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THE VORTEX FILAMENT EQUATION AND A SEMILINEAR SCHRÖDINGER EQUATION IN A HERMITIAN SYMMETRIC SPACE

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0. Introduction

We consider the initial value problem of the vortex filament equation on $\boldsymbol{R}^3$:

$$\gamma_t = \gamma_x \times \gamma_{xx} \quad (|\gamma_x| \equiv 1) \qquad (\times \text{ is the exterior product}).$$

In this paper, we will prove the existence and the uniqueness of a classical solution for the initial value problem, and generalize it to the case of curves in 3-dimensional space forms. We also consider related semilinear Schrödinger equations for curves in Kähler manifolds. It is remarkable that we need symmetric spaces as manifolds for infinite time existence of solutions.

More precisely, we will get the following results.

Theorem 1.5. *The initial value problem $\gamma_t = \gamma_x \times \gamma_{xx}$ ($|\gamma_x| \equiv 1$) for closed curves in the euclidean space $\boldsymbol{R}^3$ has a unique solution on $-\infty < t < \infty$.*

Theorem 2.2. *Let M be an oriented 3-dimensional riemannian manifold with constant curvature c. The initial value problem $\gamma_t = \gamma_x \times \nabla_x \gamma_x$ ($|\gamma_x| \equiv 1$) for closed curves in M has a unique solution on $-\infty < t < \infty$ for any initial data.*

Theorem 3.5. *Let M be a Kähler manifold. The initial value problem $\xi_t = J\nabla_x \xi_x$ for closed curves in M has a unique short time solution for any initial data.*

Theorem 4.2. *Let M be a complete locally hermitian symmetric space. The initial value problem $\xi_t = J\nabla_x \xi_x$ for closed curves has a unique all time solution ($-\infty < t < \infty$) for any initial value.*

We also get

Theorem 1.3. *Hasimoto's transformation is well defined, even when the curvature vanishes at some point.*

We use the following notations: On a riemannian manifold, we denote by ∇ the covariant derivation and by R the curvature tensor. The partial derivation is denoted by ∂ or the subscript, e.g., $\partial_x\gamma$, γ_x. We denote by $(*,*)$ the pointwise inner product, by $\langle *,*\rangle$ the L_2 inner product for x-direction, by $\max|*|$ the sup-norm for x-direction, by $\|*\|$ the L_2-norm for x-direction. We use Einstein's summation convention.

We mainly consider closed curves and quasi-periodic curves. When curves are not closed, we should set some appropriate boundedness condition or boundary condition. We only treat C^∞-objects.

We will frequently use standard estimations:

$$\begin{aligned}(0.1)\qquad &\|\nabla_x{}^k\eta\| \le C_1\|\eta\|^{1-(k/n)}\|\nabla_x{}^n\eta\|^{k/n} \le C_2(\|\eta\|+\|\nabla_x{}^n\eta\|),\\ &\max|\eta|^2 \le C_3\|\eta\|(\|\eta\|+\|\nabla_x\eta\|),\end{aligned}$$

where we denote by $\nabla_x{}^k\eta$ the k-th covariant derivative $(\nabla_x)^k\eta$. For a proof of these facts, see e.g., [4]. As in estimation (0.1), we denote by C_i constants depending only on some given data.

After this work was done, the author received a preprint [7] by T. Nishiyama and A. Tani. They prove the existence and uniqueness of a vortex filament equation containing γ_{xxx}, which is more general than our equation. However, their method can be applied only on the case of $\boldsymbol{R}^3$. (Compare with Theorem 2.2).

