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UNIQUENESS IN THE CAUCHY PROBLEM FOR QUASI-HOMOGENEOUS OPERATORS WITH PARTIALLY HOLOMORPHIC COEFFICIENTS

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1. Introduction and main reults

The purpose of this work is to extend to the case of quasi-homogeneous symbols the recent results of Tataru [10], Hörmander [3] and Robbiano-Zuily [7] concerning the uniqueness of the Cauchy problem for operators with partially holomorphic coefficients. Even in the merely C^∞ coefficients case our results will be more general that those given in Isakov [4], Dehman [1] and Lascar-Zuily [6]. The method used here will be basically the same as in the proof given by [7], that is the use of the Sjöstrand theory of FBI transform to microlocalize the symbols and then symbolic calculus for anisotropic pseudo-differential operators and the Fefferman-Phong inequality.

Let us be more precise. Let n, d be two non negative integers with $n+d \geq 1$. We shall set $\mathbb{R}^{d+n} = \mathbb{R}^d \times \mathbb{R}^n$ and, for X or ζ in $\mathbb{R}^{d+n}$, $X = (x, y)$, $\zeta = (\xi, \tau)$. Here y will be the "C^∞ variables" and x the "analytic ones".

Let $m = (m_1, \dots, m_n)$, $\tilde{m} = (\tilde{m}_1, \dots, \tilde{m}_d)$ be multi-indices, such that

$$\text{(1.1)} \qquad \begin{cases} 0 < m_1 \leq \cdots \leq m_{q-1} < m_q = \cdots = m_n = M, \\ 0 < \tilde{m}_1 \leq \cdots \leq \tilde{m}_{p-1} < \tilde{m}_p = \cdots = \tilde{m}_d = \tilde{M} = M. \end{cases}$$

We set $h_j = M/m_j$, $\tilde{h}_j = M/\tilde{m}_j$. $\{\cdot, \cdot\}_0$ will denote the quasi-homogeneous Poisson bracket that is

$$\text{(1.2)} \qquad \{f, g\}_0 = \sum_{j=q}^{n} \left(\frac{\partial f}{\partial \tau_j} \frac{\partial g}{\partial y_j} - \frac{\partial f}{\partial y_j} \frac{\partial g}{\partial \tau_j} \right) + \sum_{j=p}^{d} \left(\frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} - \frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} \right).$$

If $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$, $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{N}^n$, we set

$$\text{(1.3)} \qquad |\alpha : \tilde{m}| = \sum_{j=1}^{d} \frac{\alpha_j}{\tilde{m}_j}, \quad |\beta : m| = \sum_{j=1}^{n} \frac{\beta_j}{m_j}.$$

Let $P = P(x, y, D_x, D_y)$ be the quasi-homogeneous differential operator

$$P = \sum_{|\alpha:\tilde{m}|+|\beta:m|\leq 1} a_{\alpha\beta}(x, y) D_x^\alpha D_y^\beta, \tag{1.4}$$

with symbol

$$p(x, y, \xi, \tau) = \sum_{|\alpha:\tilde{m}|+|\beta:m|\leq 1} a_{\alpha\beta}(x, y)\xi^\alpha \tau^\beta, \tag{1.5}$$

and quasi-homogeneous principal symbol

$$p_M(x, y, \xi, \tau) = \sum_{|\alpha:\tilde{m}|+|\beta:m|=1} a_{\alpha\beta}(x, y)\xi^\alpha \tau^\beta. \tag{1.6}$$

We shall assume that

$$\begin{cases} \text{the coefficients } (a_{\alpha\beta}) \text{ of } P \text{ are } C^\infty \text{ in } (x, y) \text{ and analytic in } x \\ \text{in a neighborhood of a point } (x_0, y_0) \in \mathbb{R}^{d+n}. \end{cases} \tag{1.7}$$

Let S be a C^2 hypersurface through (x_0, y_0) locally given by

$$S = \{(x, y) : \varphi(x, y) = \varphi(x_0, y_0)\}, \quad \nabla_{p,q}\varphi(x_0, y_0) \neq 0, \tag{1.8}$$

where

$$\nabla_{p,q}\varphi = \left(0, \dots, 0, \frac{\partial\varphi}{\partial x_p}, \dots, \frac{\partial\varphi}{\partial x_d}; 0, \dots, 0, \frac{\partial\varphi}{\partial y_q}, \dots, \frac{\partial\varphi}{\partial y_n}\right). \tag{1.9}$$

Our results are as follows.

Theorem A. *Let us assume*

(H.1) *transversal ellipticity*: $p_M(x_0, y_0; 0, \tau) \neq 0$, *for all* τ *in* $\mathbb{R}^n \setminus \{0\}$.

$$\begin{cases} \textit{quasi-homogeneous pseudo-convexity}: \\ \textit{let } \Xi = (x_0, y_0; (0, \tau) + i\lambda\nabla_{p,q}\varphi(x_0, y_0)), \quad \tau \in \mathbb{R}^n, \\ \textit{then } p_M(\Xi) = \{p_M, \varphi\}_0(\Xi) = 0 \textit{ implies} \\ \frac{1}{i}\left\{\overline{p}_M(X; \zeta - i\lambda\nabla_{p,q}\varphi(X)); p_M(X; \zeta + i\lambda\nabla_{p,q}\varphi(X))\right\}_0\Big|_{\substack{X=(x_0,y_0)\\ \xi=0}} > 0. \end{cases} \tag{H.2}$$

Let V *be a neighborhood of* (x_0, y_0) *and* $u \in C^\infty(V)$ *be such that*

$$\begin{cases} Pu = 0 \quad \textit{in } V \\ \operatorname{supp} u \subset \{X \in V : \varphi(X) \leq \varphi(X_0)\}. \end{cases}$$

Then there exists a neighborhood W *of* (x_0, y_0) *in which* $u \equiv 0$.

Theorem B. *Let us assume*

(H.1)′ $\begin{cases} \text{principal normality: } \left|\{\overline{p}_M; p_M\}(x,y;0,\tau)\right| \le C|\tau|_m^{M-1}|p_M(x,y;0,\tau)|, \\ \text{for all } (x,y) \text{ in a neighborhood of } (x_0,y_0) \text{ and all } \tau \text{ in } \mathbb{R}^n, \\ \text{where } |\tau|_m^{2M} = \sum_{j=1}^n |\tau_j|^{2m_j}. \end{cases}$

(H.2)′ $\begin{cases} \text{quasi-homogeneous pseudo-convexity:} \\ \text{(i) } n=0 \text{ or } n\ge 1 \text{ and, with } Z=(x_0,y_0;0,\tau),\ \tau\in\mathbb{R}^n\setminus\{0\}, \text{ then} \\ \quad p_M(Z)=\{p_M,\varphi\}_0(Z)=0 \text{ implies } \operatorname{Re}\{\overline{p}_M;\{p_M,\varphi\}_0\}_0(Z)>0. \\ \text{(ii) Let } W=(x_0,y_0;(0,\tau)+i\lambda\nabla_{p,q}\varphi(x_0,y_0)),\ \tau\in\mathbb{R}^n, \text{ then} \\ \quad p_M(W)=\{p_M,\varphi\}_0(W)=0 \text{ implies} \\ \quad \frac{1}{i}\left\{\overline{p}_M(X;\zeta-i\lambda\nabla_{p,q}\varphi(X)); p_M(X;\zeta+i\lambda\nabla_{p,q}\varphi(X))\right\}_0\Big|_{\substack{X=(x_0,y_0)\\ \xi=0}} > 0. \end{cases}$

(H.3)′ *On* $\xi = 0$, p_M *does not depend on* x.

Then the same conclusion, as in Theorem A, *holds.*

Let us make some comments on these results. The Theorems A and B contain the results of Tataru, Hörmander and Robbiano-Zuily for which we take $m = (M, \dots, M)$, $\tilde{m} = (M, \dots, M)$. In the C^∞ case ($d = 0$), the Theorems A and B extend the results of Lascar-Zuily ([6], thm 1.3) (take $m = (1, 2, \dots, 2)$), the Theorem 2.1 in Dehman [1] and contain the results of Isakov ([4], thm 1.1 and 1.2) who consider only elliptic or real symbols. Furthermore with slight modifications of notations (1.2), (1.9), Theorems A and B remain valid with $\tilde{M} < M$ or $\tilde{M} > M$ (see (1.1)).

1. Here is an application of Theorem A. Let us consider, in a neighborhood V of $(0,0)$ in $\mathbb{R}_x \times \mathbb{R}_y^n$ a second order parabolic symbol of the form

$$p(x,y;\xi,\tau) = \sum_{j,k=2}^{n} a_{jk}(x,y)\tau_j\tau_k + i\tau_1 + a(x,y)\xi^2,$$

where the coefficients (a_{jk}) are real-valued, belong to $C^\infty(\mathbb{R}_x \times \mathbb{R}_y^n)$ and are analytic in x with $a(0,0) \neq 0$. We assume that the following parabolicity condition is satisfied

$$\sum_{j,k=2}^{n} a_{jk}(x,y)\tau_j\tau_k \ge C(\tau_2^2 + \dots + \tau_n^2) \text{ for all } (x,y) \in V,\ (\tau_2,\dots,\tau_n) \in \mathbb{R}^{n-1}.$$

Then the conclusion of Theorem A holds with $S = \{(x,y) : y_n = 0\}$ (we take $\varphi(x,y) = \exp(-\lambda y_n) - 1$, for λ large).

2. Application of Theorem B. Let us consider the case where $(x,y) \in \mathbb{R} \times \mathbb{R}^n$, $S = \{\varphi(x,y) = y_1 = 0\}$ and

$$P = D_{y_1}^2 + \sum_{j,k=2}^{n-1} a_{jk}(y)D_{y_j}D_{y_k} + c(y)D_{y_n} + d(x,y)D_x^2.$$

Assume moreover that

- (a_{jk}), c are real-valued, C^∞ in y and $c(0) \neq 0$.
- d is C^∞ in (x, y), analytic in x and $d(0) \neq 0$ real.

Then, it follow that (H.1)′ is empty, (H.3)′ is trivially satisfied and $\nabla_{p,q}\varphi(0) \neq 0$. We show that (H.2)′ (i) is equivalent to

$$\forall(\tau_2, \dots, \tau_{n-1}) \in \mathbb{R}^{n-2}, \quad \sum_{j,k=2}^{n-1} \frac{\partial a_{jk}}{\partial y_1}(0)\tau_j\tau_k - \frac{\partial c/\partial y_1(0)}{c(0)} \sum_{j,k=2}^{n-1} a_{jk}(0)\tau_j\tau_k < 0.$$

For example, we can take, $P = D_{y_1}^2 - \sum_{j=2}^{n-1} D_{y_j}^2 + (1 - y_n)D_{y_n} + (1 + ix)D_x^2$.

