Casali, M.R.
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AVERAGE ORDER OF COLOURED TRIANGULATIONS : THE GENERAL CASE

MARIA RITA CASALI

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1. Introduction

In [17], F. Luo and R. Stong introduced the notion of "average edge order"

$$\mu_0(K) = \frac{3 \cdot F_0(K)}{E_0(K)}$$

K being a triangulation of a closed 3-manifold M^3 with $E_0(K)$ edges and $F_0(K)$ triangles.

The main properties of $\mu_0(K)$ and its relations with the topology of M^3, both in the closed and bounded case (which has been successively investigated by M. Tamura in [20]), are collected into the following theorems:

Theorem 1. [17] *Let K be any triangulation of a closed 3-manifold M^3. Then:*

(a) $3 \leq \mu_0(K) < 6$, *equality holds if and only if K is the triangulation of the boundary of a 4-simplex.*

(b) *If $\mu_0(K) < 4.5$, then K is a triangulation of $\mathbb{S}^3$.*

(c) *If $\mu_0(K) = 4.5$, then K is a triangulation of $\mathbb{S}^3$, $\mathbb{S}^2 \times \mathbb{S}^1$, or $\mathbb{S}^2 \tilde{\times} \mathbb{S}^1$.*

(d) *For every M^3 and for every $\epsilon > 0$, there exist triangulations K_1 and K_2 of M^3 such that $\mu_0(K_1) < 4.5 + \epsilon$ and $\mu_0(K_2) > 6 - \epsilon$.*

Theorem 2. [20] *Let K be any triangulation of a compact 3-manifold M^3, with non-empty boundary. Then:*

(a) $2 \leq \mu_0(K) < 6$, *equality holds if and only if K is the triangulation of one 3-simplex.*

(b) *If $\mu_0(K) < 3$, then K is a triangulation of $\mathbb{D}^3$.*

(c) *If $\mu_0(K) = 3$, then K is a triangulation of $\mathbb{D}^3$, $\mathbb{D}^2 \times \mathbb{S}^1$, or $\mathbb{D}^2 \tilde{\times} \mathbb{S}^1$.*

(d) *For every M^3 and for every rational number r with $3 < r < 6$, there exists a triangulation $\tilde{K}$ of M^3 such that $\mu_0(\tilde{K}) = r$.*

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Now, if K is a triangulation of a compact PL n-manifold M^n, with $\alpha_i(K)$ i-simplices, $0 \leq i \leq n$, it is natural to define the *average* $(n-2)$*-simplex order* as

$$\mu(K) = \frac{n \cdot \alpha_{n-1}(K)}{\alpha_{n-2}(K)}$$

The aim of the present paper is to investigate the properties of $\mu(\bar{K})$, $\bar{K}$ being a "coloured" triangulation of a compact PL n-manifold M^n. In short, this means that $\bar{K}$ is a pseudocomplex (see [14]) triangulating M^n, whose vertices are labelled by "colours" 0, 1, ..., n, so that the colouring is injective on each n-simplex of $\bar{K}$. The following statements show the existence of strong analogies with the 3-dimensional simplicial cases ; here, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ (resp. $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{n-1}$) denotes the orientable (resp. non orientable) $\mathbb{S}^{n-1}$-bundle over $\mathbb{S}^1$, while $\#_h \mathbb{D}^n$ denotes the connected sum of h copies of the n-disk $\mathbb{D}^n$ (i.e. the bounded PL-manifold obtained from $\mathbb{S}^n$ by deleting the interiors of h disjoint n-disks).

Theorem 3. *Let $\bar{K}$ be any coloured triangulation of a closed PL n-manifold M^n $(n \geq 3)$. Then:*

(a) $2 \leq \mu(\bar{K}) < 6$, *equality holds if and only if $\bar{K}$ is the standard (two n-simplices) coloured triangulation of $\mathbb{S}^n$.*

(b) *If $\mu(\bar{K}) < (2(n+1))/(n-1)$, then $\bar{K}$ is a coloured triangulation of $\mathbb{S}^n$.*

(c) *For $3 \leq n \leq 5$, if $\mu(\bar{K}) = (2(n+1))/(n-1)$, then $\bar{K}$ is a coloured triangulation of one of the following n-manifolds : $\mathbb{S}^n$, $\mathbb{S}^1 \times \mathbb{S}^{n-1}$, $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{n-1}$ or (for $n = 3$) the real projective space $\mathbb{RP}^3$.*

(d) *For every M^n and for every $\epsilon > 0$, there exists a coloured triangulation $\bar{K}_\epsilon$ of M^n such that $\mu(\bar{K}_\epsilon) < (2(n+1))/(n-1) + \epsilon$.*

Theorem 4. *Let $\bar{K}$ be any coloured triangulation of a compact PL n-manifold M^n $(n \geq 3)$, with (possibly disconnected) non-empty boundary. Then:*

(a) $(2(n+2))/(n+1) \leq \mu(\bar{K}) < 6$, *equality holds if and only if $\bar{K}$ is the standard (two n-simplices) coloured triangulation of $\mathbb{D}^n$.*

(b)
- *For $n = 3$, if $\mu(\bar{K}) \leq (15)/(4)$, then $\bar{K}$ is a coloured triangulation of $\#_h \mathbb{D}^3$, with $1 \leq h \leq 2$;*
- *For $n \in \{4, 5\}$, if $\mu(\bar{K}) \leq (2(n+1))/(n-1)$, then $\bar{K}$ is a coloured triangulation of $\#_h \mathbb{D}^n$, with $1 \leq h \leq 3$;*
- *For every $n \geq 6$, if $\mu(\bar{K}) \leq (2(n+1))/(n-1)$, then $\bar{K}$ is a coloured triangulation of $\#_h \mathbb{D}^n$, with $1 \leq h \leq 2$.*

