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DECOMPOSITION THEOREM ON INVERTIBLE SUBSTITUTIONS

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0. Introduction

The decomposition theorem of automorphisms of free group is well known, and we mention the statement in the case of rank 2.

Theorem ([1]). *Let $G\{1,2\}$ be a free group generated by symbols* 1 *and* 2. *Then any automorphism of $G\{1,2\}$ is decomposed by three automorphisms*:

$$\alpha:\begin{cases}1\to 2\\ 2\to 1\end{cases},\quad \beta:\begin{cases}1\to 12\\ 2\to 1\end{cases},\quad \gamma:\begin{cases}1\to 1\\ 2\to 2^{-1}\end{cases}.$$

Recently Zhi-Xiong Wen and Zhi-Ying Wen give the decomposition theorem of invertible substitutions of rank 2, where we say an automorphism σ is an invertible substitution if words $\sigma(1)$ and $\sigma(2)$ consist of the symbols 1 or 2.

Theorem ([2]). *Any invertible substitution is generated by three invertible substitutions*:

$$\alpha:\begin{cases}1\to 2\\ 2\to 1\end{cases},\quad \beta:\begin{cases}1\to 12\\ 2\to 1\end{cases},\quad \delta:\begin{cases}1\to 21\\ 2\to 1\end{cases}.$$

In this paper we give a simple proof of the theorem and a geometrical charactarization of invertible substitutions.

1. Proof of the theorem

Let us introduce the canonical homomorphism $\mathbf{f}: G\{1,2\}\to \boldsymbol{Z}^2$ as follows:

$$\mathbf{f}(i^{\pm 1}) := \pm \boldsymbol{e}_i,\quad i=1,2$$
$$\mathbf{f}(W) := \mathbf{f}(s_1)+\mathbf{f}(s_2)+\cdots+\mathbf{f}(s_k)\quad \text{for}\quad W=s_1s_2\cdots s_k\in G\{1,2\}$$

where $\{\boldsymbol{e}_1,\boldsymbol{e}_2\}$ be canonical basis in $\boldsymbol{R}^2$. Then we know the following property.

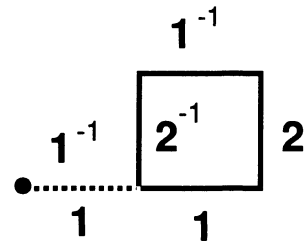

Fig. 1. $\mathcal{K}[W]$, $W = 1121^{-1}2^{-1}1^{-1}$

PROPERTY. Let us define the linear representation L_σ of σ by

$$L_\sigma = (\mathbf{f}(\sigma(1)), \mathbf{f}(\sigma(2)))\,.$$

Then the following commutative relation holds:

$$\begin{tikzcd} G\{1,2\} \arrow[r, "\sigma"] \arrow[d, "\mathbf{f}"'] & G\{1,2\} \arrow[d, "\mathbf{f}"] \\ \boldsymbol{Z}^2 \arrow[r, "L_\sigma"] & \boldsymbol{Z}^2 \end{tikzcd} \,.$$

A word $W \in G\{1,2\}$ is said to be closed if $\mathbf{f}(W) = 0$. Let $\mathcal{P}$ be the family of polygon curve with integer vertices on $\boldsymbol{R}^2$, and let us define the geometrical realization map $\mathcal{K} : G\{1,2\} \to \mathcal{P}$ by

$$\mathcal{K}[i^{\pm 1}] := \{\pm\lambda \boldsymbol{e}_i \mid 0 \le \lambda \le 1\}, \quad i = 1, 2$$

and for $W = w_1 w_2 \cdots w_k \in G\{1,2\}$

$$\mathcal{K}[w_1 w_2 \cdots w_k] := \bigcup_{i=1}^{k} \{\mathbf{f}(w_1 w_2 \cdots w_{i-1}) + \mathcal{K}[w_i]\}$$

where $\boldsymbol{x} + \mathbf{S} = \{\boldsymbol{x} + \boldsymbol{s} | \boldsymbol{s} \in \mathbf{S}\}$.
If the word W be a closed word, then the definition of $\mathcal{K}[W]$ is modified slightly as follows:

$$\mathcal{K}[W] := \mathbf{f}(U) + \mathcal{K}[W_1]$$

where U is the longest word satisfying $W = UW_1U^{-1}$.(See Fig. 1.)

Lemma 1. *For any automorphism θ, we have*

$$(*) \qquad \mathcal{K}[\theta(121^{-1}2^{-1})] = \boldsymbol{x} + \mathcal{K}[121^{-1}2^{-1}] \textit{ for some } \boldsymbol{x} \in \boldsymbol{Z}^2.$$