1. Vortex filament equation in the euclidean space

Let γ be a solution of the equation: $\gamma_t=\gamma_x\times\gamma_{xx}$. Then we see

$$\partial_t|\gamma_x|^2=2(\gamma_x,\gamma_{xt})=2(\gamma_x,\gamma_{xx}\times\gamma_{xx}+\gamma_x\times\gamma_{xxx})=0.$$

Thus, if the initial data $\gamma(0,x)$ satisfies the condition $|\gamma_x(0,x)|=1$, then the solution satisfies $|\gamma_x(t,x)|\equiv 1$. Therefore, if we set $\xi=\gamma_x$, then ξ becomes a family of curves in S^2. We rewrite the equation by means of ξ and get an equation:

$$\xi_t=(\gamma_x\times\gamma_{xx})_x=\xi\times\xi_{xx}.$$

Using the covariant derivation ∇ and the complex structure J on S^2, this equation is expressed as

$$(1.1)\qquad \xi_t=J\nabla_x\xi_x,$$

and locally as

$$z_t=\sqrt{-1}\left(z_{xx}-\frac{2\bar z}{1+|z|^2}z_x^2\right).$$

We transform solutions of equation (1.1) by means of 'development of curve', and will get a non-linear Schrödinger equation.

DEFINITION 1.1. Let c be a curve in a riemannian manifold M and $F=\{e_i\}$ a parallel orthonormal frame field along c. (A parallel orthonormal frame field is given by parallel translation along c of an orthonormal frame at a point of c.) We call such a pair *a curve c with frame field F.* For a curve with frame field, we represent its velocity vector as

$$c'(x)=u^i(x)e_i(x)$$

using Einstein's summation convention. The integral $\int u^i(x)dx$ is called *the development of c to the euclidean space.* In this paper, we do not use the development itself, but *the differential $u=(u^i)$ of the development.* If c is a closed curve in M (i.e., parametrized on $\boldsymbol{R}/\boldsymbol{Z}$,) there is an orthogonal matrix P such that $P^j{}_i e_j(x+1)=e_i(x)$. Then we have $u^i(x+1)=P^i{}_j u^j(x)$. We say that u is *quasi-periodic with correction P.*

Let ξ be a solution of (1.1). We attach to it a frame field $\{e_i\}$, and seek conditions for the differential u of its development. We fix the orientation of the frame by $Je_1=e_2$. Since $\nabla_x e_i\equiv 0$,

$$\begin{aligned}\partial_x(e_2,\nabla_t e_1)&=(e_2,\nabla_x\nabla_t e_1)=(e_2,R(\xi_x,\xi_t)e_1+\nabla_t\nabla_x e_1)\\&=(\xi_t,e_1)(\xi_x,e_2)-(\xi_x,e_1)(\xi_t,e_2)\\&=-(\nabla_x\xi_x,Je_1)u^2+(\nabla_x\xi_x,Je_2)u^1\\&=-u_x^2u^2-u_x^1u^1=-\frac{1}{2}\partial_x|u|^2.\end{aligned}$$

Thus, we can choose the frame field $\{e_i\}$, so that $\nabla_t e_1=-(1/2)|u|^2e_2$, and hence $\nabla_t e_i=-(1/2)|\xi_x|^2Je_i$. Then,

$$\begin{aligned}u_t^ie_i&=\nabla_t(u^ie_i)-u^i\nabla_t e_i=\nabla_t\xi_x+\frac{1}{2}|u|^2u^iJe_i\\&=\nabla_x\xi_t+\frac{1}{2}|u|^2u^iJe_i=J\nabla_x{}^2\xi_x+\frac{1}{2}|u|^2u^iJe_i\\&=J(u_{xx}^ie_i+\frac{1}{2}|u|^2u^ie_i).\end{aligned}\tag{1.2}$$

If the curves ξ are closed, the quasi-periodicity condition of u becomes as follows. Let $P=P(t)$ be the correction of period of u. Then,

$$(\frac{d}{dt}P^j{}_i)e_j(x+1)=\nabla_t(P^j{}_ie_j(x+1))-P^j{}_i\nabla_te_j(x+1)$$

$$=\nabla_te_i(x)+\frac{1}{2}|\xi_x|^2P^j{}_iJe_j(x+1)=0.$$

Therefore P is constant.

We can reverse this procedure. That is, the solutions of equation (1.1) have one-to-one correspondence to the solutions of equation (1.2) in the following sense.

