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A NOTE ON THE GLAUBERMAN-WATANABE CORRESPONDING BLOCKS OF FINITE GROUPS WITH NORMAL DEFECT GROUPS

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Abstract

Harris proved that there is an indecomposable bimodule with a trivial source which induces a Morita equivalence between Glauberman-Watanabe corresponding block algebras of finite groups with normal defect groups and the Glauberman correspondence of characters in corresponding blocks. We note an implication of the Puig correspondence in the conext of the Glauberman-Watanabe correspondence and then, using this, show Harris's theorem in two ways.

1. Introduction

In this article, for a prime p, let $(\mathcal{K}, \mathcal{O}, k)$ be a p-modular system where $\mathcal{O}$ is a complete discrete valuation ring having an algebraically closed residue field k of characteristic p and having a quotient field $\mathcal{K}$ of characteristic zero which will be assumed to be large enough for any of finite groups we consider in this article.

Let G be a finite group and $\mathcal{S}$ a solvable group such that $\mathcal{S}$ acts on G via automorphism and $(|G|, |\mathcal{S}|) = 1$.

Glauberman showed in [13] that there is a bijective correspondence with sign between the set of $\mathcal{S}$-invariant irreducible characters of G and the set of irreducible characters of $G^{\mathcal{S}} = C_G(\mathcal{S})$, called the *Glauberman correspondence* of ($\mathcal{K}$-)characters.

Watanabe began in [36] a block-theoretical study of the Glauberman correspondence, and gave a (p-)block correspondence under suitable assumptions, called *Glauberman-Watanabe correspondence*: she proved that if an $\mathcal{S}$-invariant block b of G has a defect group D centralized by $\mathcal{S}$, then all irreducible characters in b are $\mathcal{S}$-invariant, and by the Glauberman correspondence, all of them are mapped bijectively to the irreducible characters belonging to a single block $w(b)$ of $G^{\mathcal{S}}$ having D as a defect group and whose Brauer category is equivalent to that of b, and that Glauberman correspondence with sign gives a *perfect isometry* between the additive group of generalized characters in b and in $w(b)$. In fact, an existence of *isotypy* is proved. Here for the notions of perfect isometry and isotypy, see [6].

So, it is desirable to give a ring-theoretical explanation for this.

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In fact, until now, the case where the group is p-solvable and the case where the block has a normal defect group have been treated. For a p-solvable case, see [18] and [16]. Koshitani and Michler proved in [22] that Glauberman-Watanabe corresponding block algebras over k with normal defect groups are Morita equivalent. In fact, Koshitani noted in [21] that they are Puig equivalent, that is, having isomorphic source algebras, see [27, Definition 3.2] and hence Morita equivalent over $\mathcal{O}$ by [29, Lemma 7.8]. Then, Harris showed in [14] that there is an indecomposable $\mathcal{O}[G^{\mathcal{S}} \times G]$-module with a trivial sourc realizing a Morita equivalence and inducing the Glauberman correspondence. Recall that if the bimodule inducing a Morita equivalence between block algebras has a trivial source, then these blocks are Puig equivalent, see [31], and an isotypy is induced in the character level, see [32].

In this article, we point out that above Harris's result also follows from the Puig's theory as described in [27], [28] and [30]. In fact, we show in Theorem 4.9 that, taking as $\mathcal{S}$ a cyclic group S of prime order q (by induction, it suffices to consider this case, see Theorem 2.1 (i), (ii) and Theorem 4.11), for an S-invariant block b of G with an S-centralized normal defect group, there is some pointed group, see [27, Definition 1.1], G^S_β of $\mathcal{O}Gb$ viewed as an interior G-algebra, see [27, Definition 3.1], such that the block algebra $\mathcal{O}G^S w(b)$ is isomorphic to the embedded algebra $(\mathcal{O}Gb)_\beta$, see [30, 4.2], of G^S_β in $\mathcal{O}Gb$ as interior G^S-algebras. Then, since an indecomposable direct summand of $\mathcal{O}Gb{\downarrow}_{G^S \times G}$, which belongs to the only isomorphism class whose multiplicity in $\mathcal{O}Gb{\downarrow}_{G^S \times G}$ is not divided by q, induces a Morita equivalence between $\mathcal{O}Gb$ and $(\mathcal{O}Gb)_\beta$, it cleary induces the Glauberman correspondence, see Theorem 2.1 (3), and hence Harris's result follows.

This article consists of the following:

In Section 2, we recall the correspondences of Glauberman and Watanabe.

In Section 3, we describe an implication of the Puig correspondence quoted in Theorem 3.1 in the context of the Glauberman-Watanabe correspondence. Then as a special case we note in Corollaries 3.6 and 3.7, for $\mathcal{R} \in \{\mathcal{O}, k\}$, a vertex preserving correspondence between the indecomposable S-invariant $\mathcal{R}G$-modules and the indecomposable $\mathcal{R}G^S$-modules with some properties, characterized by the multiplicities, depending on Barker's investigations. When G^S contains a normalizer of a source, corresponding modules are the Green corresponding modules, see [35, Section 20].

In Section 4, we specialize to group algebras with normal defect groups, and, in Corollary 4.10, the Harris's result stated above is deduced by applying Puig's theorems, in particular, the structure theorem as an $(\mathcal{O}D, \mathcal{O}D)$-bimodule of the source algebra of a block with a normal defect group D.

In Section 5, we note the compatibility of the results in Section 4 with an observation by Okuyama in [26]. In fact, we note that Corollary 4.10 also follows from Broué's theorem on the Morita equivalence and the relation $(*)$ in Section 5 given in [26] with the interpretation $(**)$ in Section 5 of $(*)$ given in [34], see Remark 5.1.

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Notation and terminology. We will cite necessary facts from textbooks [19], [24] and [35] rather than original articles, and in particular, we refer to [35] for Puig's theory.

For finite groups G and S, if S acts on G via group automorphisms, we can consider the semi-direct product $G \rtimes S$ determined by that action, and denote it by GS. Let H be a subgroup of G. We denote by $\Delta H = \{(h, h) \in G \times G \mid h \in H\}$ the diagonal subgroup of $G \times G$. If S stabilizes H, the centralizer of S in H is denoted by H^S. We denote by $[H \backslash G]$ a set of left coset representatives of H in G. We denote by $G_{p'}$ the set of elements of G with the order coprime to p. A cyclic group of order r is denoted by C_r.

By a character, we always mean an ordinary character over $\mathcal{K}$. Denote by $\mathrm{Irr}(G)$ the set of all irreducible characters of G and by $\mathrm{Irr}(G)^S$ the set of all S-invariant irreducible characters of G. For $\theta \in \mathrm{Irr}(H)$, we denote $\mathrm{Irr}(G \mid \theta) = \{\chi \mid \chi \in \mathrm{Irr}(G)$ such that $[\theta, \chi{\downarrow}^G_H] \neq 0\}$, where $[\,\cdot\,,\,\cdot\,]$ is the usual inner product of characters. When $(|G|, |S|) = 1$, for $\phi \in \mathrm{Irr}(G)^S$, there exists a uniquely determined extension $\hat{\phi} \in \mathrm{Irr}(GS \mid \phi)$ of ϕ satisfying the condition $S \subset \mathrm{Ker}(\det(\hat{\phi}))$, called the *canonical extension* of ϕ, see [19, Lemma 13.3]. When S is cyclic, $\hat{S}$ acts on $\mathrm{Irr}(GS)$, see [11, Proposition 1.15 and (1.16)], where $\hat{S} = \mathrm{Hom}(S, \mathcal{K}^\times) \simeq S$ is the dual group of S, whose elements will be identified with the elements of $\mathrm{Irr}(S)$. Above action is denoted multiplicatively.

For a ring R, we denote by $R^\times$ the multiplicative group consisting of units of R, by $J(R)$ the Jacobson radical of R and by $\mathrm{Irr}(R)$ the set of all irreducible characters of R. Let $\mathcal{R} \in \{\mathcal{O}, k\}$ and $\mathcal{R}' \in \{\mathcal{K}, \mathcal{O}, k\}$. We denote by $\mathcal{R}'G$ the group algebra of G over $\mathcal{R}'$.

By modules, we mean $\mathcal{R}'$-free finitely generated left modules. For modules V and W, denote $W \mid V$ if W is isomorphic to a direct summand of V. For an $(\mathcal{R}'G, \mathcal{R}'H)$-bimodule X, we view it as an $\mathcal{R}'[G \times H]$-module in the usual way: $(g, h) \cdot x = g \cdot x \cdot h^{-1}$ where $g \in G$, $h \in H$ and $x \in X$. (We use $\cdot$ for the action of elements of group algebra on modules over that group algebra.) For a normal subgroup N of G (denoted $N \lhd G$), an $\mathcal{R}'N$-module Y and $g \in G$, we denote by Y^g the (g-)*conjugate* $\mathcal{R}'N$-module, see [35, Example 10.10]. Conjugate modules are defined also for the modules over twisted group algebras, see [10, (5.27)]. For a kG-module U, we denote by $P(U)$ the projective cover of U.

By a *block* of G or $\mathcal{R}G$, we mean a primitive idempotent b of $Z(\mathcal{R}G)$. The set of blocks of $\mathcal{R}G$ with a defect group D is denoted by $\mathrm{Bl}_{\mathcal{R}}(G \mid D)$. For a block b_0 of $\mathcal{O}G$, denote by $\overline{b_0}$ the block of kG given by the canonical image of b_0, and denote $\mathrm{Irr}(b_0) = \{\phi \in \mathrm{Irr}(G) \mid \phi(b_0) \neq 0\}$, which is also called a block and whose elements are called the characters of G *in* b_0 or $\overline{b_0}$.

We refer to [35, Section 10] for the notion of an (*interior*) G-*algebra* A over $\mathcal{R}$. We denote by $\mathrm{Res}^G_H A$ the (interior) H-algebra given by the restriction to H of the structural map. A^H is a subalgebra of A consisting of H-fixed elements. If 1_A is the unique idempotent of A^G, A is called a *primitive* G-algebra. We view group algebra $\mathcal{R}G$, block algebras $\mathcal{R}Gb$ and $\mathcal{R}$-endomorphism rings of $\mathcal{R}G$-modules as interior G-algebras in the usual way, see [35, Examples 10.3 and 10.6]. Considering $\mathcal{R}Gb$ as an $(\mathcal{R}G, \mathcal{R}G)$-bimodule, we view $\mathrm{End}_{\mathcal{R}[1\times G]}(\mathcal{R}Gb)$ as an interior G-algebra using the left $\mathcal{R}G$-module structure of $\mathcal{R}Gb$. Then, as is well-known, we have an isomorphism of interior G-algebras

$$(\sharp) \qquad \mathcal{R}Gb \to \mathrm{End}_{\mathcal{R}[1\times G]}(\mathcal{R}Gb), \quad x \mapsto \varphi_x$$

where φ_x is the $\mathcal{R}$-endomorphism of $\mathcal{R}Gb$ given by the left action of $x \in \mathcal{R}Gb$.

We refer to [28, Section 5] for the notions a *$k^\times$-group* $\hat{G}$, a *$k^\times$-subgroup* $\hat{H}$, a *$k^\times$-group homomorphism* and the *$k^\times$-group algebra* $\mathcal{R}_*\hat{G}$ over $\mathcal{R}$. See also [35, Examples 10.4, 10.8 and p. 407]. We also denote by $\mathcal{K}_*\hat{G}$ an algebra over $\mathcal{K}$ defined by $\mathcal{K} \otimes_{\mathcal{O}} \mathcal{O}_*\hat{G}$. With the obvious grading, $\mathcal{R}'_*\hat{G}$ becomes a twisted group algebra of G over $\mathcal{R}'$. For an $\mathcal{R}'_*\hat{G}$-module V and an $\mathcal{R}'_*\hat{H}$-module V', we denote by $V{\downarrow}^G_H$ the restriction of V to $\mathcal{R}'_*\hat{H}$-module and by $V'{\uparrow}^G_H$ the induced $\mathcal{R}'_*\hat{G}$-module $\mathcal{R}'_*\hat{G} \otimes_{\mathcal{R}_*\hat{H}} V'$. Similar notations for $\mathcal{K}$-characters. We denote by $m(V', V)$ the *multiplicity* of V' in $V{\downarrow}^G_H$, that is, the number of modules isomorphic to V' in a direct sum decomposition of $V{\downarrow}^G_H$, and by $n(V, V')$ the multiplicity of V in $V'{\uparrow}^G_H$. We use the similar notations for modules over ordinary group algebras, see [28, 5.12.2].

For a G-algebra A over $\mathcal{R}$, the *trace map* from A^H to A^G is defined by $\mathrm{Tr}^G_H(a) = \sum_{g\in[H\backslash G]} a^g$ for $a \in A^H$ and, for a p-subgroup P of G, the *Brauer homomorphism* Br^A_P is defined by the canonical epimorphism $A^P \to A(P) = A^P \big/ \big\{\sum_Q \mathrm{Tr}^P_Q(A^Q) + J(\mathcal{R})A^P\big\}$ where Q runs over strict subgroup of P, see [35, Section 11]. When $A = \mathcal{R}G$, $A(P)$ will be identified with $kC_G(P)$, see [35, Proposition 37.5].