The proofs follow from Carleman estimates with an exponential weight $e^{-\lambda\psi}$ and these estimates follow from Gårding type inequalities on the operator $P_\lambda = e^{\lambda\psi} P e^{-\lambda\psi}$. The problem is that all our conditions are made on the set $\{\xi = 0\}$. So we have to microlocalize our symbol on this set; this is achieved by the use of Sjöstrand's theory of the FBI transform [8], [9]. We then use the C^∞-machinery (the Hörmander-Weyl calculus, the Fefferman-Phong inequality, see [2]) to prove a Carleman estimate using some techniques of Lerner [5].

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2. The partial FBI transformation

In this section we collect some material essentially taken from [9], [7]. We introduce the partial Fourier-Bros-Iagolnitzer (FBI) transformation. It is defined for u in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n)$ by

$$Tu(z, y, \lambda) = C(\lambda) \int_{\mathbb{R}^d} e^{-(\lambda/2)(x-z)^2} u(x, y)dx \tag{2.1}$$

where $z \in \mathbb{C}^d$, $y \in \mathbb{R}^n$, $\lambda \geq 1$, $C(\lambda) = 2^{-d/2}(\lambda/\pi)^{3d/4}$ and $z^2 = \sum_{j=1}^d (z^j)^2$, $z = (z^j) \in \mathbb{C}^d$.

The function Tu is C^∞ on $\mathbb{R}^{2d} \times \mathbb{R}^n \times [1, \infty[$ and entire-holomorphic in $z \in \mathbb{C}^d$ for all (y, λ) in $\mathbb{R}^n \times [1, \infty[$. Let us set

$$\Phi(z) = \frac{1}{2}(\operatorname{Im} z)^2, \quad z \text{ in } \mathbb{C}^d, \tag{2.2}$$

$$\Lambda_\Phi = \left\{(z, \xi) \in \mathbb{C}^{2d} : \xi = \frac{2}{i}\frac{\partial \Phi}{\partial x}(z)\right\} = \left\{(z, \xi) \in \mathbb{C}^{2d} : \xi = -\operatorname{Im} z\right\}, \tag{2.3}$$

$$K_T(x, \xi) = (x - i\xi, \xi), \quad (x, \xi) \in T^*\mathbb{R}^d. \tag{2.4}$$

Then $K_T : T^*\mathbb{R}^d \to \Lambda_\Phi$ is a diffeomorphism.

In the sequel we shall also work with the partial FBI transformation T_η associated with the phase $(i/2)(1+\eta)(x-z)^2$ where η is a small non negative real number,

$$T_\eta u(z,y,\lambda) = C(\lambda)\int_{\mathbb{R}^d} e^{-(\lambda/2)(1+\eta)(x-z)^2} u(x,y)dx. \tag{2.5}$$

Let

$$K_{T_\eta}(x,\xi) = \Big(x - \frac{i\xi}{1+\eta};\xi\Big). \tag{2.6}$$

Let us introduce some notations. For $k \in \mathbb{N}$ we set

$$L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^k(\mathbb{R}^n)) = L^2\Big((\mathbb{C}^d, e^{-2\lambda(1+\eta)\Phi(x)}L(dx)); H^k(\mathbb{R}^n)\Big) \tag{2.7}$$

where $L(dx)$ denotes the Lebesgue measure in $\mathbb{C}^d$ and $H^k(\mathbb{R}^n)$ the usual Sobolev space.

If $k = 0$ we shall set for short

$$L^2_{(1+\eta)\Phi}\big(\mathbb{C}^d, H^0(\mathbb{R}^n)\big) = L^2_{(1+\eta)\Phi}, \tag{2.8}$$

$$|||u|||_k^2 = \int_{\mathbb{R}^d}\int_{\mathbb{R}^n} (\lambda + |\tau|_m)^{2k}|\hat{u}(\zeta)|^2 d\zeta. \tag{2.9}$$

Then we have

Proposition 2.1 (see [9]). i) *T_η is an isometry from $L^2(\mathbb{R}^d, H^k(\mathbb{R}^n))$ to $L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^k(\mathbb{R}^n))$.*
ii) *$T^*_\eta T_\eta$ is the identity on $L^2(\mathbb{R}^n)$, where T^*_η is the adjoint of T_η.*
iii) *$T_\eta T^*_\eta$ is the projection from $L^2_{(1+\eta)\Phi}$ to $L^2_{(1+\eta)\Phi} \cap \mathcal{H}(\mathbb{C}^d)$ where $\mathcal{H}$ denotes the space of holomorphic functions. In particular $T_\eta T^*_\eta v = v$ if $v = Tw$ where w is in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n)$.*

3. Transfer to the complex domain and the localization procedure

Let $p = \sum_{|\alpha:\tilde{m}|+|\beta:m|\le 1} a_{\alpha\beta}(x,y)\xi^\alpha\tau^\beta$, $(x,y) \in \mathbb{R}^d \times \mathbb{R}^n$, be a polynomial with coefficients in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$.

Assume moreover that

$$\begin{cases} \text{there exists } C_0 > 0 \text{ such that if we set } \omega_1 = \{z \in \mathbb{C}^d : |z| < C_0\} \\ \text{and } \omega_2 = \{y \in \mathbb{R}^n : |y| < C_0\}, \text{ then for all } (\alpha,\beta) \in \mathbb{N}^d \times \mathbb{N}^n, \\ |\alpha:\tilde{m}| + |\beta:m| \le 1, \text{ we have } a_{\alpha\beta} \in C^\infty(\omega_2, \mathcal{H}(\omega_1)). \end{cases} \tag{3.1}$$

Let $P = Op^\omega_\lambda(p)$ be the semi-classical Weyl quantized operator with symbol p, for $u \in C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$,

$$Pu(x,y) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(X-\tilde{X})\zeta} p\Big(\frac{X+\tilde{X}}{2};\lambda\zeta\Big)u(\tilde{X})d\tilde{X}d\zeta. \tag{3.2}$$

Let ψ be a real quadratic polynomial on $\mathbb{R}^d \times \mathbb{R}^n$. For any $\lambda \geq 1$, we shall denote P_λ the differential operator defined by

$$P_\lambda = e^{\lambda\psi} P e^{-\lambda\psi}. \tag{3.3}$$

It follows that

$$P_\lambda u(X) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(X-\tilde{X})\zeta} P\Big(\frac{X+\tilde{X}}{2}; \lambda\zeta + i\lambda\psi'\Big(\frac{X+\tilde{X}}{2}\Big)\Big) u(\tilde{X}) d\tilde{X} d\zeta. \tag{3.4}$$

Proposition 3.1 (see [7]). *For v in $C_0^\infty(\mathbb{R}^d \times \mathbb{R}^n)$, we have $TP_\lambda v = \tilde{P}_\lambda T v$ where*

$$\tilde{P}_\lambda T v(X, \lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(y-\tilde{y})\tau} \Big(\iint_{\xi = -\operatorname{Im}((x+\tilde{x})/2)} \omega \Big) d\tilde{y} d\tau \tag{3.5}$$

where

$$\omega = e^{i\lambda(x-\tilde{x})\xi} p\Big(\frac{x+\tilde{x}}{2} + i\xi, \frac{y+\tilde{y}}{2}; \lambda\zeta + i\lambda\psi'\Big(\frac{x+\tilde{x}}{2} + i\xi; \frac{y+\tilde{y}}{2}\Big)\Big) \\ Tv(\tilde{x}, \tilde{y}, \lambda) d\tilde{x} \wedge d\xi. \tag{3.6}$$

Let δ is a positive real number such that $2\delta < C_0$ where C_0 is defined in (3.1) and v is a C^∞ function such that $\operatorname{supp} v \subset \{X \in \mathbb{R}^d \times \mathbb{R}^n : |X| \leq \delta\}$. Let $\tilde{P}_\lambda$ be defined in Proposition 3.1.

Case of Theorem A.

Theorem 3.2 (see [7]). *There exists $\chi \in C_0^\infty(\mathbb{C}^{2d})$, $\chi(x, \xi) = 1$ if $|x| + |\xi| \leq \delta$, $\chi(x, \xi) = 0$ if $|x| + |\xi| \geq 2\delta$ such that if we set, for $\eta \in]0, 1]$,*

$$\tilde{Q}_\lambda T v(X, \lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{d+n} \iint e^{i\lambda(y-\tilde{y})\tau} \Big(\iint_{\xi = (1+\eta)\operatorname{Im}((x+\tilde{x})/2)} \chi\Big(\frac{x+\tilde{x}}{2}; \xi\Big) \omega \Big) d\tilde{y} d\tau \tag{3.7}$$

where ω is defined in (3.6), *then*

$$\tilde{P}_\lambda T v = \tilde{Q}_\lambda T v + \tilde{R}_\lambda T v + \tilde{g}_\lambda \tag{3.8}$$

where with, for any N in $\mathbb{N}$,

$$\|\tilde{R}_\lambda T v\|_{L^2_{(1+\eta)\Phi}} \leq \frac{C_N}{\lambda^N} \|Tv\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^M_\lambda(\mathbb{R}^n))} \tag{3.9}$$

$$\|\tilde{g}_\lambda\|_{L^2_{(1+\eta)\Phi}} \leq C e^{-(\lambda/3)\eta\delta^2} \|v\|_{L^2(\mathbb{R}^d, H^M_\lambda(\mathbb{R}^n))} \tag{3.10}$$

where

$$\|w\|_{H^M_\lambda(\mathbb{R}^n)} = \sum_{\sum_{j=1}^n h_j\beta_j \le M} \lambda^{M-\sum_{j=1}^n h_j\beta_j} \|D^\beta w\|_{L^2(\mathbb{R}^n)}. \tag{3.11}$$

Case of Theorem B.

Recall that we have assumed

$$\text{on } \xi = 0,\ p_M \text{ does not depend on } x. \tag{3.12}$$

In the case we have

$$p_M(X; \lambda\zeta + i\lambda\psi'(X)) = p'_M(y, \tau) + p'_{M-1}(X, \zeta) \tag{3.13}$$

where p'_M is a polynomial of order M in τ and p'_{M-1} is a polynomial of order M in ζ, but of order $M-1$ in τ.