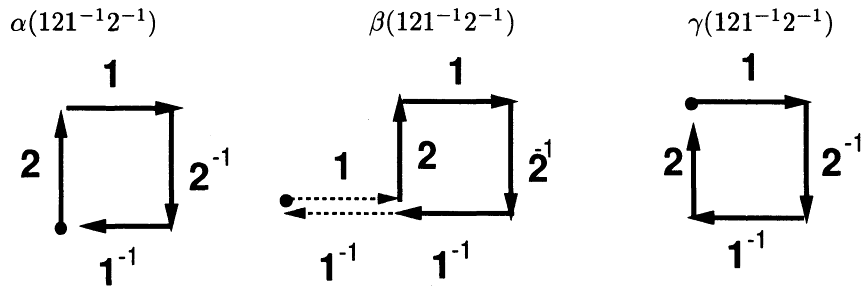

Fig. 2. $\mathcal{K}[\sigma(121^{-1}2^{-1})]$, $\sigma = \alpha,\ \beta,\ \gamma$

Proof. From Nielsen's theorem, any automorphism σ is decomposed by generators α, β and γ. On the other hand, it is easy to see that each generator of automorphisms satisfies $(*)$ property. Therefore any composition of generators also has $(*)$ property. (See Fig. 2.) □

Sublemma 1. *Let σ be an invertible substitution and let a linear representation L_σ of σ be*

$$L_\sigma = \begin{pmatrix} a & c \\ b & d \end{pmatrix}.$$

Assume that $\det L_\sigma = \pm 1$ *and* $\max\{a, b, c, d\} = 1$. *Then the invertible substitution σ is determined by the composition of α, β and δ as follows*:

list of L_σ	*list of σ*		
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow$	$\alpha\alpha : \begin{cases} 1 \to 1 \\ 2 \to 2 \end{cases}$		
$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow$	$\alpha : \begin{cases} 1 \to 2 \\ 2 \to 1 \end{cases}$		
$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \Longrightarrow$	$\beta : \begin{cases} 1 \to 12 \\ 2 \to 1 \end{cases}$	*or*	$\delta : \begin{cases} 1 \to 21 \\ 2 \to 1 \end{cases}$
$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \Longrightarrow$	$\alpha\delta : \begin{cases} 1 \to 12 \\ 2 \to 2 \end{cases}$	*or*	$\alpha\beta : \begin{cases} 1 \to 21 \\ 2 \to 2 \end{cases}$
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \Longrightarrow$	$\beta\alpha : \begin{cases} 1 \to 1 \\ 2 \to 12 \end{cases}$	*or*	$\delta\alpha : \begin{cases} 1 \to 1 \\ 2 \to 21 \end{cases}$
$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \Longrightarrow$	$\alpha\delta\alpha : \begin{cases} 1 \to 2 \\ 2 \to 12 \end{cases}$	*or*	$\alpha\beta\alpha : \begin{cases} 1 \to 2 \\ 2 \to 21 \end{cases}$

The following sublemma is easily obtained from $\det L_\sigma = \pm 1$.

Sublemma 2. *Let* $\begin{pmatrix} a & c \\ b & d \end{pmatrix}$ *be a linear representation of substitution* σ. *Assume that* $\det L_\sigma = \pm 1$ *and* $\max\{a, b, c, d\} \geq 2$ *then we have*

$$\max\{a, b, c, d\} > \max\{\{a, b, c, d\} \setminus \max\{a, b, c, d\}\}.$$

Lemma 2. *Let* σ *be a substitution and let* $\sigma(1)$ *and* $\sigma(2)$ *be* $\sigma(1) = W_1$ *and* $\sigma(2) = W_2$. *Assume that*
(1) *a linear representation* L_σ *of* σ *satisfies* $a > b \geq d \geq 0$ *and* $a > c \geq d \geq 0$
(2) $\det L_\sigma = \pm 1$
(3) $\mathcal{K}[\sigma(121^{-1}2^{-1})] = \boldsymbol{x} + \mathcal{K}[121^{-1}2^{-1}]$, $\boldsymbol{x} \in \boldsymbol{Z}^2$
then there exists non empty word U *such that*

$$W_1 = W_2 U \quad or \quad U W_2.$$

Before the proof of the lemma, we give a remark of the assumption (3). The word $\sigma(121^{-1}2^{-1})$ is a closed word, therefore $\mathcal{K}$ $[\sigma(121^{-1}2^{-1})]$ is a closed curve in general. And the assumption (3) says that the closed curve consists only of the boundary of unit square.

Proof. We can introduce the orientation of $\mathcal{K}[\sigma(121^{-1}2^{-1})]$ naturally by using the order of symbols in the word. And assume det $L_\sigma = 1$, then the orientation of $\mathcal{K}[\sigma(121^{-1}2^{-1})]$ does not change from the orientation of $\mathcal{K}[121^{-1}2^{-1}]$.

(1) The case of $W_1 = 1W_1'$ and $W_2 = 2W_2'$.
Suppose $|W_1| \leq 2$, where $|W_1|$ is the length of the word W_1, then we can determine the substitution σ by

$$\sigma : \begin{cases} 1 \to 1 \\ 2 \to 2 \end{cases} \quad \text{or} \quad \sigma : \begin{cases} 1 \to 12 \\ 2 \to 2 \end{cases},$$

and these linear representations:

$$L_\sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad L_\sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

This is contradictory to the condition (1).
Let us assume that $|W_1| \geq 3$, then W_1 and W_2 must be decomposed as $W_1 = 12W_1'$ and $W_2 = 21W_2'$. By the condition (3) we can easily see from the figure of $\mathcal{K}[\sigma(121^{-1}2^{-1})]$ that W_1 is decomposed as $W_1 = UW_2$. (See Fig. 3.)