We refer to [35, Sections 13 and 14] for the notions *pointed groups* of A, the action of G on them by the conjugation, their *localness* and the relations $K_\sigma \ge H_\beta$ and K_σ *pr* H_β between pointed groups K_σ and H_β of A. For $f \in \beta$, fAf has the obvious H-algebra (interior H-algebra if A has an interior H-algebra structure) structure. A_β is the primitive (interior) H-algebra isomorphic to fAf, called an *embedded algebra* of H_β in A. For its uniqueness, see [35, Section 13]. When a local pointed group P_γ of A such that $H_\beta \ge P_\gamma$ is maximal with respect to the relation $\ge$, P_γ is called a *defect pointed group* of H_β, see [35, Section 18] and P is called a *defect group* of H_β. For the *multiplicity* $m(\beta, \sigma)$ of the point β in σ, we refer to [4, Section 2]. In particular, if A is a primitive G-algebra and $\alpha = \{1_A\}$, then $m(\beta, \alpha)$ is the number of occurrences of idempotents of β in a primitive idempotent decomposition of 1_A in A^H, see [35, p. 28].

We denote by $A(H_\beta)$ the *multiplicity algebra* and by $V_A(H_\beta)$ the *multiplicity module*, of the pointed group H_β of A, see [35, Section 13]. Denote by π_β the canonical epimorphism from A^H to $A(H_\beta)$. For another G-algebra A', if $\mathcal{F}\colon A \to A'$ is an *embedding* of G-algebra, see [35, Section 12], then $\mathcal{F}$ induces an injection $H_\beta \mapsto H_{\beta'}$, of pointed groups of A into pointed groups of A', see [35, Proposition 15.1]. $\mathcal{F}$ also induces an embedding of $N_G(H_\beta)/H(= N_G(H_{\beta'})/H)$-algbra $A(H_\beta) \to A'(H_{\beta'})$ described in [35, Proposition 15.3] and an isomorphism of $k^\times$-groups from $\widehat{N_G(H_{\beta'})/H}$, which is defined by the action of $N_G(H_{\beta'})/H$ on $A'(H_{\beta'})$, to $\widehat{N_G(H_\beta)/H}$, which is defined by the action of $N_G(H_\beta)/H$ on $A(H_\beta)$, described in [35, Proposition 15.4]. Through these, $V_A(H_\beta)$ will be identified with a direct summand of $V_{A'}(H_{\beta'})$, see [35, Proposition 15.5]. When A is a primitive G-algebra, a defect pointed group P_γ of $G_{\{1_A\}}$ is called a *defect pointed group* of A, P is called a *defect group* of A, $i \in \gamma$ is called a *source idempotent* of A and the embedded algebra A_γ of P_γ in A is called a *source algebra* of A. Moreover, $V_A(P_\gamma)$ is called the *defect multiplicity module* of A, which is indecomposable projective $k_*\hat{\overline{N}}$-module, where $\overline{N} = N_G(P_\gamma)/P$, see [35, Section 19]. A *canonical character* of a block b is the character of $DC_G(D)/D$ corresponding to the simple projective module $V_{\mathcal{R}Gb}(D_\delta)\downarrow^{N_G(D_\delta)/D}_{DC_G(D)/D}$, where D_δ is a defect pointed group of $\mathcal{R}Gb$, see [35, Section 37].

When A is an interior G-algebra over $\mathcal{R}$, its structural homomorphism φ from G to $A^\times$ can be extended to $\varphi\colon \mathcal{R}G \to A$ and we denote $y \cdot a = \varphi(y)a$ and $a \cdot y = a\varphi(y)$ for $y \in \mathcal{R}G$ and $a \in A$. Moreover if A is primitive, then there is a unique block b of $\mathcal{R}G$ such that $b \cdot 1_A = 1_A$, and in this case, we say that A is *in* b, see [30, 4.1].

When an $\mathcal{R}$-algebra E is isomorphic to $\mathrm{End}_{\mathcal{R}}(L)$ for some $\mathcal{R}$-module L, we also use $\cdot$ for the obvious action of elements of E on L.

2. The correspondences of Glauberman and Watanabe

In this section, we recall the correspondences of Glauberman and Watanabe.

Theorem 2.1 (Glauberman [13]). *For any pair $(G, \mathcal{S})$ where G is a finite group and $\mathcal{S}$ is a finite solvable group acting on G such that $(|G|, |\mathcal{S}|) = 1$, there exists a uniquely determined bijective map $\pi(G, \mathcal{S})\colon \mathrm{Irr}(G)^{\mathcal{S}} \to \mathrm{Irr}(G^{\mathcal{S}})$ satisfying the following conditions*:

(i) *For $\mathcal{T} \lhd \mathcal{S}$, $\mathrm{Irr}(G)^{\mathcal{S}}$ is mapped bijectively to $\mathrm{Irr}(G^{\mathcal{T}})^{\mathcal{S}}$ by $\pi(G, \mathcal{T})$.*

(ii) *In the situation of (i), $\pi(G, \mathcal{S}) = \pi(G^{\mathcal{T}}, \mathcal{S}/\mathcal{T}) \circ \pi(G, \mathcal{T})$.*

(iii) *If $\mathcal{S}$ is a q-group for some prime q, then, for $\phi \in \mathrm{Irr}(G)^{\mathcal{S}}$, $\pi(G, \mathcal{S})(\phi)$ is a unique constituent of $\phi\downarrow^G_{G^{\mathcal{S}}}$ with a multiplicity m_ϕ not divisible by q. In fact, ϕ determines a sign $\epsilon_\phi \in \{\pm 1\}$ such that $m_\phi \equiv \epsilon_\phi \pmod q$.*

The correspondence in Theorem 2.1 is called the *Glauberman correspondence* of ($\mathcal{K}$-)characters.

REMARK 2.2. For an $\mathcal{S}$-invariant simple projective kG-module W, there is an $\mathcal{S}$-invariant irreducible and projective $\mathcal{O}G$-lattice $W_{\mathcal{O}}$ such that $W \simeq W_{\mathcal{O}}/J(\mathcal{O})W_{\mathcal{O}}$. Thus we can consider the Glauberman correspondent $\pi(G, \mathcal{S})(\xi)$ of the character ξ afforded by W. A set $\{\pi(G, \mathcal{S})(\xi)\}$ forms a block of $G^{\mathcal{S}}$ with defect 0, see [25, Theorem in p. 517], so that there exists an irreducible and projective $\mathcal{O}G^{\mathcal{S}}$-lattice $W_{\mathcal{O}}'$ which affords the character $\pi(G, \mathcal{S})(\xi)$. We call a simple projective $kG^{\mathcal{S}}$-module isomorphic to $W_{\mathcal{O}}'/J(\mathcal{O})W_{\mathcal{O}}'$ a *Glauberman correspondent* of W.

By (i) and (ii) in Theorem 2.1, it suffices to consider the case where $\mathcal{S}$ is cyclic of prime order q, see Theorem 4.11. In the following, we always assume Condition 2.3 below.

CONDITION 2.3. G is a finite group and $S \simeq C_q$ acts on G where q is a prime not dividing the order of G. Set $\Gamma = GS$. λ is a non-trivial element of $\hat{S}$.

In fact, the correspondence in Theorem 2.1 is determined by iterated application of the correspondence in [19, Theorem 13.6, Definition 13.7] which is defined when an acting group is cyclic, see [19, Definitions 13.12 and 13.20, Theorem 13.18] (note also that, with the notations in Theorem 2.1 and Proposition 2.4, $\epsilon_\phi = \delta_\phi$ under the common assumptions, see [19, Theorem 13.14 (b)]). For a later use, we cite the following form of the Glauberman correspondence under Condition 2.3, which follows immediately from [19, Theorem 13.6], see also [11]:

Proposition 2.4 (Glauberman [13]). *For $\phi \in \mathrm{Irr}(G)^S$, let $\hat{\phi} \in \mathrm{Irr}(\Gamma)$ be the canonical extension of ϕ when q is odd, and $\hat{\phi} \in \mathrm{Irr}(\Gamma)$ an arbitrary extension of ϕ when $q = |S| = 2$. Then there is a sign δ_ϕ which makes the following equation of generalized characters hold*:

$$(\hat{\phi} - \lambda\hat{\phi})\downarrow^{\Gamma}_{G^S S} = \delta_\phi(\pi(G, S)(\phi) \times 1 - \lambda(\pi(G, S)(\phi) \times 1)).$$

Note that, when $q = 2$, above δ_ϕ depends on the choice of $\hat{\phi}$. In the above proposition, since $G^S S$ is a direct product of G^S and S, characters of $G^S S$ are denoted in the form of the product of characters of G^S and S. Note that $\lambda(\pi(G, S)(\phi) \times 1) = \pi(G, S)(\phi) \times \lambda$. In the following, we will use the notation of product for characters of direct products, too.

Watanabe began in [36] a p-block theory of the Glauberman correspondence under the condition that a defect group is centralized by S, which will be always assumed in this article too.

Theorem 2.5 (Watanabe [36]). *If an S-invariant block b of $\mathcal{O}G$ has a defect group D centralized by S, then the following holds*:

(1) *All charcters in b are S-invariant.*

(2) *There is a block $w(b)$ of $\mathcal{O}G^S$ with a defect group D such that* $\mathrm{Irr}(w(b)) = \{\pi(G, S)(\phi) \mid \phi \in \mathrm{Irr}(b)\}$.

(3) *There is a perfect isometry $\mathbb{Z}\mathrm{Irr}(b) \simeq \mathbb{Z}\mathrm{Irr}(w(b))$ mapping $\phi \in \mathrm{Irr}(b)$ to $\delta_\phi \pi(G, S)(\phi)$, where δ_ϕ is the sign described in* Proposition 2.4 (*in the case of $q = 2$, choosing $\hat{\phi}$ in*

Proposition 2.4 *so that* $\{\hat{\phi} \mid \phi \in \mathrm{Irr}(b)\} = \mathrm{Irr}(\hat{b})$ *for a block* $\hat{b}$ *of* Γ *covering* b). *In fact, the Glauberman correspondence induces an isotypy.*

The correspondence in Theorem 2.5 is called the *Glauberman-Watanabe correspondence* of (p-)blocks. We also define $w(\overline{b})$ by $\overline{w(b)}$.

Note that if S centralizes a Sylow p-subgroup of G, then the Glauberman-Watanabe correspondence induces a one-to-one correspondence between the set of S-invariant blocks of G and the set of blocks of G^S. In particular, the set of S-invariant characters of G with defect 0 and the set of characters of G^S with defect 0 correspond bijectively by the Glauberman correspondence.

The following is included in Theorem 2.5 (3).

Proposition 2.6 (Watanabe [36]). *With the notations of* Theorem 2.5, *let* $\xi \in \mathrm{Irr}(DC_G(D)/D)$ *be a canonical character of* b. *Then* ξ *is* S*-invariant and its Glauberman correspondent is a canonical character of* $w(b)$.

In the situation of Theorem 2.5, by [36, Proposition 1] and [8], b is covered by q distinct isomorphic blocks of Γ in the sense of [1] or [17]. In particular:

Lemma 2.7 (Dade [8], Watanabe [36]). *For* b *as in* Theorem 2.5 *and any block* $\hat{b}$ *of* Γ *covering* b, *it holds that* $\mathrm{Res}_G^\Gamma(\mathcal{O}\Gamma\hat{b}) \simeq \mathcal{O}Gb$ *as interior* G*-algebras and* $\mathcal{O}\Gamma\hat{b}{\downarrow}_{G\times G}^{\Gamma\times\Gamma} \simeq \mathcal{O}Gb$ *as* $\mathcal{O}[G\times G]$*-modules.*

We also recall the following, see for example [22, Lemma 2.2]:

Lemma 2.8. $N_G(P) = N_{G^S}(P)C_G(P)$ *for an* S*-centralized subgroup* P *of* G.

3. The correspondence of Puig

In this section, we note an implication of the Puig correspondence in the context of the Glauberman-Watanabe correspondence. Firstly, we recall the *Puig correspondence* of points or pointed groups and a particular case of Barker's result, see [30, 1.4.1], [4, Remarks 4.1 and 7.1, Propositions 4.4, 4.5, 7.4 and 7.5] and [35, Theorem 19.1]:

Theorem 3.1 (Puig [27] [28]). *Let* A *be a* G*-algebra. For any local pointed group* P_γ *and any subgroup* H *of* G *containg* P, *the correspondence mapping any primitive idempotent* j *of* A^H *on the* $k_*\hat{\overline{H}}$*-module* $\pi_\gamma(j)\cdot V_A(P_\gamma)$ *induces a bijection from the set of points* ϵ *of* H *on* A *such that* P_γ *is a defect pointed group of* H_ϵ *onto the set of isomorphism classes of projective indecomposable direct summands of* $V_A(P_\gamma){\downarrow}_{\overline{H}}^{\overline{G}}$ *where* $\overline{H} = N_H(P_\gamma)/P$. *Moreover, for pointed groups* K_σ *and* H_ϵ *of* A *with a defect pointed group* P_γ, *and* $l \in \sigma$ *and* $j \in \epsilon$,

(1) *$K_\sigma \geq H_\epsilon$ if and only if $\pi_\gamma(j)\cdot V_A(P_\gamma) \mid (\pi_\gamma(l)\cdot V_A(P_\gamma))\downarrow^{\overline{K}}_{\overline{H}}$, and $m(\epsilon,\sigma) = m(s_\gamma(j)\cdot V_A(P_\gamma), s_\gamma(l)\cdot V_A(P_\gamma))$.*
(2) *K_σ pr H_ϵ if and only if $\pi_\gamma(l)\cdot V_A(P_\gamma) \mid (\pi_\gamma(j)\cdot V_A(P_\gamma))\uparrow^{\overline{K}}_{\overline{H}}$. When A is an interior G-algebra constructed from the endomorphism ring of an $\mathcal{R}G$-module L, $n(l\cdot L, j\cdot L) = n(\pi_\gamma(l)\cdot V_A(P_\gamma), \pi_\gamma(j)\cdot V_A(P_\gamma))$, and moreover if $\pi_\gamma(l)\cdot V_A(P_\gamma)$ and $\pi_\gamma(j)\cdot V_A(P_\gamma)$ are simple $k_*\hat{\overline{K}}$- and $k_*\hat{\overline{H}}$-modules respectively, then $n(l\cdot L, j\cdot L) = m(j\cdot L, l\cdot L)$ (Barker).*

With the above notations, if $V_A(P_\gamma)$ is simple (and projective) $k_*\hat{\overline{G}}$-module, we can apply the Glauberman correspondence by considering an appropriate covering group of the twisted group algebra $k_*\hat{\overline{G}}$, which is a well-known argument, see for example [16] and [23], and get some informations on pointed groups on A by the Puig correspondence quoted above. Note that we may use the twisted group algebra version of the Glauberman correspondence described in [11].