Writing $p(X, \zeta) = p_M(X, \zeta) + p''_M(X, \zeta)$ where

$$p''_M(X, \zeta) = \sum_{|\alpha:\tilde{m}|+|\beta:m| \le 1-1/M} a_{\alpha\beta}(X)\xi^\alpha\tau^\beta. \tag{3.14}$$

We have

Theorem 3.3 (see [7]). *There exists $\chi \in C_0^\infty(\mathbb{C}^{2d})$, $\chi(x, \xi) = 1$ if $|x| + |\xi| \le \delta$, $\chi(x, \xi) = 0$, if $|x| + |\xi| \ge 2\delta$, such that, if we set, for $\eta \in]0, 1]$*

$$\tilde{Q}_\lambda Tv(X, \lambda) = \left(\frac{\lambda}{2\pi}\right)^{d+n} \iint e^{i\lambda(y-\tilde{y})\tau} \left(\iint_{\xi=-(1+\eta)\operatorname{Im}((x+\tilde{x})/2)} \tilde{\omega} \right) d\tilde{y}\, d\tau \tag{3.15}$$

where

$$\begin{aligned} \tilde{\omega} = e^{i\lambda(x-\tilde{x})\xi} \Big[p'_M(y, \tau) + \chi\Big(\frac{x+\tilde{x}}{2}; \xi\Big) \Big[p'_{M-1}\Big(\frac{x+\tilde{x}}{2} + i\xi, \frac{y+\tilde{y}}{2}; \zeta\Big) \\ + p''_M\Big(\frac{x+\tilde{x}}{2} + i\xi, \frac{y+\tilde{y}}{2}; \lambda\zeta + i\lambda\psi'\Big(\frac{x+\tilde{x}}{2} + i\xi; \frac{y+\tilde{y}}{2}\Big)\Big)\Big]\Big] Tv(\tilde{x}, \tilde{y}, \lambda) d\tilde{x} \wedge d\xi. \end{aligned} \tag{3.16}$$

Then we have, with $\tilde{P}_\lambda$ introduced in Proposition 3.1,

$$\tilde{P}_\lambda Tv = \tilde{Q}_\lambda Tv + \tilde{R}_\lambda Tv + \tilde{g}_\lambda \tag{3.17}$$

with

$$\|\tilde{R}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}} \le \frac{C_N}{\lambda^N} \|Tv\|_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H^{M-1}_\lambda(\mathbb{R}^n))} \tag{3.18}$$

$$\|\tilde{g}_\lambda\|_{L^2_{(1+\eta)\Phi}} \le Ce^{-(\lambda/3)\eta\delta^2} \|v\|_{L^2(\mathbb{R}^d, H^{M-1}_\lambda(\mathbb{R}^n))}. \tag{3.19}$$

4. Back to the real domain

Let v be in $\mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n)$ and $w = T_\eta^* T v$, then it follows that

$$w = T_\eta^* T v \in \mathcal{S}(\mathbb{R}^d \times \mathbb{R}^n), \quad T_\eta w = T v. \tag{4.1}$$

We deduce from Proposition 3.1

$$\tilde{Q}_\lambda T v = \tilde{Q}_\lambda T_\eta w = T_\eta Q_\lambda \omega, \tag{4.2}$$

where Q_λ is an operator on $\mathbb{R}^d \times \mathbb{R}^n$, pseudo-differential in x, differential in y.
Moreover denoting by σ^ω the Weyl symbol

$$\sigma^\omega(Q_\lambda)(x, \xi; y, \tau) = \sigma^\omega(\tilde{Q}_\lambda)(K_{T_\eta}(x, \xi); y, \tau), \tag{4.3}$$

where

$$\begin{cases} \sigma^\omega(Q_\lambda)(X, \zeta) - \chi\Big(x - \dfrac{i}{1+\eta}\xi; \xi\Big) p\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \lambda\zeta \\ \qquad\qquad + i\lambda\psi'\Big(x + \dfrac{i\eta}{1+\eta}\xi, y\Big)\Big) \quad \text{(thm A)} \\ \sigma^\omega(Q_\lambda)(X, \zeta) = p'_M(y, \tau) + \chi\Big(x - \dfrac{i}{1+\eta}\xi; \xi\Big)\Big[p'_{M-1}\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \zeta\Big) \\ \qquad + p''_M\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \lambda\zeta + i\lambda\psi'\Big(x + \dfrac{i\eta}{1+\eta}\xi, y\Big)\Big)\Big] \quad \text{(thm B)} \end{cases} \tag{4.4}$$

and

$$Q_\lambda u(X, \lambda) = \Big(\frac{\lambda}{2\pi}\Big)^{n+d} \iint e^{i\lambda(X-\tilde{X})\zeta} \sigma^\omega(Q_\lambda)\Big(\frac{X+\tilde{X}}{2}; \lambda\zeta\Big) u(\tilde{X}) d\tilde{X} d\zeta.$$

Moreover, we have

$$\sigma^\omega(Q_\lambda)(X, \zeta) = q_M(X, \zeta) + q_{M-1}(X, \zeta), \tag{4.5}$$

where

$$\begin{cases} q_M(X, \zeta) = \chi\Big(x - \dfrac{i}{1+\eta}\xi; \xi\Big) p_M\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \lambda\zeta \\ \qquad\qquad + i\lambda\psi'\Big(x + \dfrac{i\eta}{1+\eta}\xi, y\Big)\Big) \quad \text{(thm A)} \\ q_M(X, \zeta) = p'_M(y, \tau) + \chi\Big(x - \dfrac{i}{1+\eta}\xi, \xi\Big) p'_{M-1}\Big(x + \dfrac{i\eta}{1+\eta}\xi, y; \zeta\Big) \quad \text{(thm B)} \end{cases} \tag{4.6}$$

and

$$q_{M-1}(X,\zeta) = \chi\Big(x - \frac{i}{1+\eta}\xi, \xi\Big) \times p''_M\Big(x + \frac{i\eta}{1+\eta}\xi, y; \lambda\zeta + i\lambda\psi'\Big(x + \frac{i\eta}{1+\eta}\xi, y\Big)\Big). \tag{4.7}$$

5. The estimates in case of Theorem A

We are now prepared to prove Carleman estimates for Q_λ. Without loss of generality we may assume that $(x_0, y_0) = 0$ and $\varphi(0) = 0$. Let, for $Z = (x_1, \dots, x_d; y_1, \dots, y_n)$,

$$|Z|^{2M}_{(m,\tilde{m})} = |x_1|^{2\tilde{m}_1} + \cdots + |x_d|^{2\tilde{m}_d} + |y_1|^{2m_1} + \cdots + |y_n|^{2m_n}. \tag{5.1}$$

Lemma 5.1. *There exist positive constants C, η_0 such that for all η in $]0, \eta_0]$ and if we set*

$$\psi(X) = \varphi'(0)X + \frac{1}{2}\varphi''(0)X \cdot X - \frac{1}{2C^2}|X|^2 + \frac{C}{2}(\varphi'(0)X)^2,$$

then

$$C|q_M(X,\zeta)|^2 + \frac{1}{i}\{\overline{q}_M, q_M\}(X,\zeta) \geq \frac{1}{C}(\lambda + |\lambda\tau|_m)^{2M}, \tag{5.2}$$

for $|X| + |\xi| \leq 1/C^2$ and λ so large.

By homogeneity, (5.2) is still true with the same ψ if we replace ψ by $\rho\psi$ where ρ is a positive constant.

Proof. We first take C so large that $\chi = 1$ if $|X| + |\xi| \leq 1/C^2$. It follows then from (4.6) that

$$\begin{aligned} q_M(X,\zeta) &= p_M\Big(x + \frac{i\eta}{1+\eta}\xi, y; \lambda\zeta + i\lambda\psi'\Big(x + \frac{i\eta}{1+\eta}\xi, y\Big)\Big), \\ &= p_M\big(X; \lambda\zeta + i\lambda\psi'(X)\big) + \frac{\eta}{C^2}\mathcal{O}\big((\lambda + |\lambda\tau|_m)^M\big), \end{aligned}$$

and

$$\left\{ \begin{aligned} \{\overline{q}_M, q_M\}|_{\xi=0} = \big\{\overline{p}_M(X; \lambda\zeta - i\lambda\psi'(X)); p_M(X; \lambda\zeta + i\lambda\psi'(X))\big\}\big|_{\xi=0} \\ + \eta\mathcal{O}\big((\lambda + |\lambda\tau|_m)^{2M}\big). \end{aligned} \right. \tag{5.3}$$

We shall also write

$$\{\overline{q}_M, q_M\}(X,\zeta) = \{\overline{q}_M, q_M\}|_{\xi=0}(X,\zeta) + \frac{1}{C^2}\mathcal{O}((\lambda + |\lambda\tau|_m)^{2M}), \tag{5.4}$$

and

$$p_M(X;\lambda\zeta+i\lambda\psi'(X)) = p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0} + \Big(\frac{1}{C^2}+\lambda^{-1/(M-1)}\Big)\mathcal{O}((\lambda+|\lambda\tau|_m)^M). \tag{5.5}$$

Then

$$q_M(X,\zeta) = p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0} + \Big(\frac{1}{C^2}+\lambda^{-1/(M-1)}\Big)\mathcal{O}((\lambda+|\lambda\tau|_m)^M), \tag{5.6}$$

and

$$\{\overline{q}_M, q_M\}(X,\zeta) = \Big\{\overline{p}_M(X;\lambda\zeta-i\lambda\nabla_{p,q}\psi(X)), p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))\Big\}|_{\xi=0} + \Big(\eta+\frac{1}{C^2}+\lambda^{-1/(M-1)}\Big)\mathcal{O}((\lambda+|\lambda\tau|_m)^{2M}). \tag{5.7}$$

Furthermore, we have

$$\begin{aligned} &\frac{C}{4}\Big|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\Big|^2 \\ &+\frac{1}{2i}\Big\{\overline{p}_M(X;\lambda\zeta-i\lambda\nabla_{p,q}\psi(X)); p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))\Big\}|_{\xi=0} \\ &\geq \frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M}, \quad \text{for } |X|\leq\frac{1}{C^2} \text{ and } \tau \text{ in } \mathbb{R}^n. \end{aligned} \tag{5.8}$$

Indeed, (5.8) is equivalent to

$$\begin{aligned} &\frac{C}{4}\Big|p_M(X;\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\Big|^2 \\ &+\frac{\lambda}{2i}\Big\{\overline{p}_M(X;\zeta-i\lambda\nabla_{p,q}\psi(X)); p_M(X;\zeta+i\lambda\nabla_{p,q}\psi(X))\Big\}|_{\xi=0} \\ &\geq \frac{1}{C}(\lambda+|\tau|_m)^{2M}, \quad \text{for } |X|\leq\frac{1}{C^2}. \end{aligned}$$