(c) *For every M^n and for every $\epsilon > 0$, there exists a coloured triangulation $\bar{K}_\epsilon$ of M^n such that*

$$\mu(\bar{K}_\epsilon) < \begin{cases} \dfrac{15}{4} + \epsilon & if\ n = 3 \\ \dfrac{2(n+1)}{n-1} + \epsilon & for\ every\ n \geq 4 \end{cases}$$

Both Theorem 3 and Theorem 4 will be proved in the third section, by making use of combinatorial properties of coloured triangulations (see section 2) and of the relationships with a known PL-manifold invariant, called *regular genus* (see [10] and [11] for definitions and basic properties, [7], [8], [6] and [4] for further developments).

2. Combinatorics of coloured triangulations of manifolds

If M^n is a compact PL n-manifold[1] and K is any simplicial triangulation of M^n, then the vertices of the first barycentric subdivision K' of K may be labelled in a canonical way by the elements of the colour-set $\Delta_n = \{0, 1, \dots, n\}$, so that the following conditions hold:

i) each n-simplex of K' has exactly one c-labelled vertex, for every $c \in \Delta_n$;

ii) each n-labelled vertex is internal in K'.

In fact, it is sufficient to assign every vertex of K' the dimension of the simplex of K whose barycenter is that vertex.

The resulting "labelled" complex K' is nothing but a particular example of coloured triangulation of M^n. Actually, a *coloured triangulation* of M^n may be defined as a pair $(\bar{K}, \xi)$, where:

- $\bar{K}$ is a pseudocomplex[2] (see [14]) triangulating M^n, with vertex set $S_0(\bar{K})$;
- $\xi : S_0(\bar{K}) \to \Delta_n$ is a map (*vertex-labelling*) satisfying the above conditions i) and ii).

For example, Figure 1 (resp. Figure 2) shows a coloured triangulation of the orientable (resp. non-orientable) surface of genus one with one boundary component. Note that, in both cases, the depicted pseudocomplex is not a simplicial triangulation of the associated surface.

In the existing literature, coloured triangulations of n-manifolds are usually visualized by means of $(n+1)$-*coloured graphs*, or *n-gems* (see [9], [1], [15], [16], [5], [21] and their bibliography); however, the present paper works directly with coloured triangulations, making suitable translations from known results of the combinatorial theory, when it is necessary in order to analize the properties of the

[1] See [19] or [12] for basic notions on piecewise-linear (PL) category.

[2] Remember that a pseudocomplex is a ball-complex which differs from a simplicial complex because its "h-simplices" may intersect in more than one face: thus, even if PL-manifolds may be represented both by simplicial triangulations and by coloured triangulations, there is no inclusive relation between the two classes.

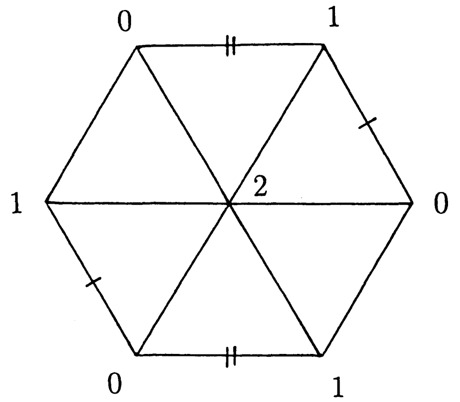

Fig. 1.

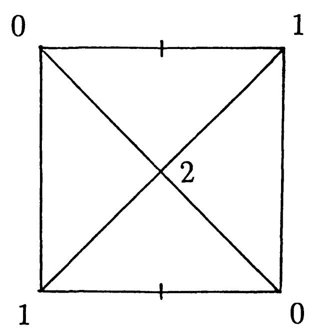

Fig. 2.

average $(n-2)$-simplex order.

From now on, let $(\bar{K},\xi)$ be a coloured triangulation of M^n. For simplicity, the vertex-labelling ξ will be often understood ; thus, the coloured triangulation $(\bar{K},\xi)$ will be simply denoted by the symbol $\bar{K}$ of its underlying pseudocomplex. For every $h>0$, an $(n-h)$-simplex of $\bar{K}$ is said to be a *boundary* $(n-h)$*-simplex* (resp. an *internal* $(n-h)$*-simplex*) if it is contained (resp. if it is not contained) in the boundary of $\bar{K}$; on the other hand, an n-simplex of $\bar{K}$ is said to be a *boundary* n*-simplex* (resp. an *internal* n*-simplex*) if it has a boundary $(n-1)$-face (resp. if its $(n-1)$-faces are internal in $\bar{K}$).

Let $p=\alpha_n(\bar{K})$ be the total number of n-simplices of $\bar{K}$, $\mathring{p}\leq p$ the number of internal n-simplices and $\bar{p}=p-\mathring{p}\geq 0$ the number of boundary n-simplices. For every $\mathcal{B}\subset\Delta_n$, with cardinality $\#\mathcal{B}=h>0$, we will denote by $g_{\mathcal{B}}$ (resp. $\mathring{g}_{\mathcal{B}}$) (resp. $\bar{g}_{\mathcal{B}}$) the number of $(n-h)$-simplices of $\bar{K}$ (resp. internal $(n-h)$-simplices) (resp. boundary $(n-h)$-simplices) which do not contain c-labelled vertices, for any $c\in\mathcal{B}$; in particular, if $\mathcal{B}=\{i,j\}$ (resp. $\mathcal{B}=\{i,j,k\}$), we will often write $g_{i,j}$, $\mathring{g}_{i,j}$ and

$\bar{g}_{i,j}$ (resp. $g_{i,j,k}$, $\mathring{g}_{i,j,k}$ and $\bar{g}_{i,j,k}$) instead of $g_{\mathcal{B}}$, $\mathring{g}_{\mathcal{B}}$ and $\bar{g}_{\mathcal{B}}$.