For V and V' as in Lemma 3.2 (1) below, we call a $k_*\widehat{G^S}$-module isomorphic to V' a *Glauberman correspondent* of V. When A is an S-invariant ordinary block algebra with defect 0, this usage coincides with that of Remark 2.2. In Lemma 3.2 (3), we view $V\downarrow^G_N$ as a kN-module in the canonical way, see [35, Example 10.9].

Lemma 3.2. *Let $\tilde{A}$ be a Γ-algebra which is simple as a k-algebra.* (*Then the unique simple $\tilde{A}$-module $\tilde{Z}$ has the $k_*\hat{\Gamma}$-module structure associated with the Γ-algebra structure of $\tilde{A}$, see* [35, *Example* 10.8].) *Let N be an S-invariant normal subgroup of G such that $G = G^S N$. Assume that the NS-algebra structure of $\tilde{A}$ is interior, whose structural map commutes with the action of Γ. Then*:
(1) *Assume that a direct summand $\tilde{V}$ of $\tilde{Z}$ is a simple and projective $k_*\hat{\Gamma}$-module such that $V = \tilde{V}\downarrow^\Gamma_G$ is a simple* (*and projective*) *$k_*\hat{G}$-module. Then there exists, unique up to isomorphism, an indecomposable direct summand V' of $V\downarrow^G_{G^S}$ such that $q \nmid m(V', V)$. Furthermore, V' is a simple* (*and projective*) *$k_*\widehat{G^S}$-module, and $m(V', V) \equiv \pm 1 \pmod{q}$.*
(2) *Assume that a direct summand U of $\tilde{Z}\downarrow^\Gamma_{G^S}$ is simple and projective $k_*\widehat{G^S}$-module and there exists an S-invariant simple* (*and projective*) *direct summand U'' of $U\uparrow^G_{G^S}$ such that $q \nmid n(V, V')$* (*see the first paragraph of the proof of* (2) *for this condition*). *Then $n(U'', U) \equiv \pm 1 \pmod{q}$, and any indecomposable S-invariant direct summand of $U\uparrow^G_{G^S}$ which is not isomorphic to U'' has a multiplicity divisible by q.*
(3) *Let W be an indecomposable direct summand of $V\downarrow^G_N$ for V in* (1). *Then*:
 (A) *W is a simple, projective and S-invariant kN-module. The isomorphism classes of the indecomposable direct summands of $V\downarrow^G_N$ are the isomorphism classes of g-conjugate of W where g runs over G* (*so that are S-invariant*).
 (B) *Let V' and W' be Glauberman correspondents of V and W, respectively. Then*:
  (a) *The isomorphism classes of the indecomposable direct summands of $V'\downarrow^{G^S}_{N^S}$ are the isomorphism classes of c-conjugate of W' where c runs over G^S.*

(b) *If* $V{\downarrow}^G_N$ *is indecomposable* (*so that* $W = V{\downarrow}^G_N$), *then* $V'{\downarrow}^{G^S}_{N^S}$ *is indecomposable* (*so that* $W' \simeq V'{\downarrow}^{G^S}_{N^S}$).

Proof. The Γ-algebra structure of $\tilde{A}$ determines a $k^\times$-group $\hat{\Gamma}$ with a structural short exact sequence: $1 \to k^\times \xrightarrow{\tau} \hat{\Gamma} \xrightarrow{d} \Gamma \to 1$. Recall that $\hat{\Gamma}$ may be taken explicitly as $\hat{\Gamma} = \{(a, x) \in \tilde{A}^\times \times \Gamma \mid a'^a = a'^x \text{ for any } a' \in \tilde{A}\}$ with $d((a, x)) = x$ for $(a, x) \in \hat{\Gamma}$ and $\tau(\sigma) = (\sigma 1_{\tilde{A}}, 1_\Gamma)$ for $\sigma \in k^\times$. There is a split monomorphism $\iota\colon NS \to \widehat{NS}$ of $d|_{\widehat{NS}}$ mapping y to $(y \cdot 1_{\tilde{A}}, y)$. Then $\iota(NS)$ is a normal subgroup of $\hat{\Gamma}$. Since $\iota(NS)$ intersects trivially with $\tau(k^\times)$, the canonical epimorphisms of groups $\pi\colon \hat{\Gamma} \to \hat{\Gamma}/\iota(NS)$ and $\overline{\pi}\colon \Gamma \to \Gamma/NS$ determine the $k^\times$-group $\hat{\Gamma}/\iota(NS)$ whose structural short exact sequence is given by the following commutative diagram:

$$\begin{tikzcd}
1 \arrow[r] & k^\times \arrow[r, "\tau"] \arrow[d, equal] & \hat{\Gamma} \arrow[r, "d"] \arrow[d, "\pi"] & \Gamma \arrow[r] \arrow[d, "\overline{\pi}"] & 1 \\
1 \arrow[r] & k^\times \arrow[r, "\overline{\tau}"] & \hat{\Gamma}/\iota(NS) \arrow[r, "\overline{d}"] & \Gamma/NS \arrow[r] & 1.
\end{tikzcd}$$

There is a subgroup $(\hat{\Gamma}/\iota(NS))^*$ of $\hat{\Gamma}/\iota(NS)$ satisfying $\overline{d}((\hat{\Gamma}/\iota(NS))^*) = \Gamma/NS$ with a central subgroup $\overline{Z} = \mathrm{Ker}(\overline{d}^*) = (\hat{\Gamma}/\iota(NS))^* \cap \overline{\tau}(k^\times) \simeq C_r$ where $\overline{d}^*$ is the restriction of $\overline{d}$ to $(\hat{\Gamma}/\iota(NS))^*$ and r is an integer determined by $|\Gamma/NS| = rp^n$ $(p \nmid r)$, see [35, Proposition 10.5]. Note that $q \nmid r$. Then $\Gamma^* = \pi^{-1}((\hat{\Gamma}/\iota(NS))^*)$ is a finite subgroup of $\hat{\Gamma}$ containing $\iota(NS)$, satisfying $d(\Gamma^*) = \Gamma$, and having a central subgroup $Z = \mathrm{Ker}(d^*) = \Gamma^* \cap \tau(k^\times) \simeq C_r$ where d^* is the restriction of d to Γ^*. For a subgroup H of Γ, define a subgroup H^* of Γ^* by $H^* = d^{*-1}(H)$. If H is contained in G and S-invariant, then $\iota(S)$ acts coprimely on H^* since $q \nmid r|H|$ and we have, for $(a_h, h) \in H^*$, $(s \cdot 1_{\tilde{A}}, s)^{-1}(a_h, h)(s \cdot 1_{\tilde{A}}, s) = (a_h^s, h^s) \in \Gamma^* \cap d^{-1}(H) = H^*$. Note that $\Gamma^* = G^*\iota(S)$. Moreover, $(H^*)^{\iota(S)} = (H^S)^*$. In fact, for $(a_c, c) \in (H^S)^*$, $(s \cdot 1_{\tilde{A}}, s)^{-1}(a_c, c)(s \cdot 1_{\tilde{A}}, s) = (a_c^s, c)$ is equivalent to (a_c, c) modulo Z, that is, $a_c^s = \mu a_c$ for some $\mu \in k^\times$ such that $\tau(\mu) \in Z$. But since the order of μ divides r, μ must be 1_k. Hence, $(H^*)^{\iota(S)} \geq (H^S)^*$. The converse inclusion is clear. The inclusion $H^* \to \hat{H}$ induces an isomorphism

$$\mathcal{R}'H^*e \simeq \mathcal{R}'_*\hat{H}, \tag{3.2.1}$$

where e is the central idempotent of $\mathcal{R}'Z$ corresponding to the faithful linear representation of Z determined by $z \mapsto \mu_z \in k^\times$ where $z = \tau(\mu_z) \in Z$, see [28, Proposition 5.15 (i)] or [10]. In the following, through the isomorphism (3.2.1), we will freely identify $\mathcal{R}'_*\hat{H}$-modules and $\mathcal{R}'H^*$-modules not annihilated by e. Similarly, we will identify the characters of $\mathcal{K}_*\hat{H}$ and the characters of $\mathcal{K}H^*$ covering ζ where $\zeta \in \mathrm{Irr}(Z)$ is the character corresponding to e, see [10, Proposition 8.1]. Note that any character which is a constituent of the characters obtained by restriction or induction of an irreducible character of H^* covering ζ to groups containing Z covers ζ.

Let $\tilde{V}_{\mathcal{O}}$ be the $\mathcal{O}_*\hat{\Gamma}$-module lifting $\tilde{V}$, and let ψ be the $\mathcal{K}$-character of $\mathcal{K}_*\hat{\Gamma}$ corresponding to $\mathcal{K}\otimes_{\mathcal{O}}\tilde{V}_{\mathcal{O}}$. Then $\phi=\psi\downarrow^{\Gamma}_{G}$ is the S-invariant irreducible character corresponding to V. Note that, by [10, Proposition 9.2], the characters of $\mathcal{K}_*H$ with defect 0, see [10, Definition 9.1], correspond to the characters of H^* with defect 0 covering ζ.

We also denote λ for $\lambda\circ\iota|_S^{-1}\in\widehat{\iota(S)}$ (viewing $\iota|_S$ a map to $\iota(S)$). Then with the notations of Proposition 2.4 for appropriate groups, if $\delta_\phi=1$ (in the case of $q=2$, when we take $\hat{\phi}=\psi$), it holds that

$$\psi\downarrow^{\Gamma^*}_{G^{*\iota(S)}\iota(S)}-\psi'=\lambda\left(\psi\downarrow^{\Gamma^*}_{G^{*\iota(S)}\iota(S)}-\psi'\right)$$

where $\psi'\in\mathrm{Irr}(G^{*\iota(S)}\iota(S))$ is some extension of $\phi'=\pi(G^*,\iota(S))(\phi)$ (when $q=2$, $\psi'=\phi'\times 1$). Note that $G^{*\iota(S)}\iota(S)=(G^SS)^*$. Hence, since λ acts regularly on $\mathrm{Irr}(G^{*\iota(S)}\iota(S))$, blocks of $G^{*\iota(S)}$ are covered by q isomorphic blocks of $G^{*\iota(S)}\iota(S)$ and projective modules are uniquely determined by its corresponding characters, see [24, III, Exercise 16], we see that

$$\tilde{V}\downarrow^{\Gamma}_{G^SS}\simeq\tilde{V}'\oplus\left(\bigoplus_{i=0}^{q-1}\lambda^i\mathcal{X}\right) \tag{3.2.2}$$

for a simple projective $k_*\widehat{G^SS}$-module $\tilde{V}'$ corresponding to ψ' and some projective $k_*\widehat{G^SS}$-module $\mathcal{X}$. Here $\lambda^i\mathcal{X}$ is a projective $k_*\widehat{G^SS}$-module corresponding to $\lambda^i\Phi\in\mathrm{Irr}(G^{*\iota(S)}\iota(S))$, denoting $\Phi\in\mathrm{Irr}(G^{*\iota(S)}\iota(S))$ the character corresponding to $\mathcal{X}$. We use similar notations in (3.2.3) below. If $\delta_\phi=-1$, then

$$\psi\downarrow^{\Gamma^*}_{G^{*\iota(S)}\iota(S)}+\psi'=\lambda\left(\psi\downarrow^{\Gamma^*}_{G^{*\iota(S)}\iota(S)}+\psi'\right)$$

and so we see that, for some projective $k_*\widehat{G^SS}$-module $\mathcal{Y}$,

$$\tilde{V}\downarrow^{\Gamma}_{G^SS}\simeq\left(\bigoplus_{j=1}^{q-1}\lambda^j\tilde{V}'\right)\oplus\left(\bigoplus_{i=0}^{q-1}\lambda^i\mathcal{Y}\right). \tag{3.2.3}$$

Hence, we see that (1) holds for a module V' corresponding to $\tilde{V}'\downarrow^{G^S\times S}_{G^S}$. We also refer to the proof of [11, Theorem 6.13] for the above argument.