Proposition 1.2. *Let $\xi^o(x)$ be a curve in S^2 with frame field $F^o=\{e_i^o(x)\}$, and $u^o(x)$ the differential of its development.*

1) *Let $\xi(t,x)$ be a solution of initial value problem* (1.1) *with initial data ξ^o. We extend $e_i^o(x)$ to $e_i(t,x)$ along by the ODE:*

$$\nabla_te_i=-\frac{1}{2}|\xi_x|^2Je_i.$$

Then, for each t_0, $F=\{e_i(t_0,x)\}$ is a frame field along $\xi(t_0,x)$. And, the family $u(t,x)$ of the differential of the development of $\xi(t,x)$ is a solution of initial value problem:

$$u_t=J(u_{xx}+\frac{1}{2}|u|^2u). \tag{1.3}$$

Moreover, if ξ is a family of closed curves, then the correction of period of u is constant.

2) *Conversely, let $u(t,x)$ be a solution of* (1.3) *with initial data u^o. We extend $\{\xi^o,e_i^o\}$ to $\{\xi(t,x),e_i(t,x)\}$ by the system of ODEs:*

$$\xi_t=Ju_x^ie_i,\qquad \nabla_te_i=-\frac{1}{2}u^2Je_i.$$

Then $\xi(t,x)$ is a solution of initial value problem (1.1). *Moreover, if ξ^o is closed and if u is quasi-periodic with constant correction, then ξ yields a family of closed curves.*

If we regard the $\boldsymbol{R}^2$-valued function u as a complex valued function $u^1+\sqrt{-1}u^2$, then u satisfies a so-called non-linear Schrödinger equation:

$$u_t=\sqrt{-1}(u_{xx}+\frac{1}{2}|u|^2u). \tag{1.4}$$

This transformation of solutions coincides with a transformation found by Hasimoto ([3]). Hasimoto's transformation is defined by

$$u=\kappa\exp(\sqrt{-1}\int\tau\,dx)$$

where κ and τ are the curvature and the torsion of γ. However, we should note that this expression itself is not defined when the curvature κ vanishes at some point. We can restate Proposition 1.2 as follows.

Theorem 1.3. *Hasimoto's transformation is well defined, even when the curvature vanishes at some point.*

Since equation (1.4) is well understood ([1]), we have

Theorem 1.4. *The initial value problem of the semilinear Schrödinger equation (1.1) $\xi_t = J\nabla_x \xi_x$ for closed curves in S^2 has a unique solution on $-\infty < t < \infty$ for any initial data.*

Theorem 1.5. *The initial value problem $\gamma_t = \gamma_x \times \gamma_{xx}$ $(|\gamma_x| \equiv 1)$ for closed curves in the euclidean space $\boldsymbol{R}^3$ has a unique solution on $-\infty < t < \infty$.*

Proof. To come back to $\boldsymbol{R}^3$ from S^2, we need to check the closedness condition $\oint \xi \, dx = 0$ on $\boldsymbol{R}^3$.

$$\begin{aligned}\frac{d}{dx}\int_{S^1} \xi \, dx &= \int_{S^1} \xi_t \, dx = \int_{S^1} J\nabla_x \xi_x \, dx \\ &= \int_{S^1} \xi \times \xi_{xx} \, dx = -\int_{S^1} \xi_x \times \xi_x \, dx = 0.\end{aligned}$$

The uniqueness follows from the ODE: $\gamma_t = \xi \times \xi_x$ with respect to t. Q.E.D.

REMARK 1.6. Let $v = v(x)$ be a solutioo of the ODE:

$$v'' + \frac{1}{2}|v|^2 v = av, \tag{1.5}$$

where a is a real constant. Then the function $u(t,x) = \exp(a\sqrt{-1}t)v(x)$ is a solution of equation (1.4). If we transform this solution to a solution $\xi(t,x)$ in S^2, we have curves moving by isometries. Moreover, the corresponding curves in $\boldsymbol{R}^3$ are elastic curves ([2]).

2. Vortex filament equation in 3-dimensional space forms

In this section, we generalize results in Section 1 to oriented 3-dimensional riemannian manifolds (M,g) with constant curvature c. We consider initial value problem:

$$\gamma_t = \gamma_x \times \nabla_x \gamma_x \qquad (|\gamma_x| \equiv 1). \tag{2.1}$$

Since

$$\begin{aligned}\partial_t |\gamma_x|^2 &= 2(\gamma_x, \nabla_t \gamma_x) = 2(\gamma_x, \nabla_x \gamma_t) \\ &= 2(\gamma_x, \nabla_x(\gamma_x \times \nabla_x \gamma_x)) = 2(\gamma_x, \gamma_x \times \nabla_x{}^2 \gamma_x) = 0,\end{aligned}$$

if the additional condition $|\gamma_x| \equiv 1$ is satisfied at $t=0$, then it is satisfied for all t. Therefore, γ_x becomes a unit vector field.