(2) The case of $W_1 = VW_1'$ and $W_2 = VW_2'$, $V \neq \emptyset$.
Assume that $W_2' = \emptyset$ then W_1 is decomposed as $W_1 = W_2U$.
Assume that $W_2' \neq \emptyset$, then we can find V such that $W_1 = V1W_1''$ and $W_2 = V2W_2''$,

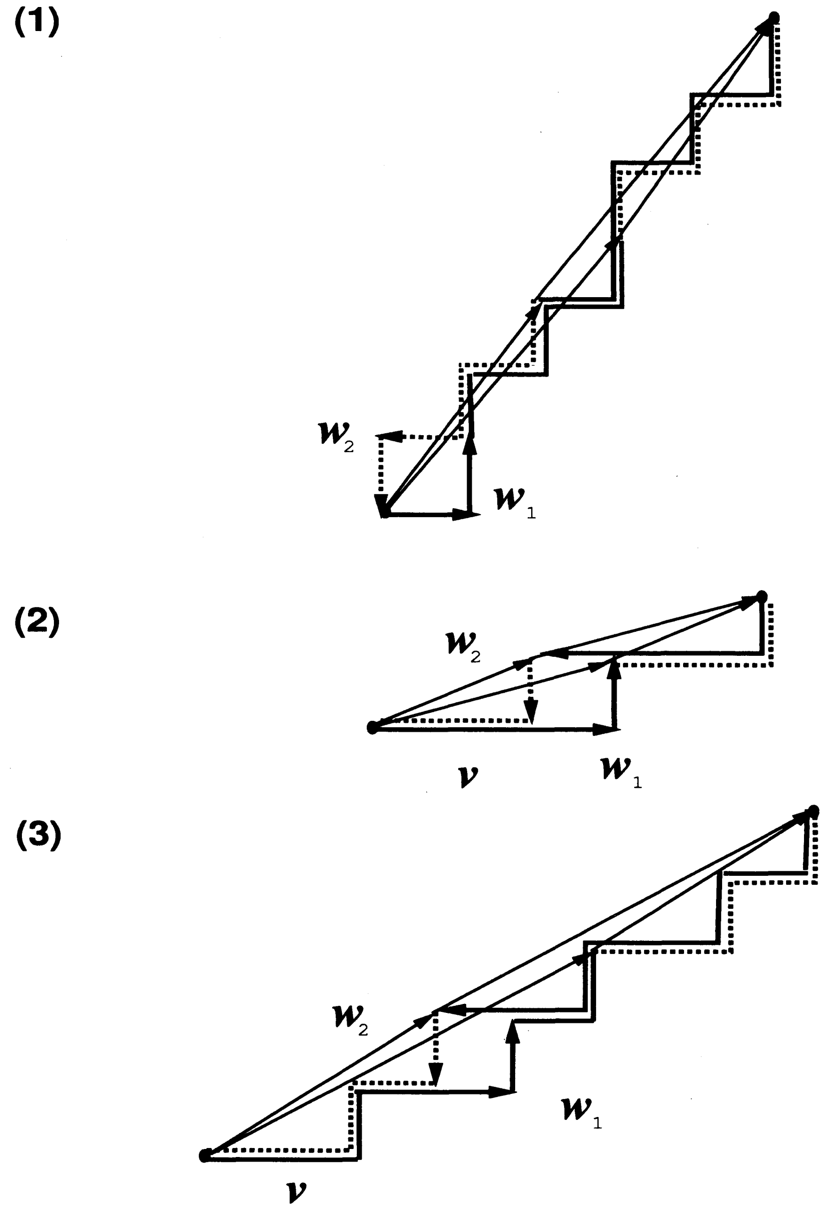

Fig. 3. $\mathcal{K}[\sigma(121^{-1}2^{-1})]$

and moreover we see that W_1'' is not empty by the condition (1). Therefore by analogous discussion of case (1) we see that there exist U such that $W_1 = UW_2$. (See Fig. 3.)

We can consider the case of det $L_\sigma = -1$ by the same manner. $\square$

Lemma 3. *Let σ is an invertible substitution which satisfies the condition* (1) *of* Lemma 2. *Then σ can be decomposed by $\sigma = \tau \circ \theta_i$ $(i \in \{1, 2\})$ with some invertible substitution τ, where θ_i is given by*

$$\theta_1 = \beta : \begin{cases} 1 \to 12 \\ 2 \to 1 \end{cases}, \quad \theta_2 = \delta : \begin{cases} 1 \to 21 \\ 2 \to 1 \end{cases}.$$

Proof. By Lemma 1, the invertible substitution σ satisfies the condition (3) of Lemma 2 and σ also satisfies the condition (2) from invertibility. So the word W_1 is decomposed as $W_1 = W_2U$ or UW_2 by Lemma 2.
Let us assume that $W_1 = W_2U$. Define the substitution τ as follows:

$$\tau : \begin{cases} 1 \to W_2 \\ 2 \to U \end{cases},$$

then we see that σ is decomposed as $\sigma = \tau \circ \theta_1$. Both σ and θ_1 are invertible, therefore τ is also invertible.
The case of $W_1 = UW_2$ is discussed analogously. $\square$

