int |h_L^{(j)}(x)|\, dx \le (\text{const.}) L^{2d}$$

so that

$$\int_{\mathbf{R}^2} dx\, dy |\partial_{\bar{z}} h_L(x+iy)| \cdot |y|^{-2} \le (\text{const.}) L^{2d}. \tag{4.7}$$

Substituting (4.5), (4.6), (4.7) into (4.4) we have

$$\begin{aligned} &\mathbf{E}[|\mathrm{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)|] \\ &\le (\text{const.}) \sum_{m \in B_p(L) \setminus \mathrm{Int}\, B_p(L)} L^{2d} l_L^{(d-1)(d+1)} (e^{-\gamma' l'_L} + l_L^d l_L'^{-(p/2+d)}). \end{aligned}$$

Hence

$$\begin{aligned} &\frac{L^d}{l_L^d} \sum_{k \in \mathrm{int}\, B_p(L)} \mathbf{E}[|\mathrm{Tr}(\chi_k(h_L(H) - h_L(H_{L,p}))\chi_k)|] \\ &\le (\text{const.}) \frac{L^d}{l_L^d} \cdot l_L^d \cdot l'_L l_L^{d-1} \cdot L^{2d} l_L^{(d-1)(d+1)} \cdot l_L^d l_L'^{-(p/2+d)} \\ &= (\text{const.}) L^{3d} l_L^{(d-1)+(d-1)(d+1)+d} l'_L \cdot l_L'^{-(p/2+d)}. \end{aligned} \tag{4.8}$$

Here we take

$$l_L = L^\alpha,\ l'_L = L^\beta, \quad 0 < \beta < \alpha < 1.$$

In order to have RHS of (4.8) $= o(1)$, α, β must satisfy

$$3d + \alpha(d^2 + 2d - 2) + \beta\left(1 - \frac{p}{2} - d\right) < 0,$$

which is possible when $p > 2d^2 + 2d - 2$. □

REMARK 4.3. The above argument also proves (2.11) without using Lemma 3.2.

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NOTE ADDED IN PROOF. Recently, Combes, Germinet, and Klein succeeded to prove Minami's estimate in the continuum Schrödinger operators.

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