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A NOTE ON TODOROV SURFACES

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Abstract

Let S be a *Todorov surface*, i.e., a minimal smooth surface of general type with $q = 0$ and $p_g = 1$ having an involution i such that S/i is birational to a $K3$ surface and such that the bicanonical map of S is composed with i.

The main result of this paper is that, if P is the minimal smooth model of S/i, then P is the minimal desingularization of a double cover of $\mathbb{P}^2$ ramified over two cubics. Furthermore it is also shown that, given a Todorov surface S, it is possible to construct Todorov surfaces S_j with $K^2 = 1, \dots, K_S^2 - 1$ and such that P is also the smooth minimal model of S_j/i_j, where i_j is the involution of S_j. Some examples are also given, namely an example different from the examples presented by Todorov in [9].

1. Introduction

An *involution* of a surface S is an automorphism of S of order 2. We say that a map is *composed with an involution* i of S if it factors through the double cover $S \to S/i$. Involutions appear in many contexts in the study of algebraic surfaces. For instance in most cases the bicanonical map of a surface of general type is non-birational only if it is composed with an involution.

Assume that S is a smooth minimal surface of general type with $q = 0$ and $p_g \neq 0$ having bicanonical map ϕ_2 composed with an involution i of S such that S/i is non-ruled. Then, according to [10, Theorem 3], $p_g(S) = 1$, $K_S^2 \leq 8$ and S/i is birational to a $K3$ surface (Theorem 3 of [10] contains the assumption $\deg(\phi_2) = 2$, but the result is still valid assuming only that ϕ_2 is composed with an involution).

Todorov ([9]) was the first to give examples of such surfaces. His construction is as follows. Consider a Kummer surface Q in $\mathbb{P}^3$, i.e., a quartic having as only singularities 16 nodes a_i. The double cover of Q ramified over the intersection of Q with a general quadric and over the 16 nodes of Q is a surface of general type with $q = 0$, $p_g = 1$ and $K^2 = 8$. Then, choose $a_1, \dots, a_6$ in general position and let G be the intersection of Q with a general quadric through j of the nodes $a_1, \dots, a_6$. The double cover of Q ramified over $Q \cap G$ and over the remaining $16 - j$ nodes of Q is a surface of general type with $q = 0$, $p_g = 1$ and $K^2 = 8 - j$.

Imposing the passage of the branch curve by a 7-th node, one can obtain a surface with $K^2 = p_g = 1$ and $q = 0$. This is the so-called *Kunev surface*. Todorov ([8]) has

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shown that the Kunev surface is a bidouble cover of $\mathbb{P}^2$ ramified over two cubics and a line.

I refer to [5] for an explicit description of the moduli spaces of Todorov surfaces.

We call *Todorov surfaces* smooth surfaces S of general type with $p_g = 1$ and $q = 0$ having bicanonical map composed with an involution i of S such that S/i is birational to a $K3$ surface.

In this paper we prove the following:

Theorem 1. *Let S be a Todorov surface with involution i and P be the smooth minimal model of S/i. Then*:
a) *there exists a generically finite degree* 2 *morphism $P \to \mathbb{P}^2$ ramified over two cubics*;
b) *for each $j \in \{1, \ldots, K_S^2 - 1\}$, there is a Todorov surface S_j, with involution i_j, such that $K_{S_j}^2 = j$ and P is the smooth minimal model of S_j/i_j.*

The idea of the proof is the following. First we verify that the evenness of the branch locus $B' + \sum A_i \subset P$ implies that each nodal curve A_j can only be contained in a Dynkin graph G of type A_{2n+1} or D_n. Then we use a Saint-Donat result to show that A_j can be chosen such that the linear system $|B' - \mathsf{G}|$ is free. This implies b). Finally we conclude that there is a free linear system $|B'_0|$ with $B_0'^2 = 2$, which gives a).

Notation and conventions. We work over the complex numbers; all varieties are assumed to be projective algebraic. For a projective smooth surface S, the *canonical class* is denoted by K, the *geometric genus* by $p_g := h^0(S, \mathcal{O}_S(K))$, the *irregularity* by $q := h^1(S, \mathcal{O}_S(K))$ and the *Euler characteristic* by $\chi = \chi(\mathcal{O}_S) = 1 + p_g - q$.