Notice that in the case of Lemma 3 the linear representation L_τ of τ satisfies

$$Ł_\tau = \begin{pmatrix} c & a-c \\ d & b-d \end{pmatrix} \quad \text{and} \quad a - c < a.$$

Therefore the following relation holds:

$$\max(\text{elements of } Ł_\sigma) > \max(\text{elements of } Ł_\tau).$$

Theorem 1. *Any invertible substitution of rank* 2 *is decomposed by three invertible substitutions*:

$$\alpha : \begin{cases} 1 \to 2 \\ 2 \to 1 \end{cases}, \quad \beta : \begin{cases} 1 \to 12 \\ 2 \to 1 \end{cases}, \quad \delta : \begin{cases} 1 \to 21 \\ 2 \to 1 \end{cases}.$$

Proof. Take any invertible substitution σ. By Sublemma 1 if max(elements of L_σ)$= 1$ then σ is decomposed by α, β and δ. Consider the case of max(elements of L_σ)≥ 2. By Sublemma 2 we take i_1, $j_1 \in \{0, 1\}$ satisfying

$$L_{\alpha^{i_1}\circ\sigma\circ\alpha^{j_1}} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}, \quad a > b \geq d \geq 0 \text{ and } a > c \geq d \geq 0.$$

By Lemma 3 there exist substitutions τ_1' and θ_{p_1} such that

$$\alpha^{i_1} \circ \sigma \circ \alpha^{j_1} = \tau_1' \circ \theta_{p_1}.$$

Therefore the substitution σ is decomposed as

$$\sigma = \alpha^{i_1} \circ \tau_1' \circ \theta_{p_1} \circ \alpha^{j_1}.$$

For $\tau_1 := \alpha^{i_1} \circ \tau_1'$ let us continue the same procedure. Then there exists τ_n such that $\max(\text{elements of } L_{\tau_n}) = 1$, and the substitution σ is decomposed as

$$\sigma = \tau_n \circ \theta_{p_n} \circ \alpha^{j_n} \circ \cdots \circ \theta_{p_2} \circ \alpha^{j_2} \circ \theta_{p_1} \circ \alpha^{j_1}.$$

where $p_k \in \{1, 2\}$ and $j_k \in \{0, 1\}$. □

Let us give a remark related to the uniqueness of decompositions. Define the invertible substitution Θ by

$$\Theta = \beta \circ \alpha \circ \delta \ (= \delta \circ \alpha \circ \beta).$$

and replace every substitutions $\beta \circ \alpha \circ \delta$ and $\delta \circ \alpha \circ \beta$ in the decomposition of σ by Θ. Then the substitution σ is decomposed uniquely by α, β, δ and Θ in our procedure. In fact, except the case of $W_1 = W_2UW_2$ we can determine which we take $\sigma = \tau \circ \theta_1$ or $\sigma = \tau \circ \theta_2$. In the case of $W_1 = W_2UW_2$, σ can be decomposed as

$$\sigma = \tau \circ \delta \circ \alpha \circ \beta = \tau \circ \beta \circ \alpha \circ \delta.$$

Using the same discussion, we have the following result.

Theorem 2 (geometrical charactarization of invertible substitutions). *Let σ be a substitution. Then σ is invertible if and only if*

$$\mathcal{K}[\sigma(121^{-1}2^{-1})] = \boldsymbol{x} + \mathcal{K}[121^{-1}2^{-1}] \quad \textit{for some} \quad \boldsymbol{x} \in \boldsymbol{Z}^2$$

Proof. If σ is invertible then by Lemma 1

$$\mathcal{K}[\sigma(121^{-1}2^{-1})] = \boldsymbol{x} + \mathcal{K}[121^{-1}2^{-1}] \quad \text{for some} \quad \boldsymbol{x} \in \boldsymbol{Z}^2.$$

Oppositely, assume that

$$(**) \qquad \mathcal{K}[\sigma(121^{-1}2^{-1})] = \boldsymbol{x} + \mathcal{K}[121^{-1}2^{-1}] \quad \text{for} \quad \boldsymbol{x} \in \boldsymbol{Z}^2$$

then we know $W_1 = W_2U$ or UW_2 by Lemma 2. In the case of $W_1 = W_2U$ (resp. $W_1 = UW_2$) determine the substitution τ (resp. τ') such that

$$\tau : \begin{cases} 1 \to W_2 \\ 2 \to U \end{cases} \quad \left(\text{resp.}\tau' : \begin{cases} 1 \to W_2 \\ 2 \to U \end{cases}\right)$$

then $\sigma = \tau \circ \theta_1$ (resp. $\sigma = \tau' \circ \theta_2$) and τ satisfies $(**)$ property. Continue the procedure, the substitution σ is decomposed by α, β and δ. So σ is invertible. □

2. Interval exchange transformations and invertible substitutions

In this section, we discuss about the dynamical system called an interval exchange transformation associated with a substitution.