We see, setting $\Gamma=\lambda/(\lambda+|\tau|_m)$, $W=(X, Z+i\Gamma\nabla_{p,q}\psi(X))$ and

$$Z=\Big(0,\dots,0;\tau_1/(\lambda+|\tau|_m)^{h_1},\dots,\tau_n/(\lambda+|\tau|_m)^{h_n}\Big)$$

that (5.8) is equivalent to

$$
\begin{aligned}
(5.9)\quad & \frac{C}{4}|p_M(W)|^2 + \Gamma\,\mathrm{Im}\Bigg(\sum_{j=1}^{n}(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial y_j}(W)\\
&+\sum_{k=1}^{d}(\lambda+|\tau|_m)^{1-\tilde{h}_k}\frac{\partial\overline{p}_M}{\partial\xi_k}(W)\frac{\partial p_M}{\partial x_k}(W)\Bigg)\\
&+\Gamma^2\,\mathrm{Re}\Bigg(\sum_{j=1}^{n}\sum_{s=q}^{n}\frac{\partial^2\psi}{\partial y_s\partial y_j}(X)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial\tau_s}(W)(\lambda+|\tau|_m)^{1-h_j}\\
&+\sum_{j=1}^{n}\sum_{k=p}^{d}\frac{\partial^2\psi}{\partial x_k\partial y_j}(X)(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial\xi_k}(W)\\
&+\sum_{j=1}^{d}\sum_{k=q}^{n}\frac{\partial^2\psi}{\partial y_k\partial x_j}(X)(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial\tau_k}(W)\\
&+\sum_{j=1}^{d}\sum_{k=p}^{d}\frac{\partial^2\psi}{\partial x_k\partial x_j}(X)(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(W)\frac{\partial p_M}{\partial\xi_k}(W)\Bigg) \ge \frac{1}{C}, \quad \text{for } |X| \le \frac{1}{C^2}.
\end{aligned}
$$

We prove (5.9) by contradiction. If it is false one can find sequences X_k, λ_k, τ_k, Γ_k with $|X_k| \le 1/k^2$, $\lambda_k \ge e^k$ and τ_k in $\mathbb{R}^n$, such that, by definition ψ,

$$
\begin{aligned}
(5.10)\quad & \frac{k}{4}|p_M(W_k)|^2 + \Gamma_k\,\mathrm{Im}\Bigg(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial y_j}(W_k) + \sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial x_j}(W_k)\Bigg)\\
&+\Gamma_k^2\,\mathrm{Re}\Bigg(\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k) + \sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_s}(W_k)\\
&+2\sum_{s=q}^{n}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)\Bigg)\\
&+k\Gamma_k^2\Bigg(\Bigg|\sum_{j=p}^{d}\frac{\partial\varphi}{\partial x_j}(0)\frac{\partial p_M}{\partial\xi_j}(W_k)\Bigg|^2 + \Bigg|\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Bigg|^2\Bigg)\\
&-\frac{\Gamma_k^2}{k^2}\Bigg(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_j}(W_k) + \sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Bigg)\\
&+2k\Gamma_k^2\,\mathrm{Re}\Bigg[\Bigg(\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Bigg)\Bigg(\sum_{s=p}^{d}\frac{\partial\varphi}{\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_s}(\overline{W}_k)\Bigg)\Bigg] + A_k \le \frac{1}{k}
\end{aligned}
$$

where

$$
(5.11)\qquad |A_k| \le C_0 k\lambda_k^{-1/(M-1)} \le C_0 k e^{-k/(M-1)}, \quad C_0 \text{ is independent of } k.
$$

Since $\Gamma_k + |Z_k|_{(m,\tilde{m})} = 1$, taking subsequences, we may assume that

$$\Gamma_k \to \Gamma^0 \text{ and } Z_k \to Z^0 \text{ with } \Gamma^0 + |Z^0|_{(m,\tilde{m})} = 1. \tag{5.12}$$

CASE 1. $\Gamma^0 \neq 0$.

If we divide both members of (5.10) by k, we get with $W^0 = (0; Z^0 + i\Gamma\nabla_{p,q}\varphi(0))$

$$p_M(W^0) = \{p_M, \varphi\}_0(W^0) = 0.$$

Removing all positive terms in (5.10) and letting k go to $+\infty$, we get

$$\begin{aligned}
&\Gamma^0 \operatorname{Im}\Big(\sum_{j=q}^{n} \frac{\partial \overline{p}_m}{\partial \tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial y_j}(W^0) + \sum_{j=p}^{d} \frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial x_j}(W^0)\Big) \\
&+(\Gamma^0)^2 \operatorname{Re}\Big(\sum_{s,j=q}^{n} \frac{\partial^2\varphi}{\partial y_s \partial y_j}(0)\frac{\partial \overline{p}_M}{\partial \tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial \tau_s}(W^0) \\
&+ \sum_{j,s=p}^{d} \frac{\partial^2\varphi}{\partial x_j \partial x_s}(0)\frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial \xi_s}(W^0) \\
&+2\sum_{s=q}^{n}\sum_{j=p}^{d} \frac{\partial^2\varphi}{\partial y_s \partial x_j}(0)\frac{\partial \overline{p}_M}{\partial \xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial \tau_s}(W^0)\Big) \leq 0
\end{aligned}$$

which contradicts the hypothesis (H.2) in theorem A.

CASE 2. $\Gamma^0 = 0$.

Since $\Gamma^0 + |Z^0|_{(m,\tilde{m})} = 1$, we have $Z^0 \neq 0$ and $W^0 = (0, Z^0)$. If we divide both members of (5.10) by k, we get $p_M(W^0) = 0$ which is contradiction with (H.1) in Theorem A.

Now (5.6), (5.7) and (5.8) imply (5.2) if η is small enough and C, λ so large. This ends the proof of Lemma 5.1. □

From now on C is fixed according to Lemma 5.1.

Let $\tilde{\theta}_0 \in C^\infty(\mathbb{C}^{2d})$ be such that $0 \leq \tilde{\theta}_0 \leq 1$ and

$$\begin{cases} \tilde{\theta}_0(x,\xi) = 1 & \text{if } |x| + |\xi| \leq \dfrac{\eta}{1+\eta} \cdot \dfrac{1}{4C^2}, \\ \tilde{\theta}_0(x,\xi) = 0 & \text{if } |x| + |\xi| \geq \dfrac{\eta}{1+\eta} \cdot \dfrac{1}{2C^2}, \\ \tilde{\theta}_0 \text{ is almost analytic on } \Lambda_{(1+\eta)\Phi}. \end{cases} \tag{5.13}$$

Let us set, with K_{T_η} defined in (2.6),

$$\theta_0 = \tilde{\theta}_0|_{\Lambda_{(1+\eta)\Phi}} \circ K_{T_\eta}. \tag{5.14}$$

It is easy to see that $\theta_0 \in C^\infty(\mathbb{R}^{2d})$ and there exists $\varepsilon_0 \in]0, 1/(2C^2)[$ such that

$$(5.15)\qquad \theta_0(x,\xi) = \begin{cases} 1 & \text{if } |x| + |\xi| \le \varepsilon_0, \\ 0 & \text{if } |x| + |\xi| \ge \dfrac{1}{2C^2}. \end{cases}$$

Let $h \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le h \le 1$ and

$$(5.16)\qquad h = \begin{cases} 1 & \text{if } |y| \le \dfrac{1}{4C^2}, \\ 0 & \text{if } |y| \ge \dfrac{1}{2C^2}. \end{cases}$$

Finally let us set

$$(5.17)\qquad \theta(X,\xi) = h(y)\theta_0(x,\xi).$$

Then

$$(5.18)\qquad \theta(X,\xi) = \begin{cases} 1 & \text{if } |X| + |\xi| \le \varepsilon_0, \\ 0 & \text{if } |X| + |\xi| \ge \dfrac{1}{C^2}. \end{cases}$$

Lemma 5.2. *Let $Q = Op_\lambda^w(q_M)$. There exist positive constants C_0, C_1, λ_0 such that for every u in $\mathcal{S}(\mathbb{R}^{d+n})$ and $\lambda \ge \lambda_0$, we have*

$$(5.19)\qquad \frac{C_1}{\lambda}\big(Op_\lambda^w((1-\theta)(\lambda + |\lambda\tau|_m)^{2M})u, u\big)_{L^2} + \|Qu\|_{L^2}^2 \ge \frac{C_0}{\lambda}|||u|||_M^2.$$

Proof. We write $Q = Q_R + iQ_I$ where $Q_R = Op_\lambda^w(\operatorname{Re} q_M)$, $Q_I = Op_\lambda^w(\operatorname{Im} q_M)$. Then writing $\|\cdot\|$ for the $L^2(\mathbb{R}^{d+n})$-norm

$$(5.20)\qquad \|Qu\|^2 = \|Q_R u\|^2 + \|Q_I u\|^2 + \frac{1}{2}([Q^*, Q]u, u).$$

Now the semiclassical principal symbols of $[Q^*, Q]$ and $Q_K^* Q_K$ are $(1/i)\{\overline{q}_M, q_M\}$ and q_K^2 where $q_R = \operatorname{Re} q_M$, $q_I = \operatorname{Im} q_M$. We claim that one can find a positive constant B such that

$$(5.21)\qquad B(1-\theta)(\lambda + |\lambda\tau|_m)^{2M} + C|q_M(X,\zeta)|^2 + \frac{1}{i}\{\overline{q}_M, q_M\}(X,\zeta) \ge \frac{1}{C}(\lambda + |\lambda\tau|_m)^{2M}, \quad \text{for all } (X,\zeta) \in \mathbb{R}^{2(d+n)}.$$

Indeed Lemma 5.1 implies (5.21) if $|X|+|\xi| \le 1/C^2$, since $0 \le \theta \le 1$, and if $|X|+|\xi| \ge 1/C^2$ then, by (5.18), $\theta = 0$ and $|q_M|^2 + |\{\overline{q}_M, q_M\}| \le C_1(\lambda + |\lambda\tau|_m)^{2M}$, thus (5.21) is true if B is large enough.