A (-2)-*curve* or *nodal curve* on a surface is a curve isomorphic to $\mathbb{P}^1$ such that $C^2 = -2$. We say that a curve singularity is *negligible* if it is either a double point or a triple point which resolves to at most a double point after one blow-up.

The rest of the notation is standard in algebraic geometry.

2. Preliminaries

The next result follows from [7, (4.1), Theorem 5.2, Propositions 5.6 and 5.7].

Theorem 2 ([7]). *Let $|D|$ be a complete linear system on a smooth $K3$ surface F, without fixed components and such that $D^2 \geq 4$. Denote by φ_D the map given by $|D|$. If φ_D is non-birational and the surface $\varphi_D(F)$ is singular then there exists an elliptic pencil $|E|$ such that $ED = 2$ and one of these cases occur*:
(i) $D \equiv \mathcal{O}_F(4E + 2\Gamma)$ *where Γ is a smooth rational irreducible curve such that $\Gamma E = 1$. In this case $\varphi_D(F)$ is a cone over a rational normal twisted quartic in $\mathbb{P}^4$*;
(ii) $D \equiv \mathcal{O}_F(3E + 2\Gamma_0 + \Gamma_1)$, *where Γ_0 and Γ_1 are smooth rational irreducible curves such that $\Gamma_0 E = 1$, $\Gamma_1 E = 0$ and $\Gamma_0\Gamma_1 = 1$. In this case $\varphi_D(F)$ is a cone over a rational*

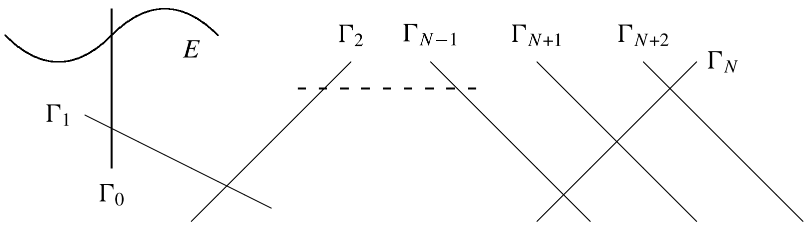

Fig. 1. Configuration (iii) b).

normal twisted cubic in $\mathbb{P}^3$;

(iii) a) $D \equiv \mathcal{O}_F(2E + \Gamma_0 + \Gamma_1)$, *where* Γ_0 *and* Γ_1 *are smooth rational irreducible curves such that* $\Gamma_0 E = \Gamma_1 E = 1$ *and* $\Gamma_0\Gamma_1 = 0$;

b) $D \equiv \mathcal{O}_F(2E + \Delta)$, *with* $\Delta = 2\Gamma_0 + \cdots + 2\Gamma_N + \Gamma_{N+1} + \Gamma_{N+2}$ $(N \geq 0)$, *where the curves* Γ_i *are irreducible rational curves as in* Fig. 1.

In both cases $\varphi_D(F)$ *is a quadric cone in* $\mathbb{P}^3$.

Moreover in all the cases above the pencil $|E|$ *corresponds under the map* φ_D *to the system of generatrices of* $\varphi_D(F)$.

3. Proof of Theorem 1

We say that a curve D is *nef* and *big* if $DC \geq 0$ for every curve C and $D^2 > 0$. In order to prove Theorem 1, we show the following:

Proposition 3. *Let* P *be a smooth* $K3$ *surface with a reduced curve* B *satisfying*:
$(*)$ $B = B' + \sum_1^t A_i$, $t \in \{9, \dots, 16\}$, *where* B' *is a nef and big curve with at most negligible singularities, the curves* A_i *are disjoint* (-2)*-curves also disjoint from* B' *and* $B \equiv 2L$, $L^2 = -4$, *for some* $L \in \operatorname{Pic}(P)$.
Then:
a) *Let* $\pi : V \to P$ *be a double cover with branch locus* B *and* S *be the smooth minimal model of* V. *Then* $q(S) = 0$, $p_g(S) = 1$, $K_S^2 = t - 8$ *and the bicanonical map of* S *is composed with the involution* i *of* S *induced by* π;
b) *If* $t \geq 10$, *then* P *contains a smooth curve* B_0' *and* (-2)*-curves* $A_1', \dots, A_{t-1}'$ *such that* $B_0'^2 = B'^2 - 2$ *and* $B_0 := B_0' + \sum_1^{t-1} A_i'$ *also satisfies condition* $(*)$.