ASSUMPTION. Let us assume that the substitution σ satisfies the following properties:

(1) $\det L_\sigma = \pm 1$

(2) the charactaristic polynomial is irreducible.

Let μ be the maximum eigenvalue of L_σ and $\binom{1}{\alpha}$ and $\binom{1}{\beta}$ be column and row eigenvectors of μ, that is,

$$L_\sigma \begin{pmatrix} 1 \\ \alpha \end{pmatrix} = \mu \begin{pmatrix} 1 \\ \alpha \end{pmatrix} \quad \text{and} \quad {}^tL_\sigma \begin{pmatrix} 1 \\ \beta \end{pmatrix} = \mu \begin{pmatrix} 1 \\ \beta \end{pmatrix}.$$

Let $\boldsymbol{l}$ be the contracting invariant line of L_σ, then $\boldsymbol{l}$ is given by

$$\boldsymbol{l} = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} 1 \\ \beta \end{pmatrix} \right) = 0 \right\}.$$

Let $\mathbf{l}_1$ and $\mathbf{l}_2$ be unit seguments spanned by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$, that is,

$$\mathbf{l}_1 := \{\lambda \boldsymbol{e}_2 \mid 0 \le \lambda \le 1\}$$
$$\mathbf{l}_2 := \{\lambda \boldsymbol{e}_1 \mid 0 \le \lambda \le 1\}.$$

Let us consider a set of unit seguments on lattice points:

$$\mathsf{S}_\beta := \left\{ (\boldsymbol{x}, \mathbf{l}) \in \boldsymbol{Z}^2 \times \{\mathbf{l}_1, \mathbf{l}_2\} \middle| \begin{array}{l} (\boldsymbol{x}, \binom{1}{\beta}) \ge 0 \\ (\boldsymbol{x} - \boldsymbol{e}_i, \binom{1}{\beta}) < 0 \text{ if } \mathbf{l} = \mathbf{l}_i \end{array} \right\}.$$

We call the union of elements of S_β the stepped curve of the line $\boldsymbol{l}$ and it is denoted by

$$\boldsymbol{S}_\beta := \bigcup_{(\boldsymbol{x}, \mathbf{l}_i) \in \mathsf{S}_\beta} (\boldsymbol{x} + \mathbf{l}_i).$$

Let us consider the finite union of S_β as follows:

$$\mathcal{G} := \left\{ \sum_{\lambda\in\Lambda} (\boldsymbol{x},\mathbf{l})_\lambda \,\middle|\, \begin{array}{l} {}^\sharp\Lambda < +\infty, (\boldsymbol{x},\mathbf{l})_\lambda \in \mathsf{S}_\beta \\ (\boldsymbol{x},\mathbf{l})_\lambda \neq (\boldsymbol{x},\mathbf{l})_{\lambda'} \text{ if } \lambda \neq \lambda' \end{array} \right\}.$$

DEFINITION. On the notation of

$$\sigma(1) = s_1 s_2 \cdots s_k,$$
$$\sigma(2) = t_1 t_2 \cdots t_l$$

and

$$L_\sigma^{-1} = (\boldsymbol{f}_1, \boldsymbol{f}_2)$$

let us define a map Σ_σ on $\mathcal{G}$ as follows:
for $r = 1, 2$

$$\Sigma_\sigma : (\mathbf{0}, \mathbf{l}_r) \mapsto \left\{ \left\{ \sum_{j; s_j = r} \left(\sum_{i=j+1}^{k} \boldsymbol{f}_{s_i}, \mathbf{l}_1 \right) \right\} + \left\{ \sum_{j'; t_{j'} = r} \left(\sum_{i=j'+1}^{l} \boldsymbol{f}_{t_i}, \mathbf{l}_2 \right) \right\} \right\}$$
$$\Sigma_\sigma(\boldsymbol{x}, \mathbf{l}_r) := L_\sigma^{-1}(\boldsymbol{x}) + \Sigma_\sigma(\mathbf{0}, \mathbf{l}_r), \boldsymbol{x} \in \boldsymbol{Z}^2$$

and

$$\Sigma_\sigma\left(\sum_p (\boldsymbol{x}_p, \mathbf{l}_{r_p})\right) := \sum_p \Sigma_\sigma(\mathbf{0}, \mathbf{l}_{r_p}).$$

The map Σ_σ is called the canonical form of σ.

REMARK. The canonical form of σ has another expression, which is for $r = 1, 2$

$$\Sigma_\sigma(\mathbf{0}, \mathbf{l}_r)$$
$$= \left\{ \left\{ \sum_{j; s_j = r} \left(-\sum_{i=1}^{j} \boldsymbol{f}_{s_i} + \boldsymbol{e}_1, \mathbf{l}_1 \right) \right\} + \left\{ \sum_{j'; t_{j'} = r} \left(-\sum_{i=1}^{j'} \boldsymbol{f}_{t_i} + \boldsymbol{e}_2, \mathbf{l}_2 \right) \right\} \right\}$$

By the definition of canonical form, Arnoux-Ito ([3]) gives following propositions.