Obviously, relation $g_{\mathcal{B}} = \mathring{g}_{\mathcal{B}} + \bar{g}_{\mathcal{B}}$ is true for every $\mathcal{B} \subset \Delta_n$, while relations $g_{\mathcal{B}} = \mathring{g}_{\mathcal{B}}$ and $\bar{g}_{\mathcal{B}} = 0$ hold if $n \notin \mathcal{B}$ or if $\partial M^n = \emptyset$. If $\partial M^n \neq \emptyset$ and $\mathcal{B} = \{n\} \cup \mathcal{B}'$, let ${}^{\partial}g_{\mathcal{B}'}$ denote the number of $(n-h)$-simplices of $\partial\bar{K}$ whose vertices are labelled by $\Delta_{n-1} - \mathcal{B}'$; then, equality $\bar{g}_{\mathcal{B}} = {}^{\partial}g_{\mathcal{B}'}$ holds.

As far as the average $(n-2)$-simplex order is concerned, it is easy to check that the total number of $(n-2)$-simplices of $\bar{K}$ is $\alpha_{n-2}(\bar{K}) = \sum_{i,j\in\Delta_n} g_{i,j}$, while the total number of $(n-1)$-simplices of $\bar{K}$ is $\alpha_{n-1}(\bar{K}) = (n+1)/2\,\mathring{p} + (n+2)/2\bar{p}$. Thus, the following fundamental formula holds, for every coloured triangulation $\bar{K}$ of a compact PL-manifold M^n:

$$\mu(\bar{K}) = n \cdot \frac{(n+1)\,\mathring{p} + (n+2)\bar{p}}{2 \cdot \sum_{i,j\in\Delta_n} g_{i,j}} \tag{1}$$

Within the representation theory of PL-manifolds by coloured triangulations (or $(n+1)$-coloured graphs), great importance is attached to the notion of regular genus, which generalizes to arbitrary dimension the genus of a surface and the Heegaard genus of a 3-manifold. For example, many results have been achieved in order to classify PL n-manifolds with "low" regular genus : see [7], [8], [6], [4] and related papers.

If $\bar{K}$ is any coloured triangulation of M^n and $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_n = n)$ is any circular permutation of Δ_n, then the 1-skeleton of the ball complex dual to $\bar{K}$ is proved to admit a particular kind of cellular embedding-called *regular embedding* [3]- onto a suitable surface F_ϵ; moreover, F_ϵ results to be orientable (resp. closed) if and only if M^n is orientable (resp. closed).

The genus (resp. half the genus) of the orientable (resp. non-orientable) surface F_ϵ is said to be the *ϵ-genus* $\rho_\epsilon(\bar{K})$ of the coloured triangulation $\bar{K}$; according to [10] and [11], the integer $\rho_\epsilon = \rho_\epsilon(\bar{K})$ may be directly computed from the combinatorial properties of $\bar{K}$ by means of the following formula:

$$\sum_{i\in\mathbb{Z}_{n+1}} \mathring{g}_{\epsilon_i,\epsilon_{i+1}} + (1-n)\frac{\mathring{p}}{2} + (2-n)\frac{\bar{p}}{2} + {}^{\partial}g_{\epsilon_0,\epsilon_{n-1}} = 2 - 2\rho_\epsilon \tag{2}$$

Finally, the *regular genus* $\mathcal{G}(M^n)$ of a PL n-manifold M^n may be defined as

$$\mathcal{G}(M^n) = \min\left\{ \rho_\epsilon(\bar{K}) \,\middle|\, \begin{array}{l} \bar{K} \text{ is a coloured triangulation of } M^n \\ \epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_n = n) \text{ is a circular permutation of } \Delta_n \end{array} \right\}$$

[3]In the closed case, this means that each embedding region is an open ball bounded by images of edges which are alternatively dual to $(\Delta_n - \{\epsilon_i\})$-labelled and $(\Delta_n - \{\epsilon_{i+1}\})$-labelled $(n-1)$-simplices of $\bar{K}$, for some $i \in \Delta_n$. On the other hand, if M^n has non-empty boundary, the notion of regular embedding has to be suitably modified, taking into account both the boundary of the surface F_ϵ and the boundary of the pseudocomplex $\bar{K}$ (see [11] for details).

Since $\rho_\epsilon(\bar{K}) = \rho_{(\epsilon^{-1})}(\bar{K})$ obviously holds, then the computation of the ϵ-*genera* $\{\rho_\epsilon(\bar{K})\}_\epsilon$ may be restricted to the subset $\bar{\mathcal{P}}_n$ of circular permutations $\epsilon = (\epsilon_0, \epsilon_1, \dots, \epsilon_n = n)$ of Δ_n, where each permutation is identified with its inverse one. Moreover, if $\bar{\mathcal{P}}'_n = \{\epsilon \in \bar{\mathcal{P}}_n \ / \ \epsilon_{n-1} = n-1\}$, then every $\epsilon \in \bar{\mathcal{P}}'_n$ induces a permutation $\epsilon_{\hat{n}} = (\epsilon_0, \epsilon_1, \dots, \epsilon_{n-1} = n-1) \in \bar{\mathcal{P}}_{n-1}$; thus, the ϵ-genus ${}^{\partial}\rho_\epsilon = \rho_{(\epsilon_{\hat{n}})}(\partial \bar{K})$ results to be well defined.