Proof. a) Let $L \equiv (1/2)B$ be the line bundle which determines π. From the double cover formulas (see e.g. [1]) and the Riemann-Roch theorem,

$$q(S) = h^1(P, \mathcal{O}_P(L)),$$

$$p_g(S) = 1 + h^0(P, \mathcal{O}_P(L)),$$

$$h^0(P, \mathcal{O}_P(L)) + h^0(P, \mathcal{O}_P(-L)) = h^1(P, \mathcal{O}_P(L)).$$

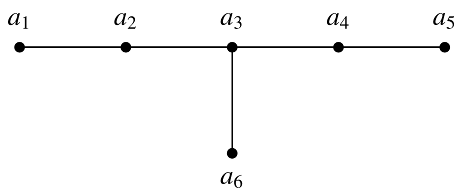

Fig. 2. E_6.

Since $2L - \sum A_i$ is nef and big, the Kawamata-Viehweg's vanishing theorem (see e.g. [3, Corollary 5.12, c)]) implies $h^1(P, \mathcal{O}_P(-L)) = 0$. Hence

$$h^1(P, \mathcal{O}_P(L)) = h^1(P, \mathcal{O}_P(K_P - L) = h^1(P, \mathcal{O}_P(-L))) = 0$$

and then $q(S) = 0$ and $p_g(S) = 1$. As

$$h^0(P, \mathcal{O}_P(2K_P + L)) = h^0(P, \mathcal{O}_P(L)) = 0,$$

the bicanonical map of S is composed with i (see [2, Proposition 6.1]).

The (-2)-curves $A_1, \ldots, A_t$ give rise to (-1)-curves in V, therefore

$$K_S^2 = K_V^2 + t = 2(K_P + L)^2 + t = 2L^2 + t = t - 8.$$

b) Denote by $\xi \subset P$ the set of irreducible curves which do not intersect B' and denote by ξ_i, $i \geq 1$, the connected components of ξ. Since $B'^2 \geq 2$, the Hodge index theorem implies that the intersection matrix of the components of ξ_i is negative definite. Therefore, following [1, Lemma I.2.12], the ξ_i's have one of the five configurations: the support of A_n, D_n, E_6, E_7 or E_8 (see e.g. [1, III.3] for the description of these graphs).

Claim 1. *Each nodal curve A_i can only be contained in a graph of type A_{2n+1} or D_n.*

Proof. Suppose that there exists an A_i which is contained in a graph of type E_6. Denote the components of E_6 as in Fig. 2. If $A_i = a_3$ or $A_i = a_6$, then $a_6 B = a_6 a_3 = 1$ or $a_3 B = 1$, contradicting $B \equiv 2L$. If $A_i = a_1$ or $A_i = a_2$, then $a_2 B = 1$ or $a_1 B = 1$, the same contradiction. By the same reason, $A_i \neq a_4$ and $A_i \neq a_5$.

Analogously one can verify that each A_i can not be in a graph of type A_{2n}, E_7 or E_8. $\square$

The possible configurations for the curves A_i in the graphs are shown in Fig. 3. Fix one of the curves A_i and denote by G the graph containing it.