Let $\mathcal{U}$ and $\mathcal{U}'$ be $\mathcal{U} = (\boldsymbol{e}_1, \mathbf{l}_1) + (\boldsymbol{e}_2, \mathbf{l}_2)$ and $\mathcal{U}' = (\mathbf{0}, \mathbf{l}_1) + (\mathbf{0}, \mathbf{l}_2)$. We define the geometrical realization map $\mathbf{K} : \mathcal{G} \to \{\text{polygons on } \boldsymbol{R}^2\}$ as follows:

$$\mathbf{K} : (\boldsymbol{x}, \mathbf{l}_r) \mapsto \boldsymbol{x} + \mathbf{l}_r \text{ for } r = 1, 2$$
$$\mathbf{K}\left[\textstyle\sum_i (\boldsymbol{x}_i, \mathbf{l}_{r_i})\right] := \textstyle\bigcup_i (\boldsymbol{x}_i + \mathbf{l}_{r_i}),$$

and let $\Pi_{\alpha,\beta}$ be a projection from $\boldsymbol{R}^2$ to the line $\boldsymbol{l}$ along $\binom{1}{\alpha}$.
Let us define domains, which is finite union of intervals on $\boldsymbol{l}$ in general, as follows:

$$\Pi_{\alpha,\beta}[\mathbf{K}(\mathbf{0},\mathbf{l}_i)] = \mathbf{D}_i^{(0)'}$$
$$\Pi_{\alpha,\beta}[\mathbf{K}(e_i,\mathbf{l}_i)] = \mathbf{D}_i^{(0)}$$
$$\mathbf{D}^{(0)} := \bigcup_{i=1,2} \mathbf{D}_i^{(0)} = \bigcup_{i=1,2} \mathbf{D}_i^{(0)'}$$

and

$$\Pi_{\alpha,\beta}[\mathbf{K}(\Sigma_\sigma(\mathbf{0},\mathbf{l}_i))] = \mathbf{D}_i^{(1)'}$$
$$\Pi_{\alpha,\beta}[\mathbf{K}(\Sigma_\sigma(e_i,\mathbf{l}_i))] = \mathbf{D}_i^{(1)}$$
$$\mathbf{D}^{(1)} := \bigcup_{i=1,2} \mathbf{D}_i^{(1)} = \bigcup_{i=1,2} \mathbf{D}_i^{(1)'}.$$

Then the following general interval exchange transformation on $\mathbf{D}^{(0)}$ and $\mathbf{D}^{(1)}$ are well-defined:

$$\begin{aligned} W_{(0)} : \mathbf{D}^{(0)} &\longrightarrow \mathbf{D}^{(0)} \\ \boldsymbol{x} &\longmapsto \boldsymbol{x} - \Pi_{\alpha,\beta}\boldsymbol{e}_i \quad \text{if} \quad \boldsymbol{x} \in \mathbf{D}_i^{(0)} \\ W_{(1)} : \mathbf{D}^{(1)} &\longrightarrow \mathbf{D}^{(1)} \\ \boldsymbol{x} &\longmapsto \boldsymbol{x} - \Pi_{\alpha,\beta}\boldsymbol{f}_i \quad \text{if} \quad \boldsymbol{x} \in \mathbf{D}_i^{(1)}, \end{aligned}$$

and the following propositions hold.

Proposition 1 ([3]).
(1) $\Sigma_\sigma\mathcal{U} \supset \mathcal{U}$ *and* $\Sigma_\sigma\mathcal{U}' \supset \mathcal{U}'$

$$\textit{Moreover, } \Sigma_\sigma\mathcal{U} - \mathcal{U} = \Sigma_\sigma\mathcal{U}' - \mathcal{U}'.$$

(2) *Assume that* $(\boldsymbol{x},\mathbf{l}_i) \in \mathsf{S}_\beta$ *then we have* $\Sigma_\sigma(\boldsymbol{x},\mathbf{l}_i) \in \mathcal{G}$.
(3) *Assume that* $(\boldsymbol{x},\mathbf{l}_i) \neq (\boldsymbol{x}',\mathbf{l}_j)$ *then we have*

$$\Sigma_\sigma(\boldsymbol{x},\mathbf{l}_i) \cap \Sigma_\sigma(\boldsymbol{x}',\mathbf{l}_j) = \emptyset.$$

Proposition 2 ([3]). *Let* $W_{(1)}|_{\mathbf{D}^{(0)}}$ *be the induced transformation of* $W_{(1)}$ *to the set* $\mathbf{D}^{(0)}$. *Then we have*
(1) $W_{(1)}|_{\mathbf{D}^{(0)}} = W_{(0)}$
(2) $W_{(1)}|_{\mathbf{D}^{(0)}}$ *has* σ*-structure, that is, for* $i = 1, 2$

$$W_{(1)}^{j-1}\mathbf{D}_1^{(0)} \subset \mathbf{D}_{s_j}^{(1)} \textit{ for } 1 \le j \le k \textit{ and } W_{(1)}^k\mathbf{D}_1^{(0)} = \mathbf{D}_1^{(0)'}$$
$$W_{(1)}^{j'-1}\mathbf{D}_2^{(0)} \subset \mathbf{D}_{t_{j'}}^{(1)} \textit{ for } 1 \le j' \le l \textit{ and } W_{(1)}^l\mathbf{D}_2^{(0)} = \mathbf{D}_2^{(0)'}.$$