We show (2). Denote by υ the character of $G^{*\iota(S)}$ with defect 0 corresponding to the simple projective $kG^{*\iota(S)}$-module U, and let $\upsilon''=\pi(G^*,\iota(S))^{-1}(\upsilon)$. We see that the condition that there exists U'' as in the statement is equivalent to the condition that υ'' is a character of G^* with defect 0. (This is also equivalent to the condition that υ'' belongs to a block of G^* which has a defect group centralized by $\iota(S)$.) In this case, U'' is a simple projective kG^*-module corresponding to υ'' and $n(U'',U)=\left[\upsilon'',\upsilon\uparrow^{G^*}_{G^{*\iota(S)}}\right]=\left[\upsilon,\upsilon''\downarrow^{G^*}_{G^{*\iota(S)}}\right]\equiv\pm1 \pmod q$.

Any $\iota(S)$-invariant direct summand of $U\uparrow_{G^{*\iota(S)}}^{G^*}$ can be written as $P(T)$ for some $\iota(S)$-invariant simple kG^*-module T. By [33, Theorem 3],

$$n(P(T), U) = m(P(U), T).$$

Since $U = P(U)$, it suffices to show that $q \mid m(U, T)$ for any $\iota(S)$-invariant simple kG^*-module T such that $T \not\simeq U''$.

Let ρ be the Brauer character of G^* corresponding to T and $\tilde{\rho}$ the function defined by $\tilde{\rho}(g^*) = \rho(g^*_{p'})$ where $g^* \in G^*$ and $g^*_{p'}$ is the p'-part of g^*. Then $\tilde{\rho}$ is a generalized character of G^* and $\tilde{\rho}|_{G^*_{p'}} = \rho$, see [24, III, Lemma 6.13]. Since $\tilde{\rho}$ is $\iota(S)$-invariant and generalized characters of G^* are uniquely expressed as linear combinations of elements of $\mathrm{Irr}(G^*)$, $\tilde{\rho}$ can be expressed as

$$\tilde{\rho} = \sum_i m_i \chi_i + \sum_j m_j \left(\sum_{s \in S} \theta_j^{\iota(s)} \right)$$

where χ_i are $\iota(S)$-invariant characters of G^*, θ_j are not $\iota(S)$-invariant characters of G^* and m_i, m_j are appropriate non-zero integers. Then, υ appears in $\chi_i\downarrow_{G^{*\iota(S)}}^{G^*}$ with a multiple of q, since χ_i and υ are not the Glauberman corresponding characters, see [24, III, Exercice 6.20]. On the other hand, υ appears with a multiple of q in $\sum_{s\in S} \theta_j^{\iota(s)}\downarrow_{G^{*\iota(S)}}^{G^*}$, since for any $s \in S$

$$\left[\upsilon, \theta_j^{\iota(s)}\downarrow_{G^{*\iota(S)}}^{G^*}\right] = \left[\upsilon^{\iota(s)^{-1}}, \left(\theta_j^{\iota(s)}\downarrow_{G^{*\iota(S)}}^{G^*}\right)^{\iota(s)^{-1}}\right] = \left[\upsilon, \theta_j\downarrow_{G^{*\iota(S)}}^{G^*}\right].$$

Hence, $\upsilon|_{G^{*\iota(S)}_{p'}}$ appears with a multiple of q in $\rho\downarrow_{G^{*\iota(S)}}^{G^*} = \left(\tilde{\rho}\downarrow_{G^{*\iota(S)}}^{G^*}\right)\Big|_{G^{*\iota(S)}_{p'}}$, and the assertion follows.

For (3), note that $G^* \triangleright N^* = Z \times \iota(N)$, $G^* = G^{*\iota(S)} N^*$ and $G^{*\iota(S)} \triangleright N^{*\iota(S)} = Z \times \iota(N^S)$. Then (A), (B) (a) and (B) (b) are just restatements of [35, Lemma 26.10] and [37, Corollary 2.4], [37, Lemma 5.3] and [20, Lemma 2.2] respectively, by the identification through the isomorphism (3.2.1) and the correspondences $\mathrm{Irr}(Z\iota(N) \mid \zeta) \to \mathrm{Irr}(\iota(N))$, $\zeta \times \xi \mapsto \xi$ and $\mathrm{Irr}(Z\iota(N^S) \mid \zeta) \to \mathrm{Irr}(\iota(N^S))$, $\zeta \times \xi' \mapsto \xi'$. □

Note that in Lemma 3.2, by the Frobenius reciprocity law, $n(V, V') = m(V', V)$ and $m(U, U'') = n(U'', U)$, and so V' satisfies the condition in (2) and $U \simeq V'$ if and only if $U'' \simeq V$.

By the definition of the canonical extension and Proposition 2.4, we see immediately the following:

Lemma 3.3. *Let q be odd. With the assumptions of* Lemma 3.2 (4) (B) (b) *and the notations in the proof of* Lemma 3.2, *the following are equivalent, denoting $\psi\downarrow_{(NS)^*}^{\Gamma^*} = \psi\downarrow_{Z\times\iota(NS)}^{\Gamma^*} = \zeta \times \eta$ and $\xi = \eta\downarrow_{\iota(N)}^{\iota(NS)}$:*

(1) $|S|^{-1}\sum_{s\in S}\overline{\iota(s)}$ *acts trivially on* $\tilde{V}'$ *where* $\overline{\iota(s)}$ *is the canonical image of* $\iota(s)\in\hat{\Gamma}$ *in* $k_*\hat{\Gamma}$.
(2) ψ *is the canonical extension of* ϕ.
(3) $\psi{\downarrow}^{\Gamma^*}_{(NS)^*}$ *is the canonical extension of* $\phi{\downarrow}^{G^*}_{N^*}$.
(4) η *is the canonical extension of* ξ.

Lemma 3.4. *Let A be a primitive interior G-algebra with a defect group P. Let H_ϵ be a pointed group of A. Assume $N_G(P)=N_H(P)C_G(P)$. Then H_ϵ has a defect group P if and only if H_ϵ has a defect pointed group P_ν where ν is any local point of P on A.*

Proof. This follows from the transitivity of defect pointed groups by the conjugation action, see [35, Corollary 18.6]. □

Proposition 3.5. *Let A be a primitive interior G-algebra over $\mathcal{R}$ which can be extended to an interior Γ-algebra $\tilde{A}$ and which has an S-centralized defect group P and a simple defect multiplicity module. Let $\alpha=\{1_A\}$ be a unique point of A^G. Then*:
(1) *There is a unique point β of A^{G^S} satisfying the following*:
 (i) *The pointed group G^S_β of A has P as a defect group.*
 (ii) $q\nmid m_\beta=m(\beta,\alpha)$.
In fact, m_β satisfies $m_\beta\equiv\pm1$ (mod q).
(2) ([4, Proposition 4.8]) G_α *pr* G^S_β.
(3) *If A is in b where b is a block of G having an S-centralized defect group D, then A_β is in $w(b)$.*

Proof. Recall from [36, Proposition 1] that there is an S-invariant defect pointed group P_γ of A by the transitivity under $N_G(P)$ of defect pointed groups of A with the group P, see [35, Corollary 18.6] and the Glauberman's lemma, see [19, Lemma 13.8]. Note that by the equality $N_G(P)=N_{G^S}(P)C_G(P)$ (Lemma 2.8) and the assumption of A being an interior G-algebra, any defect pointed group of A with the group P is also S-invariant.

(1) and (2) follow from the Puig correspondence Theorem 3.1 and Lemmas 2.8, 3.2 and 3.4. Here, in the application of Lemma 3.2, viewing $A=\mathrm{Res}^\Gamma_G(\tilde{A})$ and P_γ as a pointed group of $\tilde{A}$, we take $\overline{N}=N_G(P_\gamma)/P$, $N_\Gamma(P_\gamma)/P\simeq\overline{N}S$, $\overline{C}=PC_G(P)/P\simeq C_G(P)/Z(P)$ and $\tilde{A}(P_\gamma)$ for G, Γ, N and $\tilde{A}$ in Lemma 3.2, respectively. β is determined as the point of G^S on A corresponding to a Glauberman correspondent V' of $V=V_A(P_\gamma)=V_{\tilde{A}}(P_\gamma){\downarrow}^{\overline{N}S}_{\overline{N}}$ in $V{\downarrow}^{\overline{N}}_{\overline{N}^S}$.

We show (3). For the argument below, see [5, Section 2]. Firstly, note that we may assume $P\le D$, see [35, Proposition 37.3]. Recall that the $k\overline{C}$-module structure of $V_A(P_\gamma)$ comes from the canonical epimorphism $\pi_\gamma\colon A^P\to A(P_\gamma)\simeq\mathrm{End}_k(V_A(P_\gamma))$, which makes $A(P_\gamma)$ an interior $C_G(P)$-algebra and the fact $u\cdot\overline{a}=\overline{a}$ for any $u\in Z(P)$

and $\overline{a} \in A(P_\gamma)$, which makes $A(P_\gamma)$ an interior $\overline{C}$-algebra. The canonical epimorphism $kC_G(P) \to k\overline{C}$ induces a one-to-one correspondence, $f_P \mapsto \underline{f_P}$, between $\mathrm{Bl}_k(C_G(P) \mid Z(P))$ and $\mathrm{Bl}_k(\overline{C} \mid 1)$, see [24, V, Theorems 8.10 and 8.11]. Since π_γ factors through the Brauer homomorphism Br_P^A, see [35, Corollary 14.6], when A is in b, that is, $b \cdot 1_A = 1_A$, we see that

$$\mathrm{Br}_P^{\mathcal{R}G}(b) \cdot 1_{A(P_\gamma)} = 1_{A(P_\gamma)}. \tag{3.4.1}$$

Similary, for the block b' of G^S such that A_β is in b', we see that

$$\mathrm{Br}_P^{\mathcal{R}G^S}(b') \cdot 1_{A_\beta(P_{\gamma'})} = 1_{A_\beta(P_{\gamma'})} \tag{3.4.2}$$

where $P_{\gamma'}$ is a defect pointed group of A_β corresponding to P_γ by the embedding associated with G_β^S. Any direct summand W of $V_A(P_\gamma){\downarrow}_{\overline{C}}^{\overline{N}}$ is an S-invariant simple projective $k\overline{C}$-module, see Lemma 3.2 (3) (A), and W determines $\underline{e_P} \in \mathrm{Bl}_k(\overline{C} \mid 1)$ and $e_P \in \mathrm{Bl}_k(C_G(P) \mid Z(P))$. Note that (P, e_P) is a b-Brauer pair by (3.4.1). A Glauberman correspondent W' of W determines $w(\underline{e_P}) \in \mathrm{Bl}_k(\overline{C}^S \mid 1)$, which is the canonical image of $w(e_P) \in \mathrm{Bl}_k(C_{G^S}(P) \mid Z(P))$ since the character corresponding to W can be seen as a character in e_P. By Lemma 3.2 (3) (B), W' is isomorphic to a direct summand of a Glauberman correspondent V' of $V_A(P_\gamma)$, which can be identified with $V_{A_\beta}(P_{\gamma'})$, with the canonical $k\overline{C}$-module structures, by the construction of β. Hence we see that

$$w(e_P) \cdot A_\beta(P_{\gamma'}) \neq 0,$$

and so $(P, w(e_P))$ is a b'-Brauer pair by (3.4.2). On the other hand, since $(P, w(e_P))$ is a $w(b)$-Brauer pair by Theorem 2.5 (3), it holds that $b' = w(b)$. $\square$

Following [5, Section 2], we say that, $\mathcal{R}G$-module L is *simply defective* if the interior G-algebra $\mathrm{End}_{\mathcal{R}}(L)$ has a simple defect multiplicity module. It is known that the simple kG-modules, the full $\mathcal{O}G$-lattices of the irreducible characters of G, see [27, Proposition 1.6] or [35, p. 213], and $\mathcal{R}G$-modules with full vertex, see [3, Proposition 1.2], are simply defective.

Corollary 3.6. *Let P be an S-centralized p-subgroup of G. Then*:
(1) *Assume that X is a simply defective indecomposable S-invariant $\mathcal{R}G$-module with vertex P. Then there exists, unique up to isomorphism, an indecomposable direct summand X' of $X{\downarrow}_{G^S}^G$ satisfying the following*:
(i) *X' has P as a vertex.*
(ii) *$q \nmid m(X', X)$.*
In fact, X' is simply defective, $m(X', X) \equiv \pm 1$ (mod q) *and the set of isomorphism classes of source $\mathcal{R}P$-modules of X and X' is same.*

(2) *Assume that Y is a simply defective indecomposable $\mathcal{R}G^S$-module with vertex P and there exists an S-invariant simply defective indecomposable $\mathcal{R}G$-module Y'' satisfying the following*:

(i) *Y'' has P as a vertex.*

(ii) $q \nmid n(Y'', Y)$.

Then $n(Y'', Y) \equiv \pm 1 \pmod q$, any S-invariant indecomposable direct summand of $Y\uparrow_{G^S}^G$ with vertex P and not isomorphic to Y'' has a multiplicity divisible by q, and the set of isomorphism classes of source $\mathcal{R}P$-modules of Y and Y'' is same.