Then we can apply the Gårding inequality in the following context. Let

$$g = dx^2 + dy^2 + d\xi^2 + \sum_{j=1}^{n} \frac{\lambda^2 d\tau_j^2}{(\lambda + |\lambda\tau|_m)^{2h_j}}.$$

This is a metric which is temperate and slowly varying in the sense of Hörmander [2]. Let $a \in S((\lambda + |\lambda\tau|_m)^k, g)$, $k \in \mathbb{N}$, be a symbol such that $\operatorname{Re} a \geq \delta(\lambda + |\lambda\tau|_m)^{2k}$, and $A = Op_\lambda^w(a)$. Then there exists $\lambda_0 > 0$ such that for every u in $\mathcal{S}(\mathbb{R}^{d+n})$ and every $\lambda \geq \lambda_0$

$$\operatorname{Re}(Au, u)_{L^2} \geq \frac{\delta}{2} |||u|||_k^2. \tag{5.22}$$

Thus we may apply (5.22) with, for a, the left hand side of (5.21). It follows that for $\lambda \geq \lambda_0$

$$\begin{aligned} &B\big(Op_\lambda^w((1-\theta)(\lambda + |\lambda\tau|_m)^{2M})u, u\big) + C\|Q_R u\|^2 + C\|Q_I u\|^2 \\ &+\lambda([Q^*, Q]u, u) \geq \frac{1}{2C}|||u|||_M^2. \end{aligned}$$

Now, we deduce from (5.20) that

$$2\lambda\|Qu\|_{L^2}^2 \geq C\big(\|Q_R u\|^2 + \|Q_I u\|^2 + \lambda([Q^*, Q]u, u)\big) \quad \text{if } 2\lambda \geq C,$$

and Lemma 5.2 follows. □

Proposition 5.3. *Let Q_λ be defined in* (4.4). *Then one can find positive constants C_0, C_1, λ_0 such that for u in $\mathcal{S}(\mathbb{R}^{d+n})$ and $\lambda \geq \lambda_0$*

$$\frac{C_1}{\lambda}\big(Op_\lambda^w((1-\theta)(\lambda + |\lambda\tau|_m)^{2M})u, u\big)_{L^2} + \|Q_\lambda u\|_{L^2}^2 \geq \frac{C_0}{\lambda}|||u|||_M^2. \tag{5.23}$$

Proof. Writing $Q_\lambda = Q + Q_{M-1}$ where $Q_{M-1} = Op_\lambda^w(q_{M-1})$ defined in (4.7), then

$$\|Qu\|_{L^2}^2 \leq 2\|Q_\lambda u\|_{L^2}^2 + 2\|Q_{M-1}u\|_{L^2}^2,$$

and

$$Q_{M-1} \in Op_\lambda^w\big(S((\lambda + |\lambda\tau|_m)^{M-1}, g)\big),$$

we deduce that

$$\|Qu\|_{L^2}^2 \leq 2\|Q_\lambda u\|_{L^2}^2 + \frac{C}{\lambda^2}|||u|||_M^2. \tag{5.24}$$

It follows from Lemma 5.2 and (5.24)

$$\frac{C_1}{\lambda}\big(Op^w_\lambda((1-\theta)(\lambda+|\lambda\tau|_m)^{2M})u,u\big)_{L^2}+2\|Q_\lambda u\|^2_{L^2}+\frac{C}{\lambda^2}|||u|||^2_M\geq\frac{C_0}{\lambda}|||u|||^2_M,$$

and Proposition 5.3 follows. □

We are now ready to prove the following estimate.

Proposition 5.4 (see [7]). *Let $\tilde{Q}_\lambda$ be defined in* Theorem 3.2. *Then there exist positive constants C_1, C_2, λ_0, such that for $v\in C_0^\infty(\mathbb{R}^{d+n})$,* $\operatorname{supp} v\subset\big\{X:|X|\leq 1/(4C^2)\big\}$ *and $\lambda\geq\lambda_0$*

$$\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d,H^M_\lambda(\mathbb{R}^n))}\leq C_1\lambda\|\tilde{Q}_\lambda Tv\|^2_{L^2_{(1+\eta)\Phi}}+C_2e^{-\lambda\sigma}|||v|||^2_M,\tag{5.25}$$

where $\sigma>0$ depends only on η and C.

Proof. We apply Proposition 5.3 to $u=T^*_\eta Tv$ which is in $\mathcal{S}(\mathbb{R}^{d+n})$. It follows from Proposition 2.1

$$|||u|||_M=\|T_\eta u\|_{L^2_{(1+\eta)\Phi}(H^M_\lambda)}=\|Tv\|_{L^2_{(1+\eta)\Phi}(H^M_\lambda)},\tag{5.26}$$

$$\|Q_\lambda u\|_{L^2}=\|T_\eta Q_\lambda T^*_\eta Tv\|_{L^2_{(1+\eta)\Phi}}=\|\tilde{Q}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}}.\tag{5.27}$$

Let us set $R=Op^w_\lambda((1-\theta)(\lambda+|\lambda\tau|_m)^{2M})$. Then Proposition 4.6 in [7] show that for any integer N one can find a positive constant C_N such that

$$|(Ru,u)_{L^2}|\leq\frac{C_N}{\lambda^N}\|Tv\|^2_{L^2_{(1+\eta)\Phi}(H^M_\lambda)}+\mathcal{O}(e^{-\lambda\sigma}|||v|||^2_M),\quad\sigma>0.\tag{5.28}$$

It follows from (5.23), (5.26), (5.27) and (5.28) that Proposition 5.4 is proved. □

Theorem 5.5. *Let $\tilde{P}_\lambda$ be the operator occuring in* Proposition 3.1. *One can find positive constants C_1, C_2, λ_0, σ such that for $v\in C_0^\infty(\mathbb{R}^{d+n})$,* $\operatorname{supp} v\subset\{X:|X|\leq 1/(4C^2)\}$ *and $\lambda\geq\lambda_0$ we have*

$$\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d,H^M_\lambda(\mathbb{R}^n))}\leq C_1\lambda\|\tilde{P}_\lambda Tv\|^2_{L^2_{(1+\eta)\Phi}}+C_2e^{-\lambda\sigma}|||v|||^2_M.\tag{5.29}$$

Proof. This follows from Proposition 5.4 and Theorem 3.2. □

6. The estimates in case of Theorem B

Let $Q_M=Op^w_\lambda(q_M)$ where q_M is defined in (4.5). We have

$$\|Q_M u\|^2_{L^2}=\|Q_R u\|^2_{L^2}+\|Q_I u\|^2_{L^2}+\frac{1}{2}([Q^*_M,Q_M]u,u),\tag{6.1}$$

where $Q_M = Q_R + iQ_I$, $Q_R^* = Q_R$ and $Q_I^* = Q_I$.

Let us introduce the following Hörmander's metrics

$$\text{(6.2)} \qquad \begin{cases} g_1 = dx^2 + dy^2 + \displaystyle\sum_{j=1}^{d} \frac{\lambda^2 d\xi_j^2}{(\lambda + |\lambda\tau|_m)^{2\tilde{h}_j}} + \sum_{j=1}^{n} \frac{\lambda^2 d\tau_j^2}{(\lambda + |\lambda\tau|_m)^{2h_j}}, \\ g_2 = dx^2 + dy^2 + d\xi^2 + \displaystyle\sum_{j=1}^{n} \frac{\lambda^2 d\tau_j^2}{(\lambda + |\lambda\tau|_m)^{2h_j}}. \end{cases}$$

Then it is easy to see from (4.5) that

$$\text{(6.3)} \qquad q_M(X, \zeta) = p'_M(y, \tau) + \tilde{\chi}(x, \xi)(r_{M-1}(X, \zeta) + \eta s_{M-1}(X, \zeta)),$$

where

$$\text{(6.4)} \qquad \begin{cases} \tilde{\chi}(x, \xi) = \chi\Big(x - \dfrac{i}{1+\eta}\xi, \xi\Big); r_{M-1}(X, \zeta) = p'_{M-1}(X, \zeta), \\ r_{M-1} \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2\big), \ s_{M-1} \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2\big), \\ p'_M \in S\big((\lambda + |\lambda\tau|_m)^M, g_1\big). \end{cases}$$

We shall write $Q_M = P'_M + R_{M-1} + \eta S_{M-1}$ where $\sigma^\omega(P'_M) = p'_M(y, \tau)$, $\sigma^\omega(R_{M-1}) = \tilde{\chi} r_{M-1}$, and $\sigma^\omega(S_{M-1}) = \tilde{\chi} s_{M-1}$. Let us set

$$\text{(6.5)} \qquad L = P'_M + R_{M-1}.$$

Since R_{M-1} and S_{M-1} belong to $Op_\lambda^w\big(S(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2)\big)$ and p'_M depends only on (y, τ), it is easy to see that

$$\text{(6.6)} \qquad [Q_M^*, Q_M] - [L^*, L] \in \frac{\eta}{\lambda} Op_\lambda^w\big(S(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2)\big).$$

We shall set $\sigma^\omega(L) = \ell_1 + \ell_2 = \ell$ where

$$\text{(6.7)} \qquad \begin{cases} \ell_1 = p'_M(y, \tau) + (\tilde{\chi} r_{M-1})|_{\xi=0}, \\ \ell_2 = \tilde{\chi} r_{M-1} - (\tilde{\chi} r_{M-1})|_{\xi=0}. \end{cases}$$

Then

$$\text{(6.8)} \qquad \ell_1 \in S\big((\lambda + |\lambda\tau|_m)^M, g_1\big), \ \ell_2 \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1}, g_2\big).$$

We shall also write

$$\text{(6.9)} \qquad \sigma^\omega([L^*, L]) = \frac{1}{\lambda}(d_1 + d_2) \text{ where } d_1 = \frac{1}{i}\{\bar{\ell}, \ell\}|_{\xi=0}.$$

Then since p'_M depends only on (y, τ), we have

$$d_1 \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{2M-1}, g_1\big), \;\; d_2 \in S\big(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2\big). \tag{6.10}$$

Lemma 6.1. *There exists a positive constant C such that if we set*

$$\psi(X) = \varphi'(0)X + \frac{1}{2}\varphi''(0)X \cdot X - \frac{1}{2C^2}|X|^2 + \frac{C}{2}(\varphi'(0)X)^2$$

then

$$C^3|\ell_1(X, \tau)|^2 + d_1(X, \tau) \geq \frac{1}{C}\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, \tag{6.11}$$

for $|X| \leq 1/C^2$ and τ in $\mathbb{R}^n$. Moreover, by homogeneity, (6.11), *with possibly other constants, is still true with the same ψ if we replace ψ by $\rho\psi$ where ρ is a positive constant.*

Proof. We first take C so large that $\tilde{\chi} = 1$ if $|x| + |\xi| \leq 1/C^2$. Then from (6.7) and (6.9), we have

$$\begin{cases} \ell_1(X, \tau) = p_M(X; \lambda\zeta + i\lambda\psi'(X))|_{\xi=0}, \\ d_1(X, \tau) = \dfrac{1}{i}\{\overline{p}_M(X, \lambda\zeta - i\lambda\psi'(X)); p_M(X, \lambda\zeta + i\lambda\psi'(X))\}|_{\xi=0}. \end{cases}$$