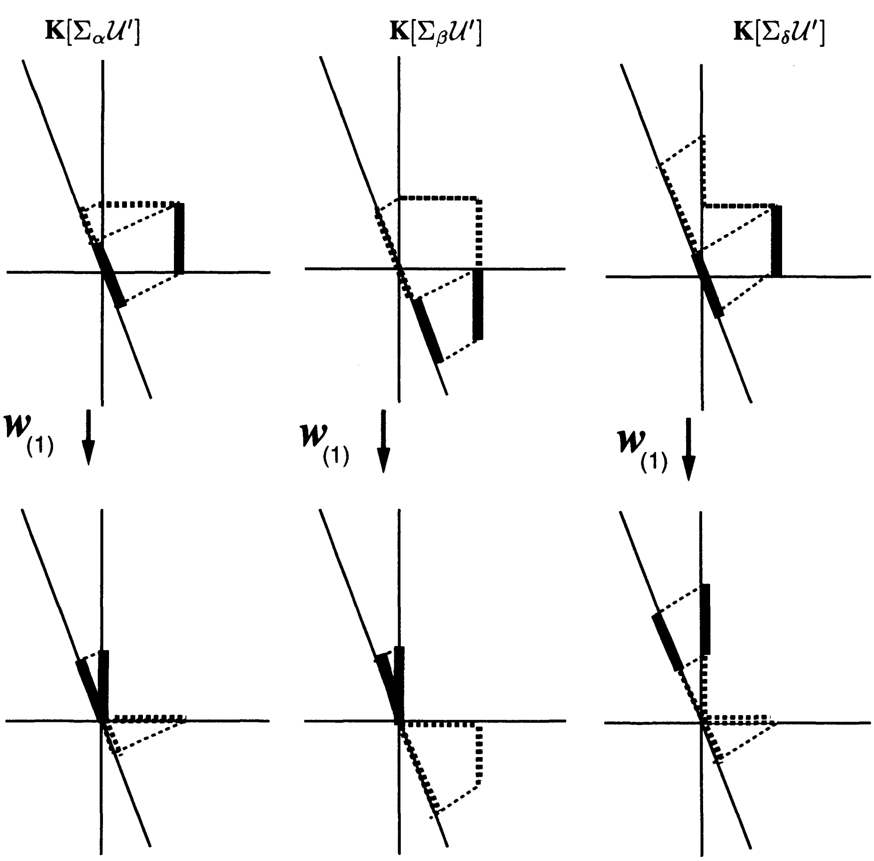

Fig. 4. $W_{(1)}$

Using the decomposition theorem in section one, we obtain the following other charactarization of invertible substitutions.

Theorem 3. *A substitution σ is an invertible substitution if and only if the interval exchange transformation $W_{(1)}$ associated with σ is 2-state interval exchange transformation.*

Proof. If σ is an invertible substitution then from the decomposition theorem the substitution σ is decomposed by the generators α, β and δ. So it is enough to show that the interval exchange transformations associated with α, β and δ are 2-state interval exchange transformations. (See Fig. 4.)

Oppositely, assume the interval exchange transformation $W_{(1)}$ assosiated with σ is 2-state interval exchange transformation. Without the loss of a generality, we assume that $L_\sigma = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ satisfies $a > b \geq d$ and $a > c \geq d$ by taking $\alpha^i \circ \sigma \circ \alpha^j$ if necessary

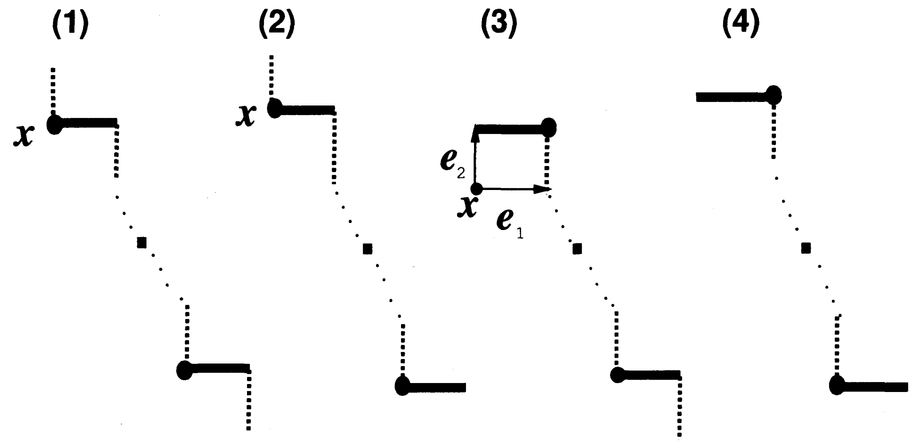

Fig. 5. $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$

where $i,\ j \in \{0,1\}$.
From the fact that $a+b>c+d$, that is,

$$\text{the number of } \mathbf{l}_1 \text{ in } \mathbf{K}[\Sigma_\sigma \mathcal{U}'] > \text{the number of } \mathbf{l}_2 \text{ in } \mathbf{K}[\Sigma_\sigma \mathcal{U}']$$