(3) (Barker) *$n(X, X') = m(X', X)$ and so X' in* (1) *satisfies the condition in* (2) *and $(X')'' \simeq X$. For Y in* (2), *$m(Y, Y'') = n(Y'', Y)$ and so $(Y'')' \simeq Y$.*

(4) *If X in* (1) *belongs to a block b of G with an S-centralized defect group, then X' belongs to the block $w(b)$ of G^S.*

Proof. Since X can be extended to an $\mathcal{R}\Gamma$-module $\tilde{X}$ by [9, Theorem 4.5], (1) follows from Proposition 3.5 (1) for $\tilde{A} = \mathrm{End}_{\mathcal{R}}(\tilde{X})$, [35, Example 13.4 and Proposition 18.11] and Lemma 2.8. (4) follows from Proposition 3.5 (3).

For (2), we consider the multiplicities of S-invariant direct summands of $Y'\uparrow_{G^S}^G$ with vertex P.

Let $\tilde{Y}$ be an extension of Y to an $\mathcal{R}G^S S$-module. Denote by P_γ be a defect pointed group of $\tilde{B} = \mathrm{End}_{\mathcal{R}}(\tilde{Y})$. Identify $\mathrm{End}_{\mathcal{R}}(Y)$ with $\mathrm{Res}_{G^S}^{G^S S}(\mathrm{End}_{\mathcal{R}}(\tilde{Y}))$ as interior G^S-algebra and the defect multiplicity module U of $\mathrm{End}_{\mathcal{R}}(Y)$ with $V_{\tilde{B}}(P_\gamma)\downarrow_{N_{G^S}(P_\gamma)/P}^{N_{G^S S}(P_\gamma)/P}$. Through the canonical embedding of interior $G^S S$-algebra $\mathcal{D}$: $\tilde{B} \to \tilde{C} = \mathrm{Ind}_{G^S S}^{\Gamma}(\tilde{B})$, see [35, p. 129], $V_{\tilde{B}}(P_\gamma)$ is identified with the direct summand $\pi_{\gamma'}(1 \otimes 1_{\tilde{B}} \otimes 1) \cdot V_{\tilde{C}}(P_{\gamma'})$ of $V_{\tilde{C}}(P_{\gamma'})\downarrow_{N_{G^S S}(P_{\gamma'})/P}^{N_\Gamma(P_{\gamma'})/P}$, where $P_{\gamma'}$ is a pointed group of $\tilde{C}$ corresponding to P_γ by $\mathcal{D}$. On the other hand, for $x \in \tilde{C}^P \gamma' \tilde{C}^P$ such that $1 \otimes 1_{\tilde{B}} \otimes 1 = \mathrm{Tr}_P^{G^S S}(x)$, that is, $1_{\tilde{C}} = \mathrm{Tr}_{G^S S}^{\Gamma}(1 \otimes 1_{\tilde{B}} \otimes 1) = \mathrm{Tr}_P^{\Gamma}(x)$, recall the following equalities and the isomorphism from the last part of the proof of [4, Proposition 7.4]:

$$\begin{aligned}
V_{\tilde{C}}(P_{\gamma'}) &= \pi_{\gamma'}(1_{\tilde{C}}) \cdot V_{\tilde{C}}(P_{\gamma'}) \\
&= \pi_{\gamma'}(\mathrm{Tr}_P^{\Gamma}(x)) \cdot V_{\tilde{C}}(P_{\gamma'}) \\
&= \mathrm{Tr}_1^{N_\Gamma(P_{\gamma'})/P}(\pi_{\gamma'}(x)) \cdot V_{\tilde{C}}(P_{\gamma'}) \\
&= \mathrm{Tr}_{N_{G^S S}(P_{\gamma'})/P}^{N_\Gamma(P_{\gamma'})/P}\left(\mathrm{Tr}_1^{N_{G^S S}(P_{\gamma'})/P}(\pi_{\gamma'}(x))\right) \cdot V_{\tilde{C}}(P_{\gamma'}) \\
&= \mathrm{Tr}_{N_{G^S S}(P_{\gamma'})/P}^{N_\Gamma(P_{\gamma'})/P}(\pi_{\gamma'}(1 \otimes 1_{\tilde{B}} \otimes 1)) \cdot V_{\tilde{C}}(P_{\gamma'}) \\
&\simeq (\pi_{\gamma'}(1 \otimes 1_{\tilde{B}} \otimes 1) \cdot V_{\tilde{C}}(P_{\gamma'}))\uparrow_{N_{G^S S}(P_{\gamma'})/P}^{N_\Gamma(P_{\gamma'})/P}.
\end{aligned}$$

Hence, with the above identifications, we have

$$U\uparrow_{N_{G^S}(P_{\gamma'})/P}^{N_G(P_{\gamma'})/P} \simeq V_{\tilde{C}}(P_{\gamma'})\downarrow_{N_G(P_{\gamma'})/P}^{N_\Gamma(P_{\gamma'})/P}.$$

Let G_σ be a pointed group of $\tilde{C}$ with a defect pointed group $P_{\gamma'}$. Note that G_σ has a defect pointed group $P_{\gamma'}$ if and only if G_σ has a defect group P, see [35, Proposition 16.7]. For $j \in \sigma$ and $s \in S$, we have

$$\begin{aligned}\pi_{\gamma'}(j^s) \cdot V_{\tilde{C}}(P_{\gamma'}) &= \pi_{\gamma'}(s^{-1} \cdot 1_{\tilde{C}})\pi_{\gamma'}(j)\pi_{\gamma'}(s \cdot 1_{\tilde{C}}) \cdot V_{\tilde{C}}(P_{\gamma'}) \\ &= \pi_{\gamma'}(s^{-1} \cdot 1_{\tilde{C}})\pi_{\gamma'}(j) \cdot V_{\tilde{C}}(P_{\gamma'})\end{aligned}$$

and it is isomorphic to s-conjugate module $(\pi_{\gamma'}(j) \cdot V_{\tilde{C}}(P_{\gamma'}))^s$ of the $k_*\widehat{N_G(P_{\gamma'})/P}$-module $\pi_{\gamma'}(j) \cdot V_{\tilde{C}}(P_{\gamma'})$. Hence, we see that the indecomposable direct summand $\pi_{\gamma'}(j) \cdot V_{\tilde{C}}(P_{\gamma'})$ of $V_{\tilde{C}}(P_{\gamma'})\downarrow^{N_\Gamma(P_{\gamma'})/P}_{N_G(P_{\gamma'})/P}$ is S-invariant if and only if G_σ is S-invariant.

On the other hand, for the interior Γ-algebra $\mathrm{End}_{\mathcal{R}}\big(\tilde{Y}\uparrow^{\Gamma}_{G^S S}\big)$, a pointed group $G_{\sigma''}$ of $\mathrm{End}_{\mathcal{R}}\big(\tilde{Y}\uparrow^{\Gamma}_{G^S S}\big)$ and $j'' \in \sigma''$, we see similarly that $G_{\sigma''}$ is S-invariant if and only if the indecomposable direct summand $j'' \cdot \big(\tilde{Y}\uparrow^{\Gamma}_{G^S S}\big)$ of the $\mathcal{R}G$-module $\big(\tilde{Y}\uparrow^{\Gamma}_{G^S S}\big)\downarrow^{\Gamma}_{G}$ is S-invariant.

Therefore, (2) follows from the condition in (2), Lemma 3.2 for $\tilde{A} = \tilde{C}(P_{\gamma'})$, Theorem 3.1 for $\tilde{C}$, the isomorphism of interior Γ-algebra $\tilde{C} \simeq \mathrm{End}_{\mathcal{R}}\big(\tilde{Y}\uparrow^{\Gamma}_{G^S S}\big)$, see [35, Example 16.4], and the isomorphism of $\mathcal{R}G$-module $\big(\tilde{Y}\uparrow^{\Gamma}_{G^S S}\big)\downarrow^{\Gamma}_{G} \simeq Y\uparrow^{G}_{G^S}$.

(3) follows from Theorem 3.1, (1) and (2). □

Note that in the proof of Corollary 3.6 (2), except in the last paragraph, we only use the condition that a defect multiplicity module of Y is simple. Hence, we may use the notations in the proof of Corollary 3.6 (2) for any indecomposable $\mathcal{R}G^S$-module Y with an S-centralized vertex P and with a simple defect multiplicity module U. Then we see that Y satisfies the condition in Corollary 3.6 (2) if and only if $\pi_{\gamma'}(1 \otimes 1_{\tilde{B}} \otimes 1) \cdot V_{\tilde{C}}(P_{\gamma'})\downarrow_{N_{G^S}(P_{\gamma'})/P} \simeq U$ satisfies the condition in Lemma 3.2 (2) for $\tilde{A} = \tilde{C}(P_{\gamma'})$. For example, if S centralizes a Sylow p-subgroup of $N_G(P_{\gamma'})/P$, then the condition is satisfied, see the first paragraph of the proof of Lemma 3.2 (2).

Corollary 3.7. *Assume that S centralizes a Sylow p-subgroup of G. Let P be any S-centralized p-subgroup of G. Then there is a one-to-one correspondence between the set of isomorphism classes of S-invariant simply defective indecomposable $\mathcal{R}G$-modules with vertex P and the set of isomorphism classes of simply defective indecomposable $\mathcal{R}G^S$-modules with vertex P. The set of isomorphism classes of source $\mathcal{R}P$-modules of the corresponding modules is same. The corresponding modules belong to the Glauberman-Watanabe corresponnding blocks.*

Proof. With the notaions in Corollary 3.6 (1), the map $X \mapsto X'$ induces a map from the former set to the latter set in the first statement.

Let Y be any simply defective indecomposable $\mathcal{R}G^S$-module with vertex P. Then Y satisfies the condition in Corollary 3.6 (2), see the remark above this corollary. Hence, with the notations in Corollary 3.6 (2), the map $Y \mapsto Y''$ induces a map from the latter set to the former set in the first statement. Note that Corollary 3.6 (2) says the uniqueness of Y'' up to isomorphism under the assumption of the existence of Y''.

By Corollary 3.6 (3), above two maps are mutually inverse maps between the sets in the first statement. $\square$

4. The Glauberman-Watanabe corresponding blocks with normal defect groups

In this section, we reprove Harris's result Corollary 4.10 below. For this, in Theorem 4.9 below, we will show that the condition that an S-invariant block algebra $A = \mathcal{O}Gb$ has an S-centralized normal defect group is a sufficient condition for a primitive interior G^S-algebra A_β determined by Proposition 3.5 being a block algebra (in this case, the simple modules in $kG\bar{b}$ and $kG^S w(\bar{b})$ correspond by the correspondence in Corollary 3.6).

CONDITION 4.1. b is an S-invariant block of $\mathcal{O}G$ with an S-centralized defect group D, and A is a primitive interior G-algebra $\mathcal{O}Gb$. β is a point of A^{G^S} determined by Proposition 3.5 and $f \in \beta$.

Proposition 4.2. *Assume* Condition 4.1. *Then* $\mathcal{O}Gb{\downarrow}^{G\times G}_{G^S\times G}$ *has, unique up to isomorphism, an indecomposable direct summand* M *satisfying the following*:
(i) *M has ΔD as a vertex.*
(ii) $q \nmid m = m(M, \mathcal{O}Gb)$.
In fact, m satisfies $m \equiv \pm 1$ (mod q), *and M is isomorphic to $f(\mathcal{O}Gb) = f\mathcal{O}G$ and is an $(\mathcal{O}G^S w(b), \mathcal{O}Gb)$-bimodule.*

Proof. As is well-known, see [31] or [2], we can identify the points of $G^S \times G$ on the interior $G\times G$-algebra $\mathrm{End}_{\mathcal{O}}(\mathcal{O}Gb)$ with the points of G^S on the interior G-algebra $A = \mathcal{O}Gb$ through (♯) in Section 1. Note also that, for a p-subgroup P of G^S, since we see $\mathrm{Res}^{G\times G}_{P\times G}(\mathrm{End}_{\mathcal{O}}(\mathcal{O}Gb))$ is relatively ΔP-projective, see [35, p. 111], it holds

$$\begin{aligned}\mathrm{Tr}^{G^S\times G}_{\Delta P}(\mathrm{End}_{\mathcal{O}}(\mathcal{O}Gb)^{\Delta P}) &= \mathrm{Tr}^{G^S\times G}_{P\times G}\big(\mathrm{Tr}^{P\times G}_{\Delta P}(\mathrm{End}_{\mathcal{O}}(\mathcal{O}Gb)^{\Delta P})\big)\\ &= \mathrm{Tr}^{G^S\times G}_{P\times G}(\mathrm{End}_{\mathcal{O}}(\mathcal{O}Gb)^{P\times G}) \simeq \mathrm{Tr}^{G^S}_{P}((\mathcal{O}Gb)^{P}).\end{aligned}$$

Hence, the isomorphism classes of the indecomposable direct summands of $\mathcal{O}Gb{\downarrow}^{G\times G}_{G^S\times G}$ with vertex ΔD correspond to the points of $(\mathcal{O}Gb)^{G^S}$ with defect group D. Since block algebras have simple defect multiplicity modules, see [35, Corollary 37.6], the hypotheses of Proposition 3.5 are satisfied by Lemma 2.7, and so the statements follow. $\square$

CONDITION 4.3. D in Condition 4.1 is normal in G.