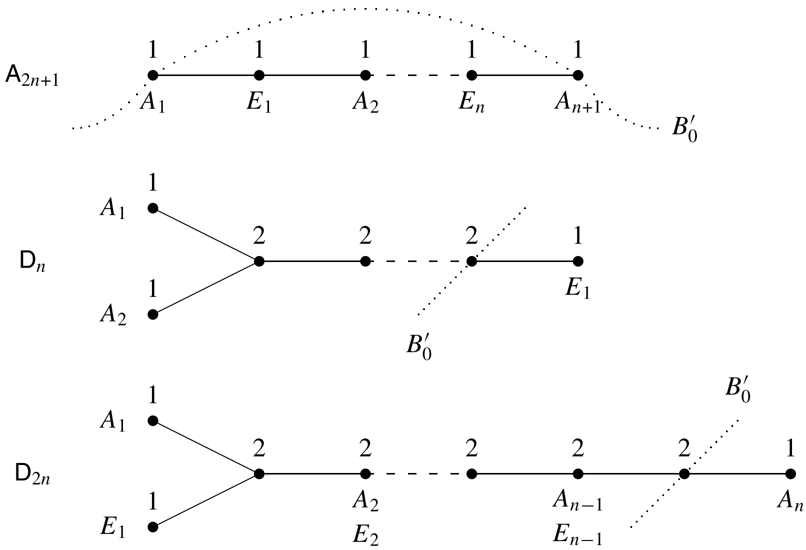

Fig. 3. The numbers represent the multiplicity and the doted curve represent a general element B_0' in $|B' - \mathsf{G}|$.

Claim 2. *We can choose A_i such that the linear system $|B' - \mathsf{G}|$ has no fixed components* (*and thus no base points, from* [7, Theorem 3.1]).

Proof. Denote by $\varphi_{|B'|}$ the map given by the linear system $|B'|$. We know that $\varphi_{|B'|}$ is birational or it is of degree 2 (see [7, Section 4]). If $\varphi_{|B'|}$ is birational or the point $\varphi_{|B'|}(\mathsf{G})$ is a smooth point of $\varphi_{|B'|}(P)$, the result is clear, since $|B' - \mathsf{G}|$ is the pullback of the linear system of the hyperplanes containing $\varphi_{|B'|}(\mathsf{G})$ and $\varphi^*_{|B'|}(\varphi_{|B'|}(\mathsf{G})) = \mathsf{G}$ (see [1, Theorems III 7.1 and 7.3]).

Suppose now that $\varphi_{|B'|}$ is non-birational and that $\varphi_{|B'|}(\mathsf{G})$ is a singular point of $\varphi_{|B'|}(P)$. Then B' is linearly equivalent to a curve with one of the configurations described in Theorem 2. Except for the last configuration, G contains at most two (-2)-curves. But $t \geq 9$, thus in these cases there exists other graph G' containing a curve A_j such that $\varphi_{|B'|}(\mathsf{G}')$ is a non-singular point of $\varphi_{|B'|}(P)$ (notice that Theorem 2 implies that $\varphi_{|B'|}(P)$ contains only one singular point).

So we can suppose that B' is equivalent to a curve with a configuration as in Theorem 2, (iii), b). None of the curves $\Gamma_0, \dots, \Gamma_N$ can be one of the curves A_j. For this note that: if $\Gamma_0 = A_j$, then $EB = E\left(B' + \sum A_i\right) = 2 + E\Gamma_0 = 3 \not\equiv 0 \pmod 2$; if $\Gamma_1 = A_j$, then $\Gamma_0 B = \Gamma_0\Gamma_1 = 1 \not\equiv 0 \pmod 2$; etc. Again this configuration can contain at most two curves A_j, the components Γ_{N+1}, Γ_{N+2}. $\square$

Let B_0' be a smooth curve in $|B' - \mathsf{G}|$. If G is an A_{2n+1} graph, then, using the notation of Fig. 3,

$$\left(B_0' + \sum_1^n E_i\right) + \sum_{n+2}^t A_i \equiv \left(B' - \sum_1^{n+1} A_i\right) + \sum_{n+2}^t A_i$$
$$\equiv B' + \sum_1^t A_i - 2\sum_1^{n+1} A_i \equiv 0 \pmod 2.$$

Therefore the curve

$$B_0 := B_0' + \sum_1^n E_i + \sum_{n+2}^t A_i$$

satisfies condition $(*)$.