and $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$ belongs in the stepped curve $\boldsymbol{S}_\beta$ from Proposition 1 (2), we see that there are no $(\boldsymbol{x},\mathbf{l}_2)$ such that $(\boldsymbol{x},\mathbf{l}_2)$ and $(\boldsymbol{x}+\boldsymbol{e}_1,\mathbf{l}_2) \in \Sigma_\sigma \mathcal{U}'$, and $\Sigma_\sigma \mathcal{U}$ has the same property by Proposition 1 (1). Let us consider 4 cases;

- The ends of $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$ are not constructed by $\mathbf{l}_2 \cdots\cdots$ (1)
- One of the ends of $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$ is constructed by $\mathbf{l}_2 \cdots\cdots$ (2) (3)
- Both of the ends of $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$ are constructed by $\mathbf{l}_2 \cdots$ (4)

(See Fig. 5.)

The case of (4) is impossible since $\Sigma_\sigma \mathcal{U}$ does not contain both $(\boldsymbol{x},\mathbf{l}_2)$ and $(\boldsymbol{x}+\boldsymbol{e}_1,\mathbf{l}_2)$ for any $\boldsymbol{x}$.
For the case of (1) and (2), if $(\boldsymbol{x},\mathbf{l}_2)$ is in $\Sigma_\sigma \mathcal{U}'$ then $(\boldsymbol{x},\mathbf{l}_1)$ is also in $\Sigma_\sigma \mathcal{U}'$ from the connectedness of $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$. So by the definition of Σ_σ we have

$$\{\boldsymbol{f}_{s_k}, \boldsymbol{f}_{s_k}+\boldsymbol{f}_{s_{k-1}}, \cdots, \sum_{i=1}^{k} \boldsymbol{f}_{s_i}\} \supset \{\boldsymbol{f}_{t_l}, \boldsymbol{f}_{t_l}+\boldsymbol{f}_{t_{l-1}}, \cdots, \sum_{i=1}^{l} \boldsymbol{f}_{t_i}\}.$$

Then there exists $\sum_{i=j}^{k} \boldsymbol{f}_{s_i}$ such that $\boldsymbol{f}_{t_l} = \sum_{i=j}^{k} \boldsymbol{f}_{s_i}$ and by operating L_σ we have

$$\mathbf{f}(t_l) = \sum_{i=j}^{k} \mathbf{f}(s_i), \quad \mathbf{f}(t_l), \mathbf{f}(s_i) \in \{\boldsymbol{e}_1, \boldsymbol{e}_2\}.$$

Therefore we have

$$\mathbf{f}(t_l) = \mathbf{f}(s_k) \quad \text{and} \quad t_l = s_k.$$

Continue the same procedure, we obtain

$$t_l = s_k, t_{l-1} = s_{k-1}, \cdots, t_1 = s_{k-l+1}.$$

This means that W_1 is decomposed as $W_1 = UW_2$.
For the case of (3), if $(\boldsymbol{x} + \boldsymbol{e}_2, \mathbf{l}_2)$ is in $\Sigma_\sigma \mathcal{U}'$ then $(\boldsymbol{x} + \boldsymbol{e}_1, \mathbf{l}_1)$ is also in $\Sigma_\sigma \mathcal{U}'$ from the connectedness of $\mathbf{K}[\Sigma_\sigma \mathcal{U}']$. So by the remark we have

$$\{\boldsymbol{f}_{s_1}, \boldsymbol{f}_{s_1} + \boldsymbol{f}_{s_2}, \cdots, \sum_{i=1}^{k} \boldsymbol{f}_{s_i}\} \supset \{\boldsymbol{f}_{t_1}, \boldsymbol{f}_{t_1} + \boldsymbol{f}_{t_2}, \cdots, \sum_{i=1}^{l} \boldsymbol{f}_{t_i}\}.$$

Then by the same procedure as the case of (1) and (2), W_1 is decomposed as $W_1 = W_2U$. Using same discussion as Lemma 3 in section one, there exists θ_i and τ which decompose σ as $\sigma = \tau \circ \theta_i$. And notice that

$$\Sigma_\sigma = \Sigma_{\theta_i} \circ \Sigma_\tau$$

we can say the substitution τ also has 2-state interval exchange transformation, since the interval exchange transformations associated with σ and θ_i are 2-state interval exchange transformations. Continue the same procedure, there exists τ_n which satisfies that

$$\max(\text{elements of } L_{\tau_n}) = 1$$

and we obtain that

$$\sigma = \tau_n \circ \theta_{p_n} \circ \alpha^{j_n} \circ \cdots \circ \theta_{p_2} \circ \alpha^{j_2} \circ \theta_{p_1} \circ \alpha^{j_1}$$

where $p_k \in \{1, 2\}$ and $j_k \in \{0, 1\}$.
So the substituiton σ is invertible. □

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