The following Lemma yields an useful relation among the total number of $(n-2)$-simplices of $\bar{K}$ and the whole set of ϵ-genera for $\bar{K}$ and (possibly) $\partial \bar{K}$.

Lemma 1. *Let $\bar{K}$ be a coloured triangulation of a compact (connected) PL n-manifold M^n, and let $h \geq 0$ be the number of connected components of its (possibly empty) boundary ∂M^n.*
i) *If $\partial M^n = \emptyset$, then*

$$\sum_{i,j \in \Delta_n} g_{i,j} = n + n(n-1)\frac{p}{4} - \frac{2}{(n-1)!} \cdot \sum_{\epsilon \in \bar{\mathcal{P}}_n} \rho_\epsilon$$

ii) *If $\partial M^n \neq \emptyset$, then*

$$\sum_{i,j \in \Delta_n} g_{i,j} = (n-h) + n(n-1)\frac{\mathring{p}}{4} + (n^2-n+2)\frac{\bar{p}}{4}$$
$$- \frac{2}{(n-1)!}\left[\sum_{\epsilon \in \bar{\mathcal{P}}'_n} (\rho_\epsilon - {}^{\partial}\rho_\epsilon) + \sum_{\epsilon \in \bar{\mathcal{P}}_n - \bar{\mathcal{P}}'_n} \rho_\epsilon\right]$$

Proof. If $\bar{K}$ is a coloured triangulation of a closed PL n-manifold M^n with $g \geq 1$ connected components, then summing up relation (2) for every connected component and for every permutation $\epsilon \in \bar{\mathcal{P}}_n$ easily yields:

$$(n-1)! \cdot \sum_{i,j \in \Delta_n} g_{i,j} + n! \cdot (1-n)\frac{p}{4} = n! \cdot g - 2\sum_{\epsilon \in \bar{\mathcal{P}}_n} \rho_\epsilon \tag{3}$$

Thus, statement (i) directly follows, when $g = 1$ is assumed.

On the other hand, if $\bar{K}$ is a coloured triangulation of a compact (connected) PL n-manifold M^n with $h \geq 1$ boundary components, then summing up relation (2) for every permutation $\epsilon \in \bar{\mathcal{P}}_n$ yields:

$$(n-1)! \cdot \sum_{i,j \in \Delta_n} \mathring{g}_{\,i,j} + \frac{n!}{2} \cdot (1-n)\frac{\mathring{p}}{2} + \frac{n!}{2} \cdot (2-n)\frac{\bar{p}}{2} + (n-2)! \cdot \sum_{i,j \in \Delta_{n-1}} {}^{\partial}g_{i,j}$$
$$= 2 \cdot \frac{n!}{2} - 2\sum_{\epsilon \in \bar{\mathcal{P}}_n} \rho_\epsilon$$

In order to evaluate $\sum_{i,j\in\Delta_n} g_{i,j}$, the term $\pm(n-1)!\cdot\sum_{i,j\in\Delta_n}\bar{g}_{i,j}$ has to be added:

$$
\begin{aligned}
&[(n-1)!\cdot\sum_{i,j\in\Delta_n}\mathring{g}_{i,j}+(n-1)!\cdot\sum_{i,j\in\Delta_n}\bar{g}_{i,j}]-(n-1)!\cdot\sum_{i,j\in\Delta_n}\bar{g}_{i,j}\\
&\qquad\qquad+\frac{n!}{2}\cdot(1-n)\frac{\mathring{p}}{2}+\frac{n!}{2}\cdot(2-n)\frac{\bar{p}}{2}+(n-2)!\cdot\sum_{i,j\in\Delta_{n-1}}{}^{\partial}g_{i,j}\\
&=n!-2\sum_{\epsilon\in\bar{\mathcal{P}}_n}\rho_\epsilon
\end{aligned}
$$

Then, since $\bar{g}_{i,j}=0$ for every $i,j\in\Delta_{n-1}$ and $\bar{g}_{i,n}=\bar{p}/2$ for every $i\in\Delta_{n-1}$, the following identity is easily obtained:

$$
\begin{aligned}
&(n-1)!\cdot\sum_{i,j\in\Delta_n}g_{i,j}+n!\cdot(1-n)\frac{\mathring{p}}{4}-n!\cdot n\frac{\bar{p}}{4}+(n-2)!\cdot\sum_{i,j\in\Delta_{n-1}}{}^{\partial}g_{i,j}\\
&=n!-2\sum_{\epsilon\in\bar{\mathcal{P}}_n}\rho_\epsilon
\end{aligned}
$$

Now, if formula (3) is applied to the boundary triangulation $\partial\bar{K}$ (with $h\geq 1$ connected components), then the previous relation becomes:

$$
\begin{aligned}
&(n-1)!\cdot\sum_{i,j\in\Delta_n}g_{i,j}+(n-1)!\cdot h+n!\cdot(1-n)\frac{\mathring{p}}{4}-(n-1)!\cdot(n^2-n+2)\frac{\bar{p}}{4}\\
&-2\sum_{\epsilon\in\bar{\mathcal{P}}'_n}{}^{\partial}\rho_\epsilon=n!-2\sum_{\epsilon\in\bar{\mathcal{P}}_n}\rho_\epsilon
\end{aligned}
$$

Hence, statement (ii) results to be proved, by a direct computation. □

3. Proofs of the main results

Instead of directly proving Theorem 3 and Theorem 4, we subdivide the proofs into steps, by making use of some preliminary Lemmas.