The case where G is a D_m graph is analogous. $\square$

Proof of Theorem 1. Let $V \to S$ be the blow-up at the isolated fixed points of the involution i and W be the minimal resolution of S/i. We have a commutative diagram

$$\begin{array}{ccc} V & \longrightarrow & S \\ {\scriptstyle\pi}\downarrow & & \downarrow \\ W & \longrightarrow & S/i. \end{array}$$

The branch locus of π is a smooth curve $B = B' + \sum_1^t A_i$, where the curves A_i are (-2)-curves which contract to the nodes of S/i. Let P be the minimal model of W and $\overline{B} \subset P$ be the projection of B. Let $L \equiv (1/2)B$ be the line bundle which determines π.

First we verify that $\overline{B}$ satisfies condition $(*)$ of Proposition 3: from [2, Proposition 6.1], $\chi(\mathcal{O}_W) - \chi(\mathcal{O}_S) = K_W(K_W + L)$, hence $K_W(K_W + L) = 0$, which implies that $\overline{B}$ has at most negligible singularities; now from [5, Theorem 5.2] we get $K_S^2 = (1/2)\overline{B'}^2$ and $1 = p_g(S) = (1/4)(K_S^2 - t) + 3$, thus $t = K_S^2 + 8$ and $\overline{B}^2 = \overline{B'}^2 - 2t = 2K_S^2 - 2t = -16$, which gives $(\overline{B}/2)^2 = -4$ and $\overline{B'}^2 \geq 2$; finally $\overline{B'}$ is nef because, on a $K3$ surface, an irreducible curve with negative self intersection must be a (-2)-curve.

Now using Proposition 3, b) and a) we obtain statement b). In particular we get also that P contains a curve B_0' and (-2)-curves A_i', $i = 1, \ldots, 9$, such that $B_0 := B_0' + \sum_1^9 A_i'$ is smooth and divisible by 2 in the Picard group. Moreover, the complete linear system $|B_0'|$ has no fixed component nor base points and $B_0'^2 = 2$. Therefore, from [7], $|B_0'|$ defines a generically finite degree 2 morphism

$$\varphi := \varphi_{|B_0'|} : P \to \mathbb{P}^2.$$

Since $g(B_0') = 2$, this map is ramified over a sextic curve β. The singularities of β are negligible because P is a $K3$ surface.

We claim that β is the union of two cubics. Let $p_i \in \beta$ be the singular point corresponding to A_i', $i = 1, \ldots, 9$. Notice that the p_i's are possibly infinitely near. Let $C \subset \mathbb{P}^2$ be a cubic curve passing through p_i, $i = 1, \ldots, 9$. As $C + \varphi_*(B_0')$ is a plane quartic, we have

$$\left(\varphi^*(C) - \sum_1^9 A_i'\right) + B_0' + \sum_1^9 A_i' \equiv \varphi^*(C + \varphi_*(B_0')) \equiv 0 \pmod 2,$$

hence also $\varphi^*(C) - \sum_1^9 A_i' \equiv 0 \pmod 2$, i.e. there exists a divisor J such that

$$2J \equiv \varphi^*(C) - \sum_1^9 A_i'.$$

Since P is a K_3 surface, the Riemann-Roch theorem implies that J is effective. From $JA_i' = 1$, $i = 1, \ldots, 9$, we obtain that the plane curve $\varphi_*(J)$ passes with multiplicity 1 through the nine singular points p_i of β. This immediately implies that $\varphi_*(J)$ is not a line nor a conic, because β is a reduced sextic. Therefore $\varphi_*(J)$ is a reduced cubic. So $\varphi_*(J) \equiv C$ and then

$$\varphi^*(\varphi_*(J)) \equiv 2J + \sum_1^9 A_i'.$$

This implies that $\varphi_*(J)$ is contained in the branch locus β, which finishes the proof of a). □

4. Examples

Todorov gave examples of surfaces S with bicanonical image $\phi_2(S)$ birational to a Kummer surface having only ordinary double points as singularities. The next sections contain an example with $\phi_2(S)$ non-birational to a Kummer surface and an example with $\phi_2(S)$ having an A_{17} double point.

4.1. *S/i* non-birational to a Kummer surface. Here we construct smooth surfaces S of general type with $K^2 = 2, 3$, $p_g = 1$ and $q = 0$ having bicanonical map of degree 2 onto a $K3$ surface which is not birational to a Kummer surface.