For a solution γ of equation (2.1), we attach to it a frame field, and seek conditions for the differential v of its development. We fix the orientation of the frame $\{e_i\}$ by $e_1 \times e_2 = e_3$. We define w by

$$\nabla_t e_i = w^j{}_i e_j .$$

Then,

$$\begin{aligned}w^j{}_{ix} &= \partial_x(\nabla_t e_i, e_j) = (\nabla_x \nabla_t e_i, e_j) = (R(\gamma_x, \gamma_t) e_i, e_j) \\ &= c(\gamma_t, e_i)(\gamma_x, e_j) - c(\gamma_x, e_i)(\gamma_t, e_j) \\ &= c(\gamma_x \times \nabla_x \gamma_x, e_i) v^j - c(\gamma_x \times \nabla_x \gamma_x, e_j) v^i.\end{aligned}$$

Therefore,

$$\begin{aligned}w^2{}_{1x} &= c(v^2 v^3_x - v^3 v^2_x) v^2 - c(v^3 v^1_x - v^1 v^3_x) v^1 \\ &= c\{((v^1)^2 + (v^2)^2) v^3_x - (v^1 v^1_x + v^2 v^2_x) v^3\} \\ &= c\{(1-(v^3)^2) v^3_x - \frac{1}{2}\partial_x(-(v^3)^2) v^3\} = c v^3_x .\end{aligned}$$

Thus we can choose $\{e_i\}$ so that $\nabla_t e_1 = c v^3 e_2 - c v^2 e_3 = c v^i e_i \times e_1$. That is,

$$\nabla_t e_i = c \gamma_x \times e_i .$$

Then, from

$$\begin{aligned}\nabla_t \gamma_x &= \nabla_t(v^i e_i) = v^i_t e_i + v^i \nabla_t e_i = v^i_t e_i + v^i \gamma_x \times e_i = v^i_t e_i , \\ \nabla_x \gamma_t &= \nabla_x(\gamma_x \times \nabla_x \gamma_x) = \gamma_x \times \nabla^2_x \gamma_x = v^i e_i \times v^j_{xx} e_j ,\end{aligned}$$

we have

$$v_t = v \times v_{xx} . \tag{2.2}$$

Note that this equation has just same expression with the case of euclidean space. However, we have to count the correction of period. Let $P=P(t)$ be the correction of period at time t. Then,

$$\begin{aligned}(\frac{d}{dt}P^j{}_i)e_j(x+1) &= \nabla_t(P^j{}_ie_j(x+1)) - P^j{}_i\nabla_te_j(x+1) \\ &= \nabla_te_i(x) - cP^j{}_i\gamma_x \times e_j(x+1) = 0.\end{aligned}$$

Thus P is constant. Moreover, when we develop v to the plain, we can check that its correction of period is constant. We summarize this transformation as:

Proposition 2.1. *Let M be an oriented 3-dimensional riemannian manifold with constant curvature c. Let γ^o be a curve in M with frame field $F^o=\{e_i^o(x)\}$, and v^o the differential of its development.*

1) *Let $\gamma(t,x)$ be a solution of initial value problem* (2.1) *with initial data γ^o. We extend $e_i^o(x)$ to $e_i(t,x)$ along γ by the ODE: $\nabla_te_i=c\gamma_x\times e_i$. Then for each t_0, $F=\{e_i(t_0,x)\}$ is a frame field along $\gamma(t_0,x)$. And, the family $v(t,x)$ of the development of $\gamma(t,x)$ is a solution of initial value problem* (2.2). *Moreover, if γ is a family of closed curves, then the correction of period of v is constant.*