Lemma 2. *Let $\bar{K}$ be a coloured triangulation of a compact (connected) PL n-manifold M^n, with $n\geq 3$. Then:*

$$\mu(\bar{K})<6$$

Proof. The Euler characteristic computation of the disjoint link of each $(n-3)$-simplex of $\bar{K}$ easily yields the following formula, which is nothing but a particular case (corresponding to $m=3$) of [2; Corollary 2]:

$$2\cdot\sum_{i,j,k\in\Delta_n}\mathring{g}_{\,i,j,k}+\sum_{i,j,k\in\Delta_n}\bar{g}_{i,j,k}=(n-1)\cdot\sum_{i,j\in\Delta_n}g_{i,j}-n(n-1)(n+1)\frac{\mathring{p}}{12}$$
$$-n(n-1)\frac{\bar{p}}{2}-n(n-1)(n-2)\frac{\bar{p}}{12}$$

Thus, formula (1) may be restated as

$$\mu(\bar{K})$$
$$=n\cdot\frac{(n+1)\,\mathring{p}\,+(n+2)\bar{p}}{n(n+1)\frac{\mathring{p}}{6}+n(n+4)\frac{\bar{p}}{6}+\frac{2}{n-1}\cdot(2\cdot\sum_{i,j,k\in\Delta_n}\mathring{g}_{\,i,j,k}+\sum_{i,j,k\in\Delta_n}\bar{g}_{i,j,k})}$$
$$=n\cdot\frac{(n+1)\,\mathring{p}\,+(n+2)\bar{p}}{n(n+1)\frac{\mathring{p}}{6}+n(n+2)\frac{\bar{p}}{6}+\left[n\frac{\bar{p}}{3}+\frac{2}{n-1}\cdot(2\cdot\sum_{i,j,k\in\Delta_n}\mathring{g}_{\,i,j,k}+\sum_{i,j,k\in\Delta_n}\bar{g}_{i,j,k})\right]}$$

Since $\bar{p}\geq 0$, $\bar{g}_{i,j,k}\geq 0$ and $\mathring{g}_{\,i,j,k}>0$ (for every $i,j,k\in\Delta_n$) obviously hold, the statement follows. □

REMARK. As already pointed out in [17] for the closed simplicial case, the claim of Lemma 2 does not hold in dimension 2, where the average vertex order results to be strictly related with the geometrical structure of the surface.

In fact, an easy Euler characteristic computation shows that, if $\bar{K}$ is a coloured triangulation of an orientable (resp. non orientable) surface F with genus $g(F)\geq 0$ (resp. $\tilde{g}(F)\geq 1$) and $h\geq 0$ boundary components, then:

$$\mu(\bar{K})=\frac{3\,\mathring{p}\,+4\bar{p}}{\sum_{i,j\in\Delta_2}g_{i,j}}=\frac{3\,\mathring{p}\,+4\bar{p}}{\chi(F)+\frac{\mathring{p}}{2}+\bar{p}}$$

Further, if every boundary component is assumed to be triangulated by exactly two 1-simplices (i.e, if $\bar{p}=2h$), the previous relation becomes:

$$\mu(\bar{K})=6-\frac{6\chi(F)+4h}{\sum_{i,j\in\Delta_2}g_{i,j}}$$

Thus, in this hypothesis (which is always trivially true in the closed case), $\mu(\bar{K})\leq 6$ holds if and only if $\chi(F)\geq -(2h)/3$, i.e. $g(F)\leq 1-(h/6)$ (resp. $\tilde{g}(F)\leq 2-(h/3)$).[4]

[4]It may be interesting to note the relationship between the present conditions and the cases of surfaces admitting *δ-regular* triangulations, with $\delta\leq 6$: see [18].

For example, if $\bar{K}_1$ (resp. $\bar{K}_2$) is the bidimensional coloured triangulation depicted in Figure 1 (resp. Figure 2), it is immediate to check that $\mu(\bar{K}_1) = 20/3 > 6$ (resp. $\mu(\bar{K}_2) = 14/3 < 6$).

Lemma 3. *Let $\bar{K}$ be any coloured triangulation of a closed PL n-manifold M^n $(n \geq 3)$. Then:*

(i) *$\mu(\bar{K}) \geq 2$, equality holds if and only if $\bar{K}$ is the standard (two n-simplices) coloured triangulation of $\mathbb{S}^n$.*

(ii) *If $M^n \neq \mathbb{S}^n$, then $\mu(\bar{K}) \geq (2(n+1))/(n-1)$.*

(iii) *$\mu(\bar{K}) = (2(n+1))/(n-1)$ implies $\rho_\epsilon(\bar{K}) = 1$ for every $\epsilon \in \bar{\mathcal{P}}_n$.*

(iv) *For every $n \geq 3$, there exists a coloured triangulation $H^{(n)}$ (resp. $\tilde{H}^{(n)}$) of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ (resp. of $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{n-1}$) with $\mu(H^{(n)}) = (2(n+1))/(n-1)$ (resp. $\mu(\tilde{H}^{(n)}) = (2(n+1))/(n-1)$).*

Proof. In the closed case, formula (1) gives $\mu(\bar{K}) = n \cdot ((n+1)p)/(2 \cdot \sum_{i,j \in \Delta_n} g_{i,j})$. Since it is obvious that $g_{i,j} \leq p/2$ for every $i, j \in \Delta_n$, the proof of statement (i) directly follows:

$$\mu(\bar{K}) \geq \frac{n(n+1) \cdot p}{2\binom{n+1}{2} \cdot \frac{p}{2}} = 2$$

and $\mu(\bar{K}) = 2$ if and only if $g_{i,j} = p/2$ for every $i, j \in \Delta_n$ (i.e. if and only if $\bar{K}$ consists of two n-simplices with identified boundary).