Now, we write

$$\begin{cases} \ell_1(X, \tau) = p_M(X; \lambda\zeta + i\lambda\nabla_{p,q}\psi(X))|_{\xi=0} + s_\lambda(\xi, \tau), \\ d_1(X, \tau) = \dfrac{1}{i}\{\overline{p}_M(X; \lambda\zeta - i\lambda\nabla_{p,q}\psi(X)); p_M(X; \lambda\zeta + i\lambda\nabla_{p,q}\psi(X))\}|_{\xi=0} \\ \qquad\qquad + r_\lambda(X, \tau), \end{cases} \tag{6.12}$$

where

$$\begin{cases} & s_\lambda \in S\big(\lambda(\lambda + |\lambda\tau|_m)^{M-1-1/(M-1)}, g_1\big) \\ \text{and} & \\ & r_\lambda \in S\big(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2-1/(M-1)}, g_1\big). \end{cases} \tag{6.13}$$

First, we shall

$$\begin{aligned} &\frac{C^3}{4}\left|p_M(X; \lambda\zeta + i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\right|^2 \\ &+\frac{1}{2i}\{\overline{p}_M(X; \lambda\zeta + i\lambda\nabla_{p,q}\psi(X)); p_M(X, \lambda\zeta + i\lambda\nabla_{p,q}\psi(X))\}\big|_{\xi=0} \\ &\geq \frac{1}{C}\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2} \text{ for } |X| \leq \frac{1}{C^2} \text{ and } \tau \text{ in } \mathbb{R}^n. \end{aligned} \tag{6.14}$$

(6.14) is equivalent to

$$\begin{aligned}&\frac{C^3}{4\lambda^2}\left|p_M(X;\zeta+i\lambda\nabla_{p,q}\psi(X)|_{\xi=0}\right|^2\\&+\frac{1}{2i\lambda}\left\{\overline{p}_M(X;\zeta-i\lambda\nabla_{p,q}\psi(X));p_M(X,\zeta+i\lambda\nabla_{p,q}\psi(X))\right\}|_{\xi=0}\\&\geq\frac{1}{C}(\lambda+|\tau|_m)^{2M-2}\text{ for }|X|\leq\frac{1}{C^2}.\end{aligned}$$

We see (6.14), setting $\Gamma=\lambda/(\lambda+|\tau|_m)$, $W=(X,Z+i\Gamma\nabla_{p,q}\psi(X))$,

$$Z=\left(0,\ldots,0;\frac{\tau_1}{(\lambda+|\tau|_m)^{h_1}},\ldots,\frac{\tau_n}{(\lambda+|\tau|_m)^{h_m}}\right)$$

that (6.14) is equivalent to

$$\begin{aligned}(6.15)\quad&\frac{C^3}{4\Gamma^2}|p_M(W)|^2+\frac{1}{\Gamma}\operatorname{Im}\left(\sum_{j=1}^{d}(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial x_j}(W)\right.\\&\left.+\sum_{j=1}^{n}(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial y_j}(W)\right)\\&+\operatorname{Re}\left(\sum_{j=1}^{d}\sum_{k=p}^{d}\frac{\partial^2\psi}{\partial x_k\partial x_j}(X)(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial x_j}(W)\right.\\&+\sum_{j=1}^{d}\sum_{k=q}^{n}\frac{\partial^2\psi}{\partial y_k\partial x_j}(X)(\lambda+|\tau|_m)^{1-\tilde{h}_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W})\frac{\partial p_M}{\partial\tau_k}(W)\\&+\sum_{j=1}^{n}\sum_{k=p}^{d}\frac{\partial^2\psi}{\partial x_k\partial y_j}(X)(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial\xi_k}(W)\\&\left.+\sum_{j=1}^{n}\sum_{k=q}^{n}\frac{\partial^2\psi}{\partial y_k\partial y_j}(X)(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W})\frac{\partial p_M}{\partial\tau_k}(W)\right)\geq\frac{1}{C},\ \text{for }|X|\leq\frac{1}{C^2}.\end{aligned}$$

We prove (6.15) by contradiction. If it is false one can find sequences X_k, λ_k, τ_j, Γ_k with $|X_k|\leq 1/k^2$, $\lambda_k\geq e^k$ and τ_k in $\mathbb{R}^n$, such that

$$\begin{aligned}(6.16)\quad&\frac{k^3}{4\Gamma_k^2}|p_M(W_k)|^2+\frac{1}{\Gamma_k}\operatorname{Im}\left(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial y_j}(W_k)\right.\\&\left.+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial x_j}(W_k)\right)+\operatorname{Re}\left(\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_s}(W_k)\right.\\&\left.+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)+2\sum_{s=q}^{M}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)\right)\end{aligned}$$

$$+k\Big(\Big|\sum_{j=p}^{d}\frac{\partial\varphi}{\partial x_j}(0)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big|^2+\Big|\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big|^2$$

$$+2\,\mathrm{Re}\Big[\Big(\sum_{j=q}^{n}\frac{\partial\varphi}{\partial\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big)\Big(\sum_{s=p}^{d}\frac{\partial\varphi}{\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_s}(\overline{W}_k)\Big)\Big]\Big)$$

$$-\frac{1}{k^2}\Big(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_j}(W_k)+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big)+B_k\le\frac{1}{k}$$

where

$$|B_k|\le\frac{C_1 k}{\Gamma_k}\lambda_k^{-1/(M-1)},\quad C_1 \text{ independent of } k. \tag{6.17}$$

Since $\Gamma_k+|Z_k|_{(m,\tilde m)}=1$, taking subsequences, we may assume that

$$\Gamma_k\to\Gamma^0 \text{ and } Z_k\to Z^0 \text{ with } \Gamma^0+|Z^0|_{(m,\tilde m)}=1. \tag{6.18}$$

CASE 1. $\Gamma^0\neq 0$.

If we divide both members of (6.16) by k^3, we get

$$p_M(W^0)=\{p_M,\varphi\}_0(W^0)=0, \tag{6.19}$$

with $W^0=(0;Z^0+i\Gamma^0\nabla_{p,q}\varphi(0))$.
Removing all positive terms in (6.16) and letting k go to $+\infty$, we get

$$\frac{1}{\Gamma^0}\mathrm{Im}\Big(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial y_j}(W^0)+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial x_j}(W^0)\Big)$$

$$+\mathrm{Re}\Big(\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\xi_s}(W^0)+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)$$

$$+2\sum_{s=q}^{n}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)\Big)\le 0$$

which contradicts the hypothesis (H.2)′ ii) in Theorem B.

CASE 2. $\Gamma^0=0$.

Since $\Gamma^0+|Z^0|_{(m,\tilde m)}=1$, we have $Z^0\neq 0$. In this case, we write

$$B_k=\frac{1}{\Gamma_k}\mathrm{Im}\Big(\sum_{j=1}^{d}(\lambda_k+|\tau_k|_m)^{1+\tilde h_j}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial x_j}(W_k)$$
$$+\sum_{j=1}^{n}(\lambda+|\tau|_m)^{1-h_j}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial y_j}W_k)\Big)+D_k \tag{6.20}$$

where

$$|D_k| \leq C_2 k \lambda_k^{-1/(M-1)}, \ C_2 \quad \text{independent of } k.$$

Therefore

$$(6.21) \quad B_k = \frac{1}{2i\Gamma_k}(\lambda_k + |\tau_k|_m)^{1-2M}\{\overline{p}_M, p_M\}(X_k; 0, \tau_k)$$
$$+\mathrm{Re}\Bigg(\sum_{s,j=q}^{n} \frac{\partial\psi}{\partial y_s}(X_k)\Big(\frac{\partial \overline{p}_M}{\partial \tau_j}(X_k, Z_k)\frac{\partial^2 p_M}{\partial \tau_j \partial y_j}(X_k, Z_k)$$
$$-\frac{\partial p_M}{\partial y_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \tau_s \partial \tau_j}(X_k, Z_k)\Big) + \sum_{s,j=p}^{d} \frac{\partial\psi}{\partial x_s}(X_k)$$
$$\Big(\frac{\partial \overline{p}_M}{\partial \xi_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \xi_s \partial x_j}(X_k, Z_k) - \frac{\partial p_M}{\partial x_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \xi_s \partial \xi_j}(X_k, Z_k)\Big)\Bigg) + D_k'$$

where

$$|D_k'| \leq C_3\Big(k\lambda_k^{-1/(M-1)} + \Gamma_k\Big), \ C_3 \text{ independent of } k.$$

We use then the assumptions (H.1)$'$ in Theorem B. We get

$$\Big|(\lambda_k + |\tau_k|_m)^{1-2M}\{\overline{p}_M, p_M\}(X_k, 0, \tau_k)\Big| \leq C'|p_m(X_k, 0, \tau_k)|(\lambda_k + |\tau_k|_m)^{-M}$$
$$\leq C'|p_M(X_k, Z_k)| \leq C'|p_M(W_k)| + C'\Gamma_k\Bigg(\Big|\sum_{j=q}^{n} \frac{\partial\psi}{\partial y_j}(X_k)\frac{\partial p_M}{\partial \tau_j}(W_k)\Big|$$
$$+\Big|\sum_{j=p}^{d} \frac{\partial\psi}{\partial x_j}(X_k)\frac{\partial p_M}{\partial \xi_j}(W_k)\Big|\Bigg) + \mathcal{O}(\Gamma_k^2).$$

Therefore

$$(6.22) \quad \Big|\frac{1}{2i}(\lambda_k + |\tau_k|_m)^{1-2M}\{\overline{p}_M, p_M\}(X_k; 0, \tau_k)\Big| \leq \frac{k^{3/2}}{4\Gamma_k}|p_M(W_k)|^2 + \frac{4(C')^2\Gamma_k}{k^{3/2}}$$
$$+C'\Gamma_k\Bigg(\Big|\sum_{j=q}^{n} \frac{\partial\psi}{\partial y_j}(X_k)\frac{\partial p_M}{\partial \tau_j}(W_k)\Big| + \Big|\sum_{j=p}^{d} \frac{\partial\psi}{\partial x_j}(X_k)\frac{\partial p_M}{\partial \xi_j}(W_k)\Big|\Bigg) + \mathcal{O}(\Gamma_k^2).$$