2) *Conversely, let v be a solution of* (2.2) *with initial data v^o. We extend $\{\gamma^o,e_i^o\}$ to $\{\gamma(t,x),e_i(t,x)\}$ by the system of ODEs: $\gamma_t=v^ie_i\times v_x^je_j$, $\nabla_te_i=cv^je_j\times e_i$. Then $\gamma(t,x)$ is a solution of initial value problem* (2.1). *Moreover, if γ^o is a closed curve and if v is quasi-periodic with constant correction, then γ is a family of closed curves.*

Theorem 2.2. *Let M be an oriented 3-dimensional riemannian manifold with constant curvature c. Initial value problem* (2.1) $\gamma_t=\gamma_x\times\nabla_x\gamma_x$ $(|\gamma_x|\equiv 1)$ *for closed curves in M has a unique solution on $-\infty<t<\infty$ for any initial data.*

Proof. By Proposition 2.1, we can transform the equation (2.1) to equation (1.1) in S^2 via equation (2.2). We solve equation (1.1) counting the correction of period, and transform the solution to a solution of the original equation. Q.E.D.

3. A semilinear Schrödinger equation in a Kähler manifold

The vortex filamenat equation in the euclidean space is reduced to a semilinear Schrödinger equation in S^2. We extend the result to curves in general Kähler manifolds (M,g). We consider a PDE:

$$\xi_t=J\nabla_x\xi_x, \tag{3.1}$$

which has just same expression as in S^2. Here, ∇ is the riemannian connection and J is the complex structure, both defined on M. This equation is locally expressed as

$$\xi_t^\alpha = \sqrt{-1}(\xi_{xx}^\alpha + \Gamma_\beta{}^\alpha{}_\gamma(\xi)\xi_x^\beta \xi_x^\gamma), \tag{3.2}$$

using a complex coordinate system.

To show existence of solutions of (3.1), we perturb it to a parabolic equation. We consider equation

$$\xi_t = (J+\varepsilon)\nabla_x \xi_x, \tag{3.3}$$

where ε is a non-negative number.

Lemma 3.1. *If ξ is a solution of initial value problem* (3.3) *for closed curves, then $\|\xi_x\|$ is non-increasing.*

Proof.

$$\frac{d}{dt}\|\xi_x\|^2 = 2\langle \xi_x, \nabla_t \xi_x\rangle = 2\langle \xi_x, \nabla_x \xi_t\rangle = 2\langle \xi_x, (J+\varepsilon)\nabla_x{}^2 \xi_x\rangle$$

$$= -2\langle \nabla_x \xi_x, (J+\varepsilon)\nabla_x \xi_x\rangle = -2\varepsilon \|\nabla_x \xi_x\|^2 \leq 0.$$

Q.E.D.

Lemma 3.2. *For any closed curve ξ^o in M, there exist positive numbers T and K with the following property: Let ε be a real number in* $[0,1]$ *and ξ a solution of* (3.3) *defined on $0 \leq t < T$ with initial value ξ^o. Then, $\|\nabla_x \xi_x\| \leq K$ on $0 \leq t < T$.*

Proof. By Lemma 3.1, the norm $\|\xi_x\|$ is bounded. We estimate $\|\nabla_x \xi_x\|$.

$$\frac{d}{dt}\|\nabla_x \xi_x\|^2 = 2\langle \nabla_x \xi_x, \nabla_t \nabla_x \xi_x\rangle = 2\langle \nabla_x \xi_x, R(\xi_t, \xi_x)\xi_x + \nabla_x{}^2 \xi_t\rangle$$

$$= -2\varepsilon \|\nabla_x{}^2 \xi_x\| + 2\langle \nabla_x \xi_x, R((J+\varepsilon)\nabla_x \xi_x, \xi_x)\xi_x\rangle$$

$$\leq C_1 \max|\xi_x|^2 \|\nabla_x \xi_x\|^2 \leq C_2 \|\xi_x\|(\|\nabla_x \xi_x\| + \|\xi_x\|)\|\nabla_x \xi_x\|^2$$

$$\leq C_3(1 + \|\nabla_x \xi_x\|^3).$$

Therefore, there exists a positive time T depending only on $\|\xi_x\|$ and $\|\nabla_x \