On the other hand, if Lemma 1(i) is applied to formula (1), the following relation is obtained:

$$\mu(\bar{K}) = \frac{n(n+1) \cdot p}{2\left[n + n(n-1)\frac{p}{4} - \frac{2}{(n-1)!} \cdot \sum_{\epsilon \in \bar{\mathcal{P}}_n} \rho_\epsilon\right]}$$

Since the existence of a null ϵ-genus is known to imply M^n being the n-sphere $\mathbb{S}^n$ (see [7]), if $M^n \neq \mathbb{S}^n$ we may assume $\rho_\epsilon \geq 1$ for every $\epsilon \in \bar{\mathcal{P}}_n$; so

$$\mu(\bar{K}) \geq \frac{n(n+1) \cdot p}{2\left[n + n(n-1)\frac{p}{4} - \frac{2}{(n-1)!} \cdot \frac{n!}{2}\right]}$$

and statements (ii) and (iii) result to be proved.

Finally, statement (iv) may be directly checked from formula (1), in case $H^{(n)}$ (resp. $\tilde{H}^{(n)}$) being the "standard" coloured triangulation of $\mathbb{S}^1 \times \mathbb{S}^{n-1}$ (resp. of $\mathbb{S}^1 \tilde{\times} \mathbb{S}^{n-1}$) constructed in [7; Corollary 1] (resp. [13; Corollary 4]): in fact, both $H^{(n)}$ and $\tilde{H}^{(n)}$ have $p = 2(n+1)$ and $g_{i,j} = n - 1$ for every $i, j \in \Delta_n$. $\square$

Lemma 4. *Let $\bar{K}$ be any coloured triangulation of a compact PL n-manifold M^n $(n \geq 3)$, with (possibly disconnected) non empty boundary. Then:*

(i) $\mu(\bar{K}) \geq 2(n+2)/(n+1)$, *equality holds if and only if $\bar{K}$ is the standard (two n-simplices) coloured triangulation of $\mathbb{D}^n$.*

(ii) • *For $n = 3$, if either $M^3 \neq \#_h\mathbb{D}^3$ or $M^3 = \#_h\mathbb{D}^3$ with $h > 2$, then $\mu(\bar{K}) > 15/4$;*

• *For every $n > 3$, if either $M^n \neq \#_h\mathbb{D}^n$ or $M^n = \#_h\mathbb{D}^3$ with $h > (2(n+1))/(n-1)$, then $\mu(\bar{K}) > (2(n+1))/(n-1)$;*

(iii) • *There exists a coloured triangulation $T_2^{(n)}$ of $\#_2\mathbb{D}^n = \mathbb{D}^1 \times \mathbb{S}^{n-1}$ with*

$$\mu(T_2^{(n)}) < \begin{cases} \dfrac{15}{4} & if\ n = 3 \\ \dfrac{2(n+1)}{n-1} & for\ every\ n \geq 4 \end{cases}$$

• *For $n \in \{4,5\}$, there exists a coloured triangulation $T_3^{(n)}$ of $\#_3\mathbb{D}^n$ with*

$$\mu(T_3^{(n)}) \leq \frac{2(n+1)}{n-1}.$$

Proof. In the bounded case, formula (1) and Lemma 1(ii) yield the following relation:

(4)

$$\mu(\bar{K}) = \frac{n}{2} \cdot \frac{(n+1)\,\mathring{p} + (n+2)\bar{p}}{(n-h) + n(n-1)\frac{\mathring{p}}{4} + (n^2-n+2)\frac{\bar{p}}{4} - \frac{2}{(n-1)!}\left[\sum_{\epsilon\in\bar{\mathcal{P}}'_n}(\rho_\epsilon - {}^\partial\rho_\epsilon) + \sum_{\epsilon\in\bar{\mathcal{P}}_n - \bar{\mathcal{P}}'_n}\rho_\epsilon\right]}$$

Since the inequalities $\rho_\epsilon \geq 0$ (resp. $\rho_\epsilon \geq^\partial \rho_\epsilon$) are known to hold for every $\epsilon \in \bar{\mathcal{P}}_n$ (resp. $\epsilon \in \bar{\mathcal{P}}'_n$) and since $n - h \leq (n-1)(\bar{p}/2)$ (use $\bar{p} \geq 2h \geq 2$), we have:

$$\begin{aligned} \mu(\bar{K}) &\geq \frac{n}{2} \cdot \frac{(n+1)\,\mathring{p} + (n+2)\bar{p}}{n(n-1)\frac{\mathring{p}}{4} + n(n+1)\frac{\bar{p}}{4}} = 2 \cdot \left(1 + \frac{p + \mathring{p}}{(n+1)p - 2\,\mathring{p}}\right) \\ &\geq 2 \cdot \left(1 + \frac{p}{(n+1)p}\right) = 2 \cdot \frac{n+2}{n+1} \end{aligned}$$

Moreover, equality holds if and only if $\mathring{p} = 0$ and $\bar{p} = 2h = 2$, i.e. if and only if $\bar{K}$ consists of two n-simplices with n common $(n-1)$-faces.