Lemma 4.4 (Fan-Puig [12]). *Let A be a primitive G-algebra with a normal defect group P and let P_γ be a defect pointed group of A. Then any point ν of P on A is a G-conjugate of γ, and so is a local point. Hence, for any subgroup H of G containing P, any pointed group H_ϵ of A has P as a defect group.*

Proof. Since $G_{\{1_A\}}$ *pr* P_γ and P is normal in G,

$$1_A \in \mathrm{Tr}_P^G(A^P\gamma A^P) \subseteq \sum_{g\in[P\backslash G]}(A^P\gamma A^P)^g = \sum_{g\in[P\backslash G]} A^P\gamma^g A^P,$$

see [12, 2.12.3]. Hence any primitive idempotent of A^P belongs to some ideal $A^P\gamma^g A^P$ of A^P by Rosenberg's lemma, see [35, Proposition 4.9]. Note that any primitive idempotent j of A^P included in $A^P\gamma^g A^P$ must belong to γ^g, see [12, 2.12.4] or [35, Lemma 14.3], and so the statement follows. □

In the normal defect case of our setting, since the Puig correspondence does not lose the information by Lemmas 2.8, 3.4 and 4.4, we have the following:

Corollary 4.5. *Assume* Conditions 4.1 *and* 4.3. *Then any indecomposable direct summand of $\mathcal{O}Gb\downarrow^{G\times G}_{G^S\times G}$ has ΔD as a vertex. Hence, we have the following indecomposable direct sum decomposition of $\mathcal{O}Gb\downarrow^{G\times G}_{G^S\times G}$:*

$$\mathcal{O}Gb\downarrow^{G\times G}_{G^S\times G} \simeq m(f\mathcal{O}G)\oplus\left(\bigoplus_i m_iM_i\right),$$

where $m \equiv \pm 1 \pmod q$ and $m_i \equiv 0 \pmod q$ for all i.

Lemma 4.6. *Assume* Conditions 4.1 *and* 4.3. *Then there are defect pointed groups D_{δ_0} and $D_{\delta'_0}$ of $\mathcal{O}Gb$ and $\mathcal{O}G^Sw(b)$, respectively, satisfying the following:*
(i) $i_0 = i'_0 f_0 = f_0 i'_0$ *for some* $i_0\in\delta_0$, $i'_0\in\delta'_0$ *and* $f_0\in\beta$.
(ii) $N_G(D_{\delta_0}) = N_{G^S}(D_{\delta'_0})DC_G(D)$. *In particular, we may take* $[DC_G(D)\backslash N_G(D_{\delta_0})] = [DC_{G^S}(D)\backslash N_{G^S}(D_{\delta'_0})]$.

Proof. Let D_δ be a defect pointed group of $\mathcal{O}Gb$, and let $D_{\delta'}$ be a defect pointed group of $\mathcal{O}G^Sw(b)$ such that $V_{\mathcal{O}G^Sw(b)}(D_{\delta'})\downarrow_{DC_{G^S}(D)/D}$ is a Glauberman correspondent of $V_{\mathcal{O}Gb}(D_\delta)\downarrow_{DC_G(D)/D}$, see Proposition 2.6. Then we have

(4.6.1) $$N_G(D_\delta)\cap G^S = N_{G^S}(D_{\delta'})$$

and

(4.6.2) $$N_G(D_\delta) = N_{G^S}(D_{\delta'})DC_G(D),$$

see [35, Proposition 37.11], [37, Lemma 3.5 (a)], Lemma 2.8 or explicitly [22, Lemma 3.1 (d) (i)].

By the isomorphism $\mathrm{End}_{\mathcal{O}[D\times G]}(M) \simeq A_\beta^D$ for M as in Proposition 4.2, Lemmas 2.8, 3.4 and 4.4 and [35, Proposition 15.1 (e)], we have an indecomposable decomposition

$$M\downarrow^{G^S\times G}_{D\times G} \simeq m_I\left(\bigoplus_{c\in[N_{G^S}(D_\delta)\backslash G^S]} I^{(c,1)}\right),$$

where I is an indecomposable $\mathcal{O}[D\times G]$-module isomorphic to $i\mathcal{O}G$ for $i\in\delta$, which is called a *source module* of b in [2], and $m_I = m(I, M)$. On the other hand, by the similar consideration for $\mathrm{End}_{\mathcal{O}[1\times G^S]}(\mathcal{O}G^S w(b)) \simeq \mathcal{O}G^S w(b)$, we have a decomposition

$$\begin{aligned} M\downarrow^{G^S\times G}_{D\times G} &\simeq (\mathcal{O}G^S w(b)\otimes_{\mathcal{O}G^S} M)\downarrow^{G^S\times G}_{D\times G} \\ &\simeq m_{I'}\left(\bigoplus_{c'\in[N_{G^S}(D_{\delta'})\backslash G^S]} I'^{(c',1)}\right)\otimes_{\mathcal{O}G^S} M \\ &\simeq m_{I'}\left(\bigoplus_{c'\in[N_{G^S}(D_{\delta'})\backslash G^S]} (I'\otimes_{\mathcal{O}G^S} M)^{(c',1)}\right), \end{aligned}$$

where I' is an indecomposable $\mathcal{O}[D\times G^S]$-module isomorphic to $i'\mathcal{O}G^S$ for $i'\in\delta'$ and $m_{I'} = m(I', \mathcal{O}G^S w(b))$. Note, firstly, that we may take $[N_{G^S}(D_\delta)\backslash G^S] = [N_{G^S}(D_{\delta'})\backslash G^S]$ by (4.6.1). Secondly,

$$\begin{aligned} m_I &= \dim_k(V_{\mathrm{End}_{\mathcal{O}}(M)}(D\times G)_\delta) \\ &= \dim_k(V_{A_\beta}(D_\delta)) \\ &= \dim_k(V_{\mathcal{O}G^S w(b)}(D_{\delta'})) \\ &= \dim_k(V_{\mathrm{End}_{\mathcal{O}}(\mathcal{O}G^S w(b))}(D\times G^S)_{\delta'}) = m_{I'}. \end{aligned}$$

Here, for the first and last equalities, see [35, Proposition 4.15 (a)] and the third equality follows from Proposition 2.6 and Lemma 3.2 (3) (B) (b). Therefore, comparing the above two decompositions of $M\downarrow^{G^S\times G}_{D\times G}$, we see that $I^{(c,1)} \simeq I'\otimes_{\mathcal{O}G^S} M$ as $\mathcal{O}[D\times G]$-modules for some $c\in[N_{G^S}(D_\delta)\backslash G^S]$. Hence we may take some $i\in\delta$, $i'\in\delta'$ and $f\in\beta$ such that $i^c = i'f = fi'$, and so D_{δ^c} and $D_{\delta'}$ satisfy the condition (i).

For (ii), it suffices to show that $N_{G^S}(D_{\delta'})^c = N_{G^S}(D_{\delta'})$ since $N_G(D_{\delta^c}) = N_G(D_\delta)^c = N_{G^S}(D_{\delta'})^c DC_G(D)$ by (4.6.2). Assume $N_{G^S}(D_{\delta'}) \neq N_{G^S}(D_{\delta'})^c$. Then, by (4.6.1),

$$N_{G^S}(D_{\delta'}) \neq N_{G^S}(D_{\delta'})^c = N_{G^S}(D_\delta)^c = N_{G^S}(D_{\delta^c}).$$

Hence, there is some $x\in N_{G^S}(D_{\delta'})$ such that $x\notin N_{G^S}(D_{\delta^c})$ and so $(i^c)^x\notin\delta^c$. Therefore, $i^c\mathcal{O}G \not\simeq (i^c)^x\mathcal{O}G$ as $\mathcal{O}[D \times G]$-modules. On the other hand, as $\mathcal{O}[D \times G]$-modules,

$$i^c\mathcal{O}G = i' f\mathcal{O}G \simeq i'^x f\mathcal{O}G = (i' f)^x\mathcal{O}G = (i^c)^x\mathcal{O}G,$$

which is a contradiction. Hence, D_{δ^c} and $D_{\delta'}$ satisfy the conditions. □

We cite Puig's theorems as lemmas, on which our proof of Theorem 4.9 depends.

Lemma 4.7 (Puig [30, Proposition 4.3]). *Let b be a block of $\mathcal{O}G$ and i a source idempotent of $\mathcal{O}Gb$. Then there is an equivalence of categories between the category of isomorphism classes of primitive interior G-algebras in b and the category of isomorphism classes of primitive $i\mathcal{O}Gi$-algebras, see* [31, 4.2]. *An object A of the former corresponds to an object $i \cdot A \cdot i$ of the latter.*

Lemma 4.8 (Puig [28] or see [35, Theorem 44.3]). *Let b be a block of $\mathcal{O}G$ with a normal defect group D, D_δ a defect pointed group of $\mathcal{O}Gb$ and $i \in \delta$. Then, for any $a_g \in (i\mathcal{O}Gi)^\times$ satisfying $a_g^{-1} \cdot u \cdot a_g = u^g \cdot i$ for any $u \in D$ where $g \in \Xi = [DC_G(D)\backslash N_G(D_\delta)]$* (*see* [35, *Proposition* 44.2] *for the existence of a_g*), *we have the following description of a source algebra $i\mathcal{O}Gi$ of $\mathcal{O}Gb$ as an $(\mathcal{O}D, \mathcal{O}D)$-bimodule*:

$$i\mathcal{O}Gi = \bigoplus_{g\in\Xi} \mathcal{O}D \cdot a_g \simeq \bigoplus_{g\in\Xi} \mathcal{O}Dg.$$

Theorem 4.9. *Assume* Conditions 4.1 *and* 4.3. *Then $(\mathcal{O}Gb)_\beta$ and $\mathcal{O}G^S w(b)$ are isomorphic as primitive interior G^S-algebras. In particular, $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ are Puig equivalent* ([21], [14]).

Proof. Second statement is clear, since $A = \mathcal{O}Gb$ and A_β are Puig equivalent by the construction.

Since A_β is in $w(b)$ by Proposition 3.5 (3), it suffices to show that, for some source idempotent i' of $\mathcal{O}G^S w(b)$,

$$i' \cdot \mathcal{O}G^S w(b) \cdot i' \simeq i' \cdot A_\beta \cdot i' \tag{4.9.1}$$

as primitive $i'\mathcal{O}G^S i'$-algebras, see Lemma 4.7.

For (4.9.1), let δ_0, δ_0', $i_0 \in \delta_0$, $i_0' \in \delta_0'$ and $f_0 \in \beta$ be points and idempotents as in the statements of Lemma 4.6. Then we can see that

$$\varphi\colon i_0' \cdot \mathcal{O}G^S w(b) \cdot i_0' \to i_0' \cdot f_0 A f_0 \cdot i_0', \quad x \mapsto f_0 x f_0$$

is well-defined and is an isomorphism of primitive $i_0'\mathcal{O}G^S i_0'$-algebras.

It is straightforward to check that φ is well-defined and is an $i'_0\mathcal{O}G^S i'_0$-algebra homomorphism by the $i'_0\mathcal{O}G^S i'_0$-algebra structures of $i'_0 \cdot \mathcal{O}G^S w(b) \cdot i'_0 = i'_0\mathcal{O}G^S i'_0$ and $i'_0 \cdot f_0 A f_0 \cdot i'_0 = i'_0 f_0 A f_0 i'_0 = f_0 i'_0 \mathcal{O}G i'_0 f_0$, which is equal to $i_0 A i_0$, see the condition (i) in Lemma 4.6. Below, we note that φ is an isomorphism of $\mathcal{O}$-spaces, using the structure theorem of the source algebra of a block with a normal defect group. Denote $\Xi = [DC_{G^S}(D)\backslash N_{G^S}(D_{\delta'_0})] = [DC_G(D)\backslash N_G(D_{\delta_0})]$, see the condition (ii) in Lemma 4.6. By Lemma 4.8, we have

$$i'_0\mathcal{O}G^S i'_0 = \bigoplus_{g\in\Xi} \mathcal{O}D \cdot a'_g$$

for $a'_g \in (i'_0\mathcal{O}G^S i'_0)^\times$ as in Lemma 4.8. Then, for any $g \in \Xi$, $a_g = \varphi(a'_g) = f_0 a'_g f_0$ is in $(i_0 A i_0)^\times$ and satisfies $a_g{}^{-1}(u \cdot i_0)a_g = u^g \cdot i_0$ for any $u \in D$, as is immediately checked. Hence we have

$$i_0 A i_0 = \bigoplus_{g\in\Xi} \mathcal{O}D \cdot a_g$$

by Lemma 4.8. Therefore, φ is an isomorphism, since φ maps an $\mathcal{O}$-basis $\{u \cdot a'_g\}_{u\in D, g\in\Xi}$ of $i'_0\mathcal{O}G^S i'_0$ □

Assume Conditions 4.1 and 4.3. Since $f\mathcal{O}G \mid \mathcal{O}Gb{\downarrow}^{G\times G}_{G^S\times G}$, $f\mathcal{O}G$ has a trivial source. By Theorem 4.9, tensoring $f\mathcal{O}G$ over $\mathcal{O}G$, that is, multiplication by the idempotent f, induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$, see [35, Theorem 9.9]. Moreover, from this and the decomposition of the restriction of $\mathcal{O}Gb$-modules to $\mathcal{O}G^S$-modules described in Corollary 4.5, $f\mathcal{O}G$ induces cleary the Glauberman correspondence, see Theorem 2.1 (3). (Note that if b, hence $w(b)$, is a principal block, then the defect multiplicity module is trivial and so $f = b$, $\mathcal{O}Gb{\downarrow}^{G\times G}_{G^S\times G} = f\mathcal{O}G$ and $\mathcal{O}Gb = \mathcal{O}G^S w(b)$.)