It follows from (6.21), (6.22) that (6.16) is equivalent to

$$(6.23) \quad \frac{1}{4}\Big(\frac{k^3}{\Gamma_k^2} - \frac{k^{3/2}}{\Gamma_k^2}\Big)|p_M(W_k)|^2$$
$$+\mathrm{Re}\Bigg(\sum_{s,j=q}^{m} \frac{\partial\psi}{\partial y_s}(X_k)\Big(\frac{\partial \overline{p}_M}{\partial \tau_j}(X_k, Z_k)\frac{\partial^2 p_M}{\partial \tau_j \partial y_j}(X_k, Z_k) - \frac{\partial p_M}{\partial y_j}(X_k, Z_k)\frac{\partial^2 \overline{p}_M}{\partial \tau_s \partial \tau_j}(X_k, Z_k)\Big)$$

$$
\begin{aligned}
&+\sum_{s,j=p}^{d}\frac{\partial\psi}{\partial x_s}(X_k)\Big(\frac{\partial\overline{p}_M}{\partial\xi_j}(X_k,Z_k)\frac{\partial^2\overline{p}_M}{\partial\xi_s\partial x_j}(X_k,Z_k)-\frac{\partial p_M}{\partial x_j}(X_k,Z_k)\frac{\partial^2\overline{p}_M}{\partial\xi_s\partial\xi_j}(X_k,Z_k)\Big)\Big)\\
&+k\Big(\Big|\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\partial\tau_j}(W_k)\Big|^2+\Big|\sum_{j=p}^{d}\frac{\partial\varphi}{\partial x_j}(0)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big|^2\\
&+2\,\mathrm{Re}\Big[\Big(\sum_{j=q}^{n}\frac{\partial\varphi}{\partial y_j}(0)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big)\Big(\sum_{s=p}^{d}\frac{\partial\varphi}{\partial x_s}(0)\frac{\partial p_M}{\partial\xi_s}(W_k)\Big)\Big]\Big)\\
&-C'\Big(\Big|\sum_{j=q}^{n}\frac{\partial\psi}{\partial y_j}(X_k)\frac{\partial p_M}{\partial\tau_j}(W_k)\Big|+\Big|\sum_{j=p}^{d}\frac{\partial\psi}{\partial x_j}(X_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big|\Big)\\
&-\frac{1}{k^2}\Big(\sum_{j=q}^{n}\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_j}(W_k)+\sum_{j=p}^{d}\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_j}(W_k)\Big)\\
&+\mathrm{Re}\Big(\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\xi_s}(W_k)+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\partial\tau_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)\\
&+2\sum_{s=q}^{n}\sum_{j=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}_k)\frac{\partial p_M}{\partial\tau_s}(W_k)\Big)+\mathcal{O}\Big(k\lambda_k^{-1/(M-1)}+\Gamma_k+\frac{1}{k^{3/2}}\Big)\le\frac{1}{k}.
\end{aligned}
$$

Dividing both members by k^3/Γ_k^2, we get, since $\Gamma_k\to 0$, $k\to+\infty$,

(6.24) $$p_M(W^0)=0 \text{ with } W^0=(0,Z^0),\quad Z^0\neq 0.$$

Now since, $(k^3/\Gamma_k^2-k^{3/2}/\Gamma_k^2)|p_M(W_k)|^2\ge 0$, dividing (6.23) by k, we get

(6.25) $$\{p_M,\varphi\}_0(W^0)=0.$$

Removing all positive terms in (6.23) and letting k go to $+\infty$, we get

$$
\begin{aligned}
&\mathrm{Re}\Big[\sum_{s,j=q}^{n}\frac{\partial\varphi}{\partial y_s}(0)\Big(\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial^2 p_M}{\partial\tau_s\partial y_j}(W^0)-\frac{\partial p_M}{\partial y_j}(W^0)\frac{\partial^2\overline{p}_M}{\partial\tau_s\partial\tau_j}(\overline{W}^0)\Big)\\
&+\sum_{s,j=p}^{d}\frac{\partial\varphi}{\partial\partial x_s}(0)\Big(\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial^2 p_M}{\partial\xi_s\partial x_j}(W^0)-\frac{\partial p_M}{\partial x_j}(W^0)\frac{\partial^2\overline{p}_M}{\partial\xi_j\partial\xi_s}(\overline{W}^0)\Big)\\
&+\sum_{j,s=p}^{d}\frac{\partial^2\varphi}{\partial x_j\partial x_s}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\xi_s}(W^0)+\sum_{s,j=q}^{n}\frac{\partial^2\varphi}{\partial y_s\partial y_j}(0)\frac{\partial\overline{p}_M}{\partial\tau_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)\\
&+2\sum_{s=q}^{n}\sum_{s=p}^{d}\frac{\partial^2\varphi}{\partial y_s\partial x_j}(0)\frac{\partial\overline{p}_M}{\partial\xi_j}(\overline{W}^0)\frac{\partial p_M}{\partial\tau_s}(W^0)\Big]\le 0
\end{aligned}
$$

which is contradiction with (H.2)′ i) in Theorem B.

It follows from (6.12), (6.13) and (6.14) that

$$\frac{C^3}{4}\Big|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\Big|^2+\frac{1}{2}d_1(X,\tau)\geq\frac{1}{C}\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2}+\frac{1}{2i}r_\lambda(X,\tau).$$

But we have

$$\begin{cases}\Big|p_M(X;\lambda\zeta+i\lambda\nabla_{p,q}\psi(X))|_{\xi=0}\Big|^2\leq 2|\ell_1(X,\tau)|^2+C'\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2-2/(M-1)}\\ \Big|\frac{1}{2i}r_\lambda(X,\tau)\Big|\leq C''\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2-1/(M-1)}.\end{cases}$$

Il follows that

$$\frac{C^3}{2}|\ell_1(X,\tau)|^2+\frac{1}{2}d_1(X,\tau)\geq\frac{1}{2C}\lambda^2(\lambda+|\lambda\tau|_m)^{2M-2},$$

for large λ and Lemma 6.1 follows. □

Lemma 6.2. *We have*

(6.26)
$$\Big(\frac{C^3+1}{\lambda^2}\Big)\Big(\|Op^w_\lambda(\mathrm{Re}\,\ell_1)u\|^2_{L^2}+\|Op^w_\lambda(\mathrm{Im}\,\ell_1)u\|^2_{L^2}\Big)+\frac{1}{\lambda^2}(Op^w_\lambda(d_1)u,u)\geq\frac{1}{2C}|||u|||^2_{M-1},$$

where $|||\cdot|||_{M-1}$ is defined (2.9), *and for large* λ.

Proof. Let us $a=(C^3/\lambda^2)|\ell_1|^2+d_1/\lambda^2$ and $a_0=a|_{x=0}$. Let $h_0\in C_0^\infty(\mathbb{R}^d)$ be such that $h_0=1$ if $|x|\leq 1/(4C^2)$, $h_0=0$ if $|x|\geq 1/(2C^2)$ and $0\leq h_0\leq 1$. Then we have

(6.27)
$$a+(1-h_0)(a_0-a)\geq\frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M-2},\ \text{ if } |y|\leq\frac{1}{2C^2}.$$

Indeed, if $|x|\leq 1/(2C^2)$, then by Lemma 6.1, a and a_0 satisfy (6.11) thus (6.27) is true. If $|x|\geq 1/(2C^2)$ then $h_0=0$ and a_0 satisfies (6.11) and (6.27) is also true.

Now denoting by t_k a symbol in the class $S((\lambda+|\lambda\tau|_m)^k,g_2)$, by (6.8) and (6.9), we have

$$a=\frac{C^3}{\lambda^2}|p'_M(y,\tau)|^2+\frac{2}{\lambda^2}\,\mathrm{Im}\Big(\frac{\partial}{\partial\tau}(p'_M(y,\tau))\frac{\partial}{\partial y}(p'_M(y,\tau))\Big)+\frac{1}{\lambda}\,\mathrm{Re}(\ell_1\cdot t_{M-1})+t_{2M-2}.$$

Thus $a-a_0=(1/\lambda)\,\mathrm{Re}(\ell_1\cdot t_{M-1})+t_{2M-2}$ so

(6.28)
$$|a-a_0|\leq\frac{|\ell_1|^2}{\lambda^2}+C'(\lambda+|\lambda\tau|_m)^{2M-2}.$$

It follows from (6.11), (6.27) and (6.28) that if $|y| \leq 1/(2C^2)$

$$\frac{(C^3+1)}{\lambda^2}|\ell_1|^2 + \frac{1}{\lambda^2}d_1 + C'(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2} \geq \frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M-2}. \tag{6.29}$$

Let $h_1 \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \leq h_1 \leq 1$, $h_1 = 0$ if $|y| \geq 1/(2C^2)$ and $h_1 = 1$ if $|y| \leq 1/(4C^2)$. Thus we have, from (6.29)

$$\Big(\frac{(C^3+1)}{\lambda^2}|\ell_1|^2 + \frac{1}{\lambda^2}d_1 + C'(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2} - \frac{1}{C}(\lambda+|\lambda\tau|_m)^{2M-2}\Big)\lambda^2 h_1^2(y) \geq 0$$

for any (X,τ) in $\mathbb{R}^{d+n}\times\mathbb{R}^n$, and this symbol belongs to $S((\lambda+|\lambda\tau|_m)^{2M}, g_1)$. Therefore we can apply the Fefferman-Phong inequality and get

$$\begin{aligned}\Big(Op_\lambda^w\Big(\frac{(C^3+1)}{\lambda^2}|\ell_1|^2h_1^2\Big)u,u\Big) + \Big(Op_\lambda^w\Big(\frac{d_1}{\lambda^2}h_1^2\Big)u,u\Big) \\ \geq \frac{1}{C}\Big(Op_\lambda^w(h_1^2(\lambda+|\lambda\tau|_m)^{2M-2})u,u\Big) \\ -C'\Big(Op_\lambda^w(h_1^2(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2})u,u\Big) - \frac{C''}{\lambda^2}|||u|||_{M-1}^2.\end{aligned} \tag{6.30}$$

We can use the symbolic calculus in $S(\cdot, g_1)$. We get

$$\begin{aligned}J = \Big(Op_\lambda^w\Big(\frac{(C^3+1)}{\lambda^2}|\ell_1|^2h_1^2\Big)u,u\Big) = \frac{(C^3+1)}{\lambda^2}\Big(\Big(Op_\lambda^w(\ell_1^Rh_1)^*Op_\lambda^w(\ell_1^Rh_1) \\ +Op_\lambda^w(\ell_1^Ih_1)^*Op_\lambda^w(\ell_1^Ih_1)\Big)u,u\Big) + \frac{1}{\lambda^2}\mathcal{O}(|||u|||_{M-1}^2)\end{aligned}$$

where $\ell_1^R = \operatorname{Re}\ell_1$ and $\ell_1^I = \operatorname{Im}\ell_1$. Thus

$$J = \frac{(C^3+1)}{\lambda^2}\Big(\|Op_\lambda^w(\ell_1^R)u\|_{L^2}^2 + \|Op_\lambda^w(\ell_1^I)u\|_{L^2}^2\Big) + \frac{1}{\lambda^2}\mathcal{O}(|||u|||_{M-1}^2) \tag{6.31}$$

because

$$Op_\lambda^w(\ell_1^K)h_1 = Op_\lambda^w(\ell_1^Kh_1) + Op_\lambda^w\big(S((\lambda+|\lambda\tau|_m)^{M-1}, g_1)\big)$$

for $K = R$ or I and $h_1u = u$ since $\operatorname{supp} u \subset \{|y| \leq 1/(4C^2)\}$. By the same way

$$Op_\lambda^w(d_1h_1^2) = Op_\lambda^w(d_1)h_1^2 + Op_\lambda^w\big(S(\lambda(\lambda+|\lambda\tau|_m)^{2M-2}, g_1)\big)$$

thus

$$(Op_\lambda^w(d_1h_1^2)u,u) = (Op_\lambda^w(d_1)u,u) + \lambda\mathcal{O}(|||u|||_{M-1}^2). \tag{6.32}$$