In order to prove statement (ii), we apply to formula (4) only the inequalities $\rho_\epsilon \geq 0$ and $\rho_\epsilon \geq^{\partial} \rho_\epsilon$, so that the following relation is obtained:

$$\begin{aligned}\mu(\bar{K}) &\geq \frac{n}{2} \cdot \frac{(n+1)\overset{\circ}{p} + (n+2)\bar{p}}{(n-h) + n(n-1)\frac{\overset{\circ}{p}}{4} + (n^2-n+2)\frac{\bar{p}}{4}} \\ &= \frac{2(n+1)}{n-1} + \frac{2\bar{p}\cdot(n^2-3n-2) - 8(n-h)(n+1)}{(n-1)[n(n-1)\overset{\circ}{p} + (n^2-n+2)\bar{p} + 4(n-h)]}\end{aligned}$$

In case $n=3$, the previous relation yields:

$$\mu(\bar{K}) \geq 4 + \frac{-\bar{p} - 24 + 8h}{3\overset{\circ}{p} + 4\bar{p} + 6 - 2h} = \frac{15}{4} + \frac{\frac{3}{4}\overset{\circ}{p} + \frac{15}{2}(h-3)}{3\overset{\circ}{p} + 4\bar{p} + 6 - 2h}$$

Thus, if $h \geq 3$, we have

$$\mu(\bar{K}) \geq \frac{15}{4} + \frac{\frac{3}{4}\overset{\circ}{p}}{3\overset{\circ}{p} + 4\bar{p}}$$

This directly yields $\mu(\bar{K}) > 15/4$, since $\overset{\circ}{p} = 0$ would imply $\bar{K}$ to be a cone over its boundary[5] (i.e. a coloured triangulation of the 3-disk), against the assumption $h \geq 3$.

For every $n \geq 4$, since $\bar{p} \geq 2h$ and $n^2 - 3n - 2 > 0$ trivially hold, we have:

$$\mu(\bar{K}) \geq \frac{2(n+1)}{n-1} + \frac{4h - 8\frac{n+1}{n-1}}{4 + (n-1)p + \frac{2}{n}(\bar{p} - 2h)} \tag{5}$$

Thus, formula (5) obviously implies that, if $h > (2(n+1))/(n-1)$ (i.e., $h > 3$ with $n \in \{4,5\}$, and $h > 2$ with every $n \geq 6$), then $\mu(\bar{K}) > (2(n+1))/(n-1)$.

On the other hand, if $M^n \neq \#_h\mathbb{D}^n$ is assumed, the main result of [8] ensures that $\rho_\epsilon \geq 1$ for every $\epsilon \in \bar{\mathcal{P}}_n$; hence, formula (4) yields:

$$\begin{aligned}\mu(\bar{K}) &\geq \frac{n}{2} \cdot \frac{(n+1)\overset{\circ}{p} + (n+2)\bar{p}}{(n-h) - (n-1) + n(n-1)\frac{\overset{\circ}{p}}{4} + (n^2-n+2)\frac{\bar{p}}{4}} \\ &\geq \frac{n}{2} \cdot \frac{(n+1)\overset{\circ}{p} + (n+2)\bar{p}}{n(n-1)\frac{\overset{\circ}{p}}{4} + (n^2-n+2)\frac{\bar{p}}{4}} \\ &= 2 \cdot \left[\frac{n+1}{n-1} + \frac{(n^2-3n-2)\bar{p}}{(n-1)[n(n-1)\overset{\circ}{p} + (n^2-n+2)\bar{p}]}\right]\end{aligned}$$

[5]Note that, if $\bar{K}$ has no internal Δ_{n-1}-labelled $(n-1)$-simplex, then $\bar{K}$ contains exactly one n-labelled vertex v_n ; moreover, $\bar{K}$ results to coincide with the "star" of v_n in $\bar{K}$.

In case $n = 3$, the previous inequality becomes

$$\mu(\bar{K}) \geq 2 \cdot \left[2 - \frac{\bar{p}}{3\overset{\circ}{p} + 4\bar{p}}\right] = 4 - \frac{1}{4} + \frac{\frac{3}{4}\overset{\circ}{p}}{3\overset{\circ}{p} + 4\bar{p}} > \frac{15}{4};$$

finally, if $n \geq 4$, the proof of statement (ii) is completed by noting that $n^2 - 3n - 2 > 0$ trivially holds.

As far as statement (iii) is concerned, it is important to check that $\bar{M}^n = \#_h \mathbb{D}^n$ (for every $n \geq 3$ and for every $h \geq 1$) admits a coloured triangulation $T_h^{(n)}$ with $\bar{p} = 2h$, $\overset{\circ}{p} = 2(h-1)(n-1)$ and $\rho_\epsilon(T_h^{(n)}) = 0$ for every $\epsilon \in \bar{\mathcal{P}}_n$. The construction of $T_h^{(n)}$ is performed by induction on h in the following way:

- Let $T_1^{(n)} = \theta_{-1}$ be the standard coloured triangulation of $\mathbb{D}^n$ consisting of two (vertex labelled) n-simplices with all $(n-1)$-faces identified, but the Δ_{n-1}-labelled ones;
- $T_{r+1}^{(n)}$ is obtained from $T_r^{(n)}$ by adding, for every $n(r-1) \leq i \leq nr - 1$, an n-dipole θ_i involving the colour set $\Delta_n - \{i \text{ mod.} n\}$[6] within the unique $(\Delta_n - \{i \text{ mod.} n\})$-labelled $(n-1)$-face of θ_{i-1}, and then by deleting the identifications of Δ_{n-1}-labelled faces contained in θ_{nr-1}.