From above and [15, Proposition 2.5], we have the following by induction, see the first paragraph of the proof of Theorem 4.11:

Corollary 4.10 (Harris [14]). *Assume the following*:

1. *G is a finite group acted by a finite solvable group $\mathcal{S}$ such that* $(|G|, |\mathcal{S}|) = 1$.
2. *b is an $\mathcal{S}$-invariant block of G with an $\mathcal{S}$-centralized normal defect group D.*

Let $1 = S_0 \leq S_1 \leq S_2 \leq \cdots \leq S_n = \mathcal{S}$ *be a composition series of $\mathcal{S}$ such that* $|S_i/S_{i-1}|$ *is a prime, for* $1 \leq i \leq n$. (*Then* $\mathrm{Irr}(b) = \mathrm{Irr}(b)^S$ *and there is a unique block $w_i(b)$ of G^{S_i} such that* $\mathrm{Irr}(w_i(b)) = \{\pi(G, S_i)(\phi) \mid \phi \in \mathrm{Irr}(b)\}$, *see* [36, Proposition 1 and Theorem 1].)

Then there is an indecomposable $\mathcal{O}[G^{S_i} \times G]$-module with a trivial source inducing a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^{S_i} w_i(b)$ and the Glauberman correspondence between $\mathrm{Irr}(b)$ *and* $\mathrm{Irr}(w_i(b))$. *In particular, $\mathcal{O}Gb$ and $\mathcal{O}G^{S_i} w_i(b)$ are Puig equivalent* ([21], [14]).

In fact, we have a formulation in terms of pointed groups, which implies Corollary 4.10:

Theorem 4.11. *With the same assumptions and the notations in* Corollary 4.10, *we have the following*:
(1) *There is a sequence* $G_{\{b\}} = G^{S_0}_{\beta_0} \geq G^{S_1}_{\beta_1} \geq G^{S_2}_{\beta_2} \geq \cdots \geq G^{S}_{\beta_n} \geq D_\delta$ *of pointed groups of* $\mathcal{O}Gb$ *such that* $(\mathcal{O}Gb)_{\beta_i} \simeq \mathcal{O}G^{S_i}w_i(b)$ *as interior* G^{S_i}*-algebras, where* δ *is any point of* $(\mathcal{O}Gb)^D$, *which is necessarily local, see* Lemma 4.4.
(2) (Watanabe) *For* $l_i \in \beta_i$, $y_i \mapsto y_i \cdot l_i$, *is an isomorphism of interior* G^{S_i}*-algebra from* $\mathcal{O}G^{S_i}w_i(b)$ *to* $l_i\mathcal{O}Gl_i$ *where* $y_i \in \mathcal{O}G^{S_i}w_i(b)$, *and so* $l_i\mathcal{O}Gl_i = \mathcal{O}G^{S_i}w_i(b)l_i$.
(3) *For* $l_i \in \beta_i$, *the* $(\mathcal{O}G^{S_i}, \mathcal{O}G)$*-bimodule* $l_i\mathcal{O}G$ *induces a Morita equivalence between* $\mathcal{O}Gb$ *and* $\mathcal{O}G^{S_i}w_i(b)$ *and the Glauberman correspondence between* $\mathrm{Irr}(b)$ *and* $\mathrm{Irr}(w_i(b))$.

Proof. Firstly, recall the definition of $w_i(b)$. Below, the action of S_i/S_{i-1} on $G^{S_{i-1}}$ is the action induced by the given action of S_i on G. For an S_i/S_{i-1}-invariant block B of $G^{S_{i-1}}$ with an S_i/S_{i-1}-centralized defect group, denote by $w_{S_i/S_{i-1}}(B)$ the Glauberman-Watanabe corresponding block of $G^{S_i} = (G^{S_{i-1}})^{S_i/S_{i-1}}$ induced by $\pi(G^{S_{i-1}}, S_i/S_{i-1})$. Taking $w_0(b) = b$ and using induction, $w_i(b)$ is defined by $w_i(b) = w_{S_i/S_{i-1}}(w_{i-1}(b))$. By induction, then, $\mathrm{Irr}(w_i(b)) = \{\pi(G, S_i)(\phi) \mid \phi \in \mathrm{Irr}(b)\}$ by Theorem 2.1 (ii) and Theorem 2.5 (2), $w_i(b)$ is S_{i+1}/S_i-invariant by Theorem 2.1 (i) for $S_i \lhd S_{i+1}$ and $w_i(b)$ has a defect group D by Theorem 2.5 (2), which is S_{i+1}/S_i-centralized.

By Theorem 4.9, there is some primitive idempotent f_i in $(\mathcal{O}G^{S_{i-1}}w_{i-1}(b))^{(G^{S_{i-1}})^{S_i/S_{i-1}}} = (\mathcal{O}G^{S_{i-1}}w_{i-1}(b))^{G^{S_i}}$ such that

$$f_i\mathcal{O}G^{S_{i-1}}w_{i-1}(b)f_i \simeq \mathcal{O}G^{S_i}w_i(b)$$

as interior G^{S_i}-algebras. Since we have an interior $G^{S_{i-1}}$-algebra isomorphism

$$\Psi_{i-1}\colon \mathcal{O}G^{S_{i-1}}w_{i-1}(b) \to l_{i-1}\mathcal{O}Gl_{i-1}$$

for $l_{i-1} \in \beta_{i-1}$ by induction (in the case $i = 1$, this is trivial), for the primitive idempotent $l_i = \Psi_{i-1}(f_i)$ of $(l_{i-1}\mathcal{O}Gl_{i-1})^{G^{S_i}}$, there is an interior G^{S_i}-algebra isomorphism

$$\Psi_i\colon l_i(l_{i-1}\mathcal{O}Gl_{i-1})l_i = l_i\mathcal{O}Gl_i \to \mathcal{O}G^{S_i}w_i(b).$$

Note that l_i is also primitive in $(\mathcal{O}Gb)^{G^{S_i}}$, see [35, Proposition 4.12], and let β_i be the point of G^{S_i} in $\mathcal{O}Gb$ containing l_i. Note that β_i is uniquely determined by the point of $(\mathcal{O}G^{S_{i-1}}w_{i-1}(b))^{(G^{S_{i-1}})^{S_i/S_{i-1}}}$ containing f_i, see [35, Proposition 15.1 (a)]. Since $G^{S_i}_{\beta_i}$ has defect group D and $N_{G^{S_{i-1}}}(D) = N_{G^{S_i}}(D)C_{G^{S_{i-1}}}(D)$, D_δ is a defect pointed group of $G^{S_i}_{\beta_i}$ where δ is any local point of $(\mathcal{O}Gb)^D$. From above, (1) follows.

Since $l_i \mathcal{O}Gl_i \simeq \mathcal{O}G^{S_i} w_i(b)$ as interior G^{S_i}-algebras in $w_i(b)$, the structural map, which is an interior G^{S_i}-algebra homomorphism, $\mathcal{O}G^{S_i} w_i(b) \to l_i \mathcal{O}Gl_i$, $y_i \mapsto y_i \cdot l_i$, is an isomorphism. Hence, we have (2).

By (2), we can take $x_{i-1} \in \mathcal{O}G^{S_{i-1}} w_{i-1}(b)$ such that $l_i = x_{i-1} \cdot l_{i-1} = l_{i-1} \cdot x_{i-1}$. Then (3) follows from the following isomorphisms of $(\mathcal{O}G^{S_i} w_i(b), \mathcal{O}Gb)$-bimodules:

$$
\begin{aligned}
& f_i(\mathcal{O}G^{S_{i-1}} w_{i-1}(b)) \otimes_{\mathcal{O}G^{S_{i-1}}} f_{i-1}(\mathcal{O}G^{S_{i-2}} w_{i-2}(b)) \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \\
& \otimes_{\mathcal{O}G^{S_2}} f_2(\mathcal{O}G^{S_1} w_1(b)) \otimes_{\mathcal{O}G^{S_1}} f_1(\mathcal{O}Gb) \\
& \simeq l_i(l_{i-1}\mathcal{O}Gl_{i-1}) \otimes_{\mathcal{O}G^{S_{i-1}}} l_{i-1}(l_{i-2}\mathcal{O}Gl_{i-2}) \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \\
& \quad \otimes_{\mathcal{O}G^{S_2}} l_2(l_1\mathcal{O}Gl_1) \otimes_{\mathcal{O}G^{S_1}} l_1(\mathcal{O}Gb) \\
& = l_i(l_{i-1}\mathcal{O}G^{S_{i-1}}l_{i-1}) \otimes_{\mathcal{O}G^{S_{i-1}}} l_{i-1}(l_{i-2}\mathcal{O}G^{S_{i-2}}l_{i-2}) \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \\
& \quad \otimes_{\mathcal{O}G^{S_2}} l_2(l_1\mathcal{O}G^{S_1}l_1) \otimes_{\mathcal{O}G^{S_1}} l_1(\mathcal{O}Gb) \\
& = l_i \otimes_{\mathcal{O}G^{S_{i-1}}} l_{i-1} \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \otimes_{\mathcal{O}G^{S_2}} l_2 \otimes_{\mathcal{O}G^{S_1}} l_1(\mathcal{O}Gb) \\
& = l_i(l_{i-1} \cdot x_{i-1}) \otimes_{\mathcal{O}G^{S_{i-1}}} l_{i-1}(l_{i-2} \cdot x_{i-2}) \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \otimes_{\mathcal{O}G^{S_2}} l_2(l_1 \cdot x_1) \otimes_{\mathcal{O}G^{S_1}} l_1(\mathcal{O}Gb) \\
& = l_i(l_{i-1} \cdot x_{i-1}) \otimes_{\mathcal{O}G^{S_{i-1}}} l_{i-1}(l_{i-2} \cdot x_{i-2}) \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \otimes_{\mathcal{O}G^{S_2}} l_2 \otimes_{\mathcal{O}G^{S_1}} l_2(\mathcal{O}Gb) \\
& = \cdots \\
& = l_i \otimes_{\mathcal{O}G^{S_{i-1}}} l_{i-1} \otimes_{\mathcal{O}G^{S_{i-2}}} \cdots \otimes_{\mathcal{O}G^{S_2}} l_2 \otimes_{\mathcal{O}G^{S_1}} l_i(\mathcal{O}Gb) \\
& \simeq l_i(\mathcal{O}Gb).
\end{aligned}
$$

□

5. Appendix

Throughtout this section, we assume Condition 4.1.

Since the Glauberman correspondence induces an isotypy between Glauberman-Watanabe corresponding blocks, it is desirable to exist a two-sided complex inducing a splendid derived equivalence and the Glauberman correspondence. In fact, in the normal defect case, this is proved in Harris's result Corollary 4.10.

On the other hand, Okuyama pointed out in [26] that there is some pairwise orthogonal (possibly zero) idempotents b_i, $0 \le i \le q-1$, of $(\mathcal{O}Gb)^{G^S}$ such that $b = \sum_{i=0}^{q-1} b_i$ and, as generalized characters of $\mathcal{K}[G^S \times G]$,

$$(*) \qquad \Phi_0 - \Phi_l = \sum_{\phi \in \mathrm{Irr}(b)} \delta_\phi(\pi(G, S)(\phi) \times \check{\phi}) \quad \text{for} \quad 1 \le l \le q-1,$$

where Φ_i is the character corresponding to $b_i\mathcal{K}G$, $\check{\phi}$ is the dual of ϕ and δ_ϕ is the sign described in Proposition 2.4, taking $\hat{\phi}$ so that $\mathrm{Irr}(\hat{b}_0) = \{\hat{\phi} \mid \phi \in \mathrm{Irr}(b)\}$ in the case of $q = 2$. Here and below, we denote by $\hat{b}_0$ a canonical extension of b in the sense of

[36, p. 555]. In fact, we can take as b_i

$$(**)\qquad b_i = \sum_{j=0}^{q-1} e_j \hat{b}_{j+i} \quad \text{for} \quad 0 \leq i \leq q-1,$$

where $\hat{b}_t$ is the block of $\mathcal{O}\Gamma$ such that $\mathrm{Irr}(\hat{b}_t) = \{\lambda^t\psi \mid \psi \in \mathrm{Irr}(\hat{b}_0)\}$ and e_t is the block of $\mathcal{O}S$ corresponding to $\lambda^t \in \mathrm{Irr}(S)$ for any integer t, see [34, Section 4]. Note that when $q = 2$, b_0 depends on the choice of $\hat{b}_0$.

We note that the normal defect case fits into this observation. That is, there is a complex $C^\bullet$ of $(\mathcal{O}G^S, \mathcal{O}G)$-bimodules which induces a splendid derived equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$ and whose character is canonically reduced to the left hand side of $(*)$ (hence which induces the Glauberman correspondence). Let $m = m(f\mathcal{O}G, \mathcal{O}Gb)$ as in Proposition 4.2. In fact, by Proposition 5.2 below, for example, we can take

$$C^\bullet : \cdots \to 0 \to b_l\mathcal{O}G \to b_0\mathcal{O}G \to 0 \to \cdots$$

where the degree of $b_0\mathcal{O}G$ is 0 and the non-trivial differential is induced by inclusion in the case $m \equiv 1 \pmod q$ and projection in the case $m \equiv -1 \pmod q$. Note that $C^\bullet \simeq f\mathcal{O}G$ or $C^\bullet \simeq f\mathcal{O}G[1]$ in the appropriate derived (or homotopy) category where we view $f\mathcal{O}G$ as a complex concentrated in degree 0.