We have also

$$(6.33)\qquad \big(Op^w_\lambda(h_1^2(\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) = |||u|||^2_{M-1} + \frac{1}{\lambda}\mathcal{O}(|||u|||^2_{M-1}),$$

$$(6.34)\qquad \big(Op^w_\lambda(h_1^2(1-h_0)(\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) = |||(1-h_0)u|||^2_{M-1} + \frac{1}{\lambda}\mathcal{O}(|||u|||^2_{M-1}),$$

and

$$(6.35)\qquad |||(1-h_0)u|||^2_{M-1} \le \frac{C_N}{\lambda^N}|||u|||^2_{M-1}, \text{ for any } N \text{ in } \mathbb{N}.$$

Thus (6.26) follows from (6.30) to (6.35). □

Lemma 6.3. *Let ℓ_2 and d_2 be defined in* (6.7) *and* (6.9). *Then there exists $\sigma > 0$ such that for any $\varepsilon > 0$ one can find a positive constant C_ε such that*

$$(6.36)\qquad \|Op^w_\lambda(\ell_2)u\|_{L^2(\mathbb{R}^{d+n})} \le \lambda\varepsilon|||u|||_{M-1} + \sqrt{\lambda}C_\varepsilon|||u|||_{M-1} + \mathcal{O}(e^{-\lambda\sigma}|||v|||_{M-1}),$$

and

$$(6.37)\qquad |(Op^w_\lambda(d_2)u,u)| \le \lambda^2\Big(\varepsilon|||u|||^2_{M-1} + \frac{C_\varepsilon}{\sqrt{\lambda}}|||u|||^2_{M-1}\Big) + \mathcal{O}(e^{-\lambda\sigma}|||v|||^2_{M-1}),$$

for any $u = T^*_\eta Tv$, $v \in C_0^\infty(\mathbb{R}^{n+d})$.

Proof. Given $\varepsilon > 0$, let $\chi(X,\xi)$ in C^∞ with $0 \le \chi \le 1$ and $\operatorname{supp}\chi \subset \{|X|+|\xi| \le \varepsilon\}$. We claim that one can find $C_\varepsilon > 0$ such that

$$(6.38)\qquad \frac{1}{\lambda}\|Op^w_\lambda(\ell_2\chi)u\|_{L^2} \le \varepsilon|||u|||_{M-1} + \frac{C_\varepsilon}{\sqrt{\lambda}}|||u|||_{M-1}.$$

This follows from the sharp Gårding inequality in the class $S(1,g_2)$. Indeed, we have $\varepsilon^2(\lambda+|\lambda\tau|_m)^{2M-2} - \xi^2\chi^2(\lambda+|\lambda\tau|_m)^{2M-2} \ge 0$. Thus

$$(6.39)\qquad \varepsilon^2\big(Op^w_\lambda((\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) - \big(Op^w_\lambda(\xi^2\chi^2(\lambda+|\lambda\tau|_m)^{2M-2})u,u\big) \ge -\frac{C_\varepsilon}{\lambda}|||u|||^2_{M-1}.$$

Since $\ell_2 \in S(\lambda(\lambda+|\lambda\tau|_m)^{M-1}, g_2)$ and $\ell_2|_{\xi=0}$, we have

$$(6.40)\qquad \|Op^w_\lambda(\ell_2\chi)u\|_{L^2} \le C\lambda\|Op^w_\lambda(\xi\chi(\lambda+|\lambda\tau|_m)^{M-1})u\|_{L^2}.$$

We deduce (6.38) from (6.39) and (6.40).

Therefore taking $\chi = \theta(x, \xi)g(y)$, such that $\chi = 1$ if $|X| + |\xi| \leq \varepsilon/2$, we write

$$\|Op_\lambda^w(\ell_2)u\|_{L^2} \leq \|Op_\lambda^w(\ell_2\chi)u\|_{L^2} + \|Op_\lambda^w((1-\chi)\ell_2)u\|_{L^2}.$$

It follows from Proposition 4.6 in [7] that

$$\|Op_\lambda^w((1-\chi)\ell_2)u\|_{L^2} \leq \frac{C_N}{\lambda^N}|||u|||_{M-1} + \mathcal{O}(e^{-\lambda\sigma}|||v|||_{M-1}). \tag{6.41}$$

Then we deduce (6.36) from (6.40) and (6.41). This gives the first part of the lemma. For the second part, we observe that $d_2 \in S(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2)$. Therefore from (6.39) and Proposition 4.6 in [7], we deduce (6.37). □

We are now ready to prove the Carleman estimate for Q_M.

Proposition 6.4. *Let $Q_M = Op_\lambda^w(q_M)$ be defined in* (4.6). *Then one can find positive constants C_0, C_1, λ_0, σ such that, for any $u = T_\eta^* T v$, $v \in C_0^\infty$,* $\operatorname{supp} v \subset \{|X| \leq 1/(4C^2)\}$ *and $\lambda \geq \lambda_0$, we have*

$$C_0|||u|||_{M-1}^2 \leq \frac{C_1}{\lambda}\|Q_M u\|_{L^2}^2 + \mathcal{O}(e^{-\lambda\sigma}|||v|||_{M-1}^2). \tag{6.42}$$

Proof. It follows from (6.3), (6.5) and (6.7) that

$$\|Op_\lambda^w(\ell_1^R)u\|_{L^2} \leq \|Q_R u\|_{L^2} + \|Op_\lambda^w(\ell_2^R)u\|_{L^2} + \eta\|Op_\lambda^w(\tilde{\chi}s_{M-1}^R)u\|_{L^2}.$$

Therefore, applying Lemma 6.3, we deduce

$$\begin{aligned}\|Op_\lambda^w(\ell_1^K)u\|_{L^2} &\leq \|Q_K u\|_{L^2} + C_1\lambda\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C_2\eta\Big)|||u|||_{M-1}\\ &\quad+\mathcal{O}\big(e^{-\lambda\sigma}|||v|||_{M-1}\big), \quad \text{for } K = R, I.\end{aligned} \tag{6.43}$$

Using (6.6), (6.9) and Lemma 6.3, we get

$$\begin{aligned}&\Big|\big((Op_\lambda^w(d_1) - \lambda[Q_M^*, Q_M])u, u\big)\Big|\\ &= \Big|\big((Op_\lambda^w(d_2) - \eta Op_\lambda^w(S(\lambda^2(\lambda + |\lambda\tau|_m)^{2M-2}, g_2))u, u\big)\Big|,\\ &\leq |(Op_\lambda^w(d_2)u, u)| + \eta\lambda^2\big|(Op_\lambda^w(S((\lambda + |\lambda\tau|_m)^{2M-2}, g_2))u, u)\big|,\\ &\leq C_1\lambda^2\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + C_2\eta\Big)|||u|||_{M-1}^2 + \mathcal{O}\big(e^{-\lambda\sigma}|||v|||_{M-1}^2\big).\end{aligned} \tag{6.44}$$

It follows from (6.43), (6.44) and Lemma 6.2 that

$$\frac{1}{2C}|||u|||_{M-1}^2 \leq \frac{2}{\lambda^2}(C^3+1)\Big(\|Q_I u\|_{L^2}^2 + \|Q_I u\|_{L^2}^2 + \frac{\lambda}{2}\big([Q_M^*, Q_M]u, u\big)\Big)$$
$$+\tilde{C}_1\Big(\varepsilon + \frac{C_\varepsilon}{\sqrt{\lambda}} + \tilde{C}_2\eta\Big)|||u|||_{M-1}^2 + \mathcal{O}\big(e^{-\lambda\sigma}|||v|||_{M-1}^2\big).$$

Taking ε and η small, then λ large, we get, by (6.1), proposition 6.4. □

Theorem 6.5. *Let* $\tilde{P}_\lambda$ *the operator occuring in* Proposition 3.1. *One can find positive constants* C_1, C_2, λ_0, ε_2, σ *such that for* $v \in C_0^\infty(\mathbb{R}^{d+n})$, $\operatorname{supp} v \subset \{|X| \leq \varepsilon_2\}$ *and* $\lambda \geq \lambda_0$ *we have*

$$\lambda\|Tv\|^2_{L^2_{(1+\eta)\Phi}(\mathbb{C}^d, H_\lambda^{M-1}(\mathbb{R}^n))} \leq C_1\|\tilde{P}_\lambda Tv\|^2_{L^2_{(1+\eta)\Phi}} + C_2 e^{-\lambda\sigma}|||v|||_{M-1}^2. \tag{6.45}$$

Proof. By Theorem 3.2, (6.45) will follow from the same estimate for $\tilde{Q}_\lambda$. Now

$$\|\tilde{Q}_\lambda Tv\|_{L^2_{(1+\eta)\Phi}} = \|Q_\lambda u\|_{L^2}$$

and by (4.5) we have $\sigma^w(Q_\lambda) = \sigma^w(Q_M) + \sigma^w(Q'_{M-1})$ where

$$Q'_{M-1} \in Op_\lambda^w(S((\lambda + |\lambda\tau|_m)^{M-1}, g_2)).$$

Thus (6.45) follows from Proposition 6.4 if λ is large enough. □

7. End of the proof of the Theorems A and B

The Theorems 5.5 and 6.5 ensure that one can find $\sigma > 0$ such that

$$\lambda^{2M-1}\|Tv\|^2_{L^2_{(1+\eta)\Phi}} \leq C_1\|\tilde{P}_\lambda Tv\|^2_{L^2_{(1+\eta)\Phi}} + C_2 e^{-\lambda\sigma}|||v|||_M^2. \tag{7.1}$$

The end of the proof, *i.e.* the passage from Carleman's inequality (7.1) to uniqueness of the Cauchy problem for the operator P, is the same as the one in Robbiano-Zuily [7].

The proof of Theorems A and B is complete.

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