Now, the proof of formula (5) directly yields

$$\mu(T_h^{(n)}) = \frac{2(n+1)}{n-1} + \frac{4h - 8\dfrac{n+1}{n-1}}{4 + (n-1)p + \dfrac{2}{n}(\bar{p} - 2h)},$$

that is

$$\mu(T_h^{(n)}) \begin{cases} < \dfrac{15}{4} & \text{if } n = 3 \text{ and } h \leq 2 \\ < \dfrac{2(n+1)}{n-1} & \text{if } h < \dfrac{2(n+1)}{n-1} \\ = \dfrac{2(n+1)}{n-1} & \text{if } h = \dfrac{2(n+1)}{n-1} \text{ (in particular, if } n = 5 \text{ and } h = 3) \end{cases} \quad \square$$

Lemma 5. *Let M^n be a compact PL n-manifold ($n \geq 3$), with $h \geq 0$ boundary components.*

(i) *For every $\epsilon > 0$ there exists a coloured triangulation $\bar{K}_\epsilon$ of M^n such that $\mu(\bar{K}_\epsilon) < (2(n+1))/(n-1) + \epsilon$.*

(ii) *In particular, if $n = 3$ and $h \geq 1$, then for every $\epsilon > 0$ there exists also a coloured triangulation $\bar{\bar{K}}_\epsilon$ of M^3 such that $\mu(\bar{\bar{K}}_\epsilon) < 15/4 + \epsilon$.*

[6] An n-dipole involving the colour set $\Delta_n - \{j\}$ is a subcomplex which triangulates $\mathbb{D}^n$ and consists of two (vertex labelled) n-simplices with all $(n-1)$-faces identified, but the $(\Delta_n - \{j\})$-labelled ones.

Proof. Let $\bar{K}$ be any fixed coloured triangulation of M^n, with average $(n-2)$-simplex order $\mu(\bar{K}) = (n \cdot \alpha_{n-1}(\bar{K}))/(\alpha_{n-2}(\bar{K}))$.

Further, let $\bar{K}'$ to be obtained from $\bar{K}$ by adding an *internal 1-dipole*, i.e. a subcomplex which triangulates $\mathbb{D}^n$ and consists of two internal n-simplices with exactly one common $(n-1)$-face (see [9] for related notions and results); as it is easy to check, $\bar{K}'$ is a new coloured triangulation of M^n with $\mu(\bar{K}') = (n \cdot [\alpha_{n-1}(\bar{K}) + (n+1)]])/(\alpha_{n-2}(\bar{K}) + (\binom{n}{2}))$. Hence, the proof of statement (i) follows from the fact that, if $\bar{K}^{(r)}$ is obtained from $\bar{K}$ by successive addition of r internal 1-dipoles, then:

$$\mu(\bar{K}^{(r)}) = \frac{n \cdot [\alpha_{n-1}(\bar{K}) + r(n+1)]}{\alpha_{n-2}(\bar{K}) + r\dbinom{n}{2}} \xrightarrow[r \to +\infty]{} 2 \cdot \frac{n+1}{n-1}.$$

On the other hand, if M^3 is a bounded 3-manifold, we may also add to $\bar{K}$ a *boundary 1-dipole*, i.e. a subcomplex which triangulates $\mathbb{D}^3$ and consists of two boundary 3-simplices with exactly one common 2-face; thus, the resulting new coloured triangulation $\bar{\bar{K}}'$ has $\mu(\bar{\bar{K}}') = (3 \cdot [\alpha_2(\bar{K}) + 5])/(\alpha_1(\bar{K}) + 4)$. Now, statement (ii) follows from the fact that, if $\bar{\bar{K}}^{(r)}$ is obtained from $\bar{K}$ by successive addition of r boundary 1-dipoles, then:

$$\mu(\bar{\bar{K}}^{(r)}) = \frac{3 \cdot [\alpha_2(\bar{K}) + 5r]}{\alpha_1(\bar{K}) + 4r} \xrightarrow[r \to +\infty]{} > \frac{15}{4}. \qquad \square$$

Proof of Theorem 3. Obviously, the proof of statement (a) (resp. (b)) (resp. (d)) is a direct consequence of Lemma 2 and Lemma 3 (i) (resp. of Lemma 3 (ii)) (resp. of Lemma 5 (i)).

On the other hand, in order to prove statement (c), it is sufficient to make use of Lemma 3 (iii) and (iv), and of the existing classifications of PL n-manifolds ($n \leq 5$) with "low" regular genus: see [6] and [4].[7] $\square$

Proof of Theorem 4. Obviously, the proof of statement (a) (resp. (b)) (resp. (c)) is a direct consequence of Lemma 2 and Lemma 4 (i) (resp. of Lemma 4 (ii) and (iii)) (resp. of Lemma 5 (i) and (ii)). $\square$

[7] As far as dimension $n = 3$ is concerned, it is useful to remember that regular genus coincides with the classical Heegaard genus for 3-manifolds ; moreover, a detailed analysis of closed 3-manifold coloured triangulations $\bar{K}$ with average edge order $\mu(\bar{K}) = 4$ was already performed in [3].

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Dipartimento di Matematica
Pura ed Applicata
Università di Modena
Via Campi 213B
I-41100 MODENA (Italy)
e-mail: CASALI@UNIMO.IT