It is known since [4] that there exist special sets of 6 nodes, called Weber hexads, in the Kummer surface $Q \in \mathbb{P}^3$ such that the surface which is the blow-up of Q at these nodes can be embedded in $\mathbb{P}^3$ as a quartic with 10 nodes. This quartic is the Hessian of a smooth cubic surface.

The space of all smooth cubic surfaces has dimension 4 while the space of Kummer surfaces has dimension 3. Thus it is natural to ask if there exist Hessian "non-Kummer" surfaces, i.e. which are not the embedding of a Kummer surface blown-up at 6 points. This is studied in [6], where the existence of "non-Kummer" quartic Hessians H in $\mathbb{P}^3$ is shown. These are surfaces with 10 nodes a_i such that the projection from one node a_1 to $\mathbb{P}^2$ is a generically $2:1$ cover of $\mathbb{P}^2$ with branch locus $\alpha_1+\alpha_2$ satisfying: α_1, α_2 are smooth cubics tangent to a nondegenerate conic C at 3 distinct points. We use this in the following construction.

Let α_1, α_2 and C be as above. Take the morphism $\pi\colon W\to\mathbb{P}^2$ given by the canonical resolution of the double cover of $\mathbb{P}^2$ with branch locus $\alpha_1+\alpha_2$. The strict transform of C gives rise to the union of two disjoint (-2)-curves $A_1, A_2\subset W$ (one of these correspond to the node a_1 from which we have projected).

Let $T\in\mathbb{P}^2$ be a general line. Let $A_3,\ldots,A_{11}\subset W$ be the disjoint (-2)-curves contained in $\pi^*(\alpha_1+\alpha_2)$. We have $\pi^*(T+\alpha_1)\equiv 0 \pmod 2$, hence, since α_1 is in the branch locus, also

$$\pi^*(T)+\sum_3^{11} A_i\equiv 0 \pmod 2.$$

The linear systems $|\pi^*(T)+A_2|$ and $|\pi^*(T)+A_1+A_2|$ have no fixed components nor base points (see [7, (2.7.3) and Corollary 3.2]). The surface S is the minimal model of the double cover of W ramified over a general element in

$$|\pi^*(T)+A_2|+\sum_2^{11} A_i$$

or

$$|\pi^*(T)+A_1+A_2|+\sum_1^{11} A_i.$$

4.2. $\phi_2(S)$ with A_{17} and A_1 singularities. This section contains a brief description of a construction of a surface S of general type having bicanonical image $\phi_2(S)\subset\mathbb{P}^3$ a quartic $K3$ surface with A_{17} and A_1 singularities. I omit the details, which were verified using the *Computational Algebra System Magma*.

Let C_1 be a nodal cubic, p an inflection point of C_1 and T the tangent line to C_1 at p. The pencil generated by C_1 and $3T$ contains another nodal cubic C_2, smooth at p. The curves C_1 and C_2 intersect at p with multiplicity 9.

Let $\rho\colon X\to\mathbb{P}^2$ be the resolution of C_1+C_2 and $\pi\colon W\to X$ be the double cover with branch locus the strict transform of C_1+C_2. Denote by $\bar{l}$ the line containing the nodes of C_1 and C_2 and by $l\subset W$ the pullback of the strict transform of $\bar{l}$. The map given by $|(\rho\circ\pi)^*(\bar{l})+l|$ is birational onto a quartic Q in $\mathbb{P}^3$ with an A_1 and A_{17} singularities (notice that l is a (-2)-curve and $((\rho\circ\pi)^*(\bar{l})+l)l=0$).

Let $B' \in |(\rho\circ\pi)^*(\bar{l})+l|$ be a smooth element and $A_1,\ldots,A_9$ be the disjoint (-2)-curves contained in $(\rho\circ\pi)^*(p)$. Let S be the minimal model of the double cover of W with branch locus $B' + \sum_1^9 A_i + l$. The surface Q is the image of the bicanonical map of S and $p_g(S) = 1$, $q(S) = 0$, $K_S^2 = 2$.

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