REMARK 5.1. From $(*)$ and Proposition 5.2 below, we see that, under Conditions 4.1 and 4.3, the character of $\mathcal{K} \otimes_{\mathcal{O}} f\mathcal{O}G$ is $\sum_{\phi\in\mathrm{Irr}(b)}(\pi(G,S)(\phi) \times \check{\phi})$. Hence, by [7, Théorème 0.2.], $f\mathcal{O}G$ induces a Morita equivalence between $\mathcal{O}Gb$ and $\mathcal{O}G^S w(b)$, and we can also get Corollary 4.10 without using Lemmas 4.6, 4.7 and 4.8, if we utilize the facts $(*)$ and $(**)$.

For a group H and an $\mathcal{O}H$-module $\mathcal{M}$, we denote by $\mathcal{M}_Q$ a maximal direct summand of $\mathcal{M}$ any of whose indecomposable direct summand has Q as a vertex. We have a more precise statement of Proposition 4.2:

Proposition 5.2. *Assume* Condition 4.1. *With the above notations, we have the following isomorphisms of* $(\mathcal{O}G^S, \mathcal{O}G)$*-bimodules: when q is odd, if* $m \equiv 1 \pmod q$ *then*

$$(5.1.1)\qquad (b_0\mathcal{O}G)_{\Delta D} \simeq f\mathcal{O}G \oplus (b_l\mathcal{O}G)_{\Delta D}$$

and if $m \equiv -1 \pmod q$ *then*

$$(5.1.2)\qquad (b_l\mathcal{O}G)_{\Delta D} \simeq f\mathcal{O}G \oplus (b_0\mathcal{O}G)_{\Delta D},$$

and when $q = 2$, depending on the choice of b_0, we have (5.1.1) *or* (5.1.2). *Hence,* *if moreover we assume* Condition 4.3, *then we have*

$$b_0\mathcal{O}G \simeq f\mathcal{O}G \oplus b_l\mathcal{O}G$$

or

$$b_l\mathcal{O}G \simeq f\mathcal{O}G \oplus b_0\mathcal{O}G.$$

Proof. Let q be odd and $m \equiv 1 \pmod q$. Similar for the case $m \equiv -1 \pmod q$, using (3.2.3) in the proof of Lemma 3.2.

Let D_δ be a defect pointed group of $\mathcal{O}\Gamma\hat{b}_0$ and $\hat{V} = V_{\mathcal{O}\Gamma\hat{b}_0}(D_\delta)$. As in the proof of Proposition 4.2, the indecomposable decomposition of $(\mathcal{O}\Gamma\hat{b}_0{\downarrow}_{G^sS\times\Gamma})_{\Delta D}$ is described by the indecomposable decomposition of $\hat{V}{\downarrow}_{G^sS}$. Note that $N_\Gamma(D) = N_{G^sS}(D)C_\Gamma(D)$. Below we will identify S with its canonical image in $C_\Gamma(D)/D$ and recall that $\hat{V}$ has a canonical $kC_\Gamma(D)/D$-module structure.

Firstly, since primitive idempotents of $\mathrm{End}_{\mathcal{O}[G^sS\times\Gamma]}(\mathcal{O}\Gamma\hat{b}_0)$ remain to be primitive idempotents of $\mathrm{End}_{\mathcal{O}[G^s\times\Gamma]}(\mathcal{O}\Gamma\hat{b}_0)$, for primitive idempotents j_1 and j_2 of $(\mathcal{O}\Gamma\hat{b}_0)^{G^sS}$ such that $j_1\mathcal{O}\Gamma\hat{b}_0 \mid (\mathcal{O}\Gamma\hat{b}_0{\downarrow}_{G^sS\times\Gamma})_{\Delta D}$ and $j_2\mathcal{O}\Gamma\hat{b}_0 \mid (\mathcal{O}\Gamma\hat{b}_0{\downarrow}_{G^sS\times\Gamma})_{\Delta D}$,

$$j_1(\mathcal{O}\Gamma\hat{b}_0){\downarrow}^{G^sS\times\Gamma}_{G^s\times\Gamma} \simeq (j_2\mathcal{O}\Gamma\hat{b}_0){\downarrow}^{G^sS\times\Gamma}_{G^s\times\Gamma}$$

if and only if

$$(\pi_\delta(j_1)\cdot\hat{V}){\downarrow}^{N_{G^sS}(D_\delta)/D}_{N_{G^s}(D_\delta)/D} \simeq (\pi_\delta(j_2)\cdot\hat{V}){\downarrow}^{N_{G^sS}(D_\delta)/D}_{N_{G^s}(D_\delta)/D}.$$

Secondly, for a primitive idempotent j of $(\mathcal{O}\Gamma\hat{b}_0)^{G^sS}$, since $j = \sum_{r=1}^{q-1} e_r j$ in $(\mathcal{O}\Gamma\hat{b}_0)^{G^sS}$ and π_δ is an interior S-algebra homomorphism, we have $e_t\cdot(j\mathcal{O}\Gamma\hat{b}_0) \neq 0$ if and only if $\overline{e_t}\cdot(\pi_\delta(j)\cdot\hat{V}) \neq 0$ for $t \in \mathbb{Z}$.

Note that an indecomposable direct summand Z of $\mathcal{O}\Gamma\hat{b}_0{\downarrow}_{G^sS\times\Gamma}$ is determined by $Z{\downarrow}_{G^s\times\Gamma}$ and e_t such that $e_t\cdot Z \neq 0$.

Since a canonical character of $\hat{b}_0$ is the canonical extension of a canonical character of b by [36, p. 558], from the isomorphism (3.2.2) in the proof of Lemma 3.2, Lemma 3.3 and above remarks, we have the following isomorphism of $(\mathcal{O}G^SS, \mathcal{O}\Gamma)$-bimodules for some $(\mathcal{O}G^S, \mathcal{O}Gb)$-bimodule $\mathcal{N}$:

$$\left(\mathcal{O}\Gamma\hat{b}_0{\downarrow}^{\Gamma\times\Gamma}_{G^sS\times\Gamma}\right)_{\Delta D} \simeq \lambda^0\widehat{f\mathcal{O}G} \oplus \left(\bigoplus_{i=0}^{q-1}\lambda^i\hat{\mathcal{N}}\right)$$

where, for an $(\mathcal{O}G^S, \mathcal{O}Gb)$-bimodule $\mathcal{M}$, we denote by $\lambda^i\hat{\mathcal{M}}$ the extension of $\mathcal{M}$ to $(\mathcal{O}G^SSe_i, \mathcal{O}\Gamma\hat{b}_0)$-bimodule.

Hence, by Lemma 5.3 for $t = 0$ below, we have

$$(b_0\mathcal{O}G)_{\Delta D} \simeq (\lambda^0\widehat{f\mathcal{O}G} \oplus \lambda^0\hat{\mathcal{N}})\downarrow^{G^S S\times\Gamma}_{G^S\times G} \simeq f\mathcal{O}G \oplus \mathcal{N}$$

and

$$(b_l\mathcal{O}G)_{\Delta D} \simeq (\lambda^{q-l}\hat{\mathcal{N}})\downarrow^{G^S S\times\Gamma}_{G^S\times G} \simeq \mathcal{N}.$$

Hence the statement follows.

Similar for the case of $q = 2$. □

Lemma 5.3. *Assume* Condition 4.1. *With the above notations and for any integer* t, *we have the following* $(\mathcal{O}G^S, \mathcal{O}G)$*-bimodule isomorphism*:

$$b_i\mathcal{O}G \simeq (e_{t-i}\mathcal{O}\Gamma\hat{b}_t)\downarrow_{G^S\times G}.$$

Proof. We have the following isomorphisms of $(\mathcal{O}G^S, \mathcal{O}G)$-bimodules:

$$e_{t-i}\mathcal{O}\Gamma\hat{b}_t = \sum_{j=0}^{q-1} e_{j+t-i}\hat{b}_{j+t}\mathcal{O}\Gamma\hat{b}_t = b_i\mathcal{O}\Gamma\hat{b}_t \simeq b_i\mathcal{O}Gb.$$

For the last isomorphism, see Lemma 2.7. □

References

[1] J.L. Alperin: *Isomorphic blocks*, J. Algebra **43** (1976), 694–698.
[2] J.L. Alperin, M. Linckelmann and R. Rouquier: *Source algebras and source modules*, J. Algebra **239** (2001), 262–271.
[3] L. Barker: *Modules with simple multiplicity modules*, J. Algebra **172** (1995), 152–158.
[4] L. Barker: *Induction, restriction and G-algebras*, Comm. Algebra **22** (1994), 6349–6383.
[5] L. Barker: *On p-soluble groups and the number of simple modules associated with a given Brauer pair*, Quart. J. Math. Oxford Ser. (2) **48** (1997), 133–160.
[6] M. Broué: *Isométries parfaites, types de blocs, catégories dérivées*, Astérisque **181**–**182** (1990), 61–92.
[7] M. Broué: *Isométries de caractères et équivalences de Morita ou dérivées*, Inst. Hautes Études Sci. Publ. Math. **71** (1990), 45–63.
[8] E.C. Dade: *Block extensions*, Illinois J. Math. **17** (1973), 198–272.
[9] E.C. Dade: *Extending group modules in a relatively prime case*, Math. Z. **186** (1984), 81–98.
[10] E.C. Dade: *Counting characters in blocks* II, J. Reine Angew. Math. **448** (1994), 97–190.
[11] E.C. Dade: *A new approach to Glauberman's correspondence*, J. Algebra **270** (2003), 583–628.
[12] Y. Fan and L. Puig: *On blocks with nilpotent coefficient extensions*, Algebr. Represent. Theory **1** (1998), 27–73.
[13] G. Glauberman: *Correspondences of characters for relatively prime operator groups*, Canad. J. Math. **20** (1968), 1465–1488.

[14] M.E. Harris: *Glauberman-Watanabe corresponding p-blocks of finite groups with normal defect groups are Morita equivalent*, Trans. Amer. Math. Soc. **357** (2005), 309–335.
[15] M.E. Harris and S. Koshitani: *An extension of Watanabe's theorem for the Isaacs-Horimoto-Watanabe corresponding blocks*, J. Algebra **296** (2006), 96–109.
[16] M.E. Harris and M. Linckelmann: *On the Glauberman and Watanabe correspondences for blocks of finite p-solvable groups*, Trans. Amer. Math. Soc. **354** (2002), 3435–3453.
[17] A. Hida and S. Koshitani: *Morita equivalent blocks in non-normal subgroups and p-radical blocks in finite groups,* J. London Math. Soc. (2) **59** (1999), 541–556.
[18] H. Horimoto: *A note on the Glauberman correspondence of p-blocks of finite p-solvable groups*, Hokkaido Math. J. **31** (2002), 255–259.
[19] I.M. Isaacs: Character Theory of Finite Groups, Academic Press, New York, 1976.
[20] I.M. Isaacs and G. Navarro: *Character correspondences and irreducible induction and restriction*, J. Algebra **140** (1991), 131–140.
[21] S. Koshitani: *A remark on Glauberman-Watanabe correspondence of p-blocks of finite groups,* preprint (2002).
[22] S. Koshitani and G.O. Michler: *Glauberman correspondence of p-blocks of finite groups*, J. Algebra **243** (2001), 504–517.
[23] Z. Lu: *Module correspondences for twisted group algebras*, Comm. Algebra **31** (2003), 3519–3527.
[24] H. Nagao and Y. Tsushima: Representations of Finite Groups, Academic Press, Boston, MA, 1989.
[25] G. Navarro: *Some open problems on coprime action and character correspondences,* Bull. London Math. Soc. **26** (1994), 513–522.
[26] T. Okuyama: *A talk at a Seminar*, Ochanomizu University, 5 November, 2005.
[27] L. Puig: *Pointed groups and construction of characters,* Math. Z. **176** (1981), 265–292.
[28] L. Puig: *Pointed groups and construction of modules*, J. Algebra **116** (1988), 7–129.
[29] L. Puig: *Nilpotent blocks and their source algebras*, Invent. Math. **93** (1988), 77–116.
[30] L. Puig: *On Thévenaz' parametrization of interior G-algebras*, Math. Z. **215** (1994), 321–335.
[31] L. Puig: On the Local Structure of Morita and Rickard Equivalences Between Brauer Blocks, Progr. Math. **178**, Birkhäuser, Basel, 1999.
[32] J. Rickard: *Splendid equivalences*: *derived categories and permutation modules*, Proc. London Math. Soc. (3) **72** (1996), 331–358.
[33] G.R. Robinson: *On projective summands of induced modules*, J. Algebra **122** (1989), 106–111.
[34] F. Tasaka: *On the isotypy induced by the Glauberman-Dade correspondence between blocks of finite groups*, J. Algebra **319** (2008), 2451–2470.
[35] J. Thévenaz: G-Algebras and Modular Representation Theory, Oxford Univ. Press, New York, 1995.
[36] A. Watanabe: *The Glauberman character correspondence and perfect isometries for blocks of finite groups*, J. Algebra **216** (1999), 548–565.
[37] T.R. Wolf: *Character correspondences in solvable groups*, Illinois J. Math. **22** (1978), 327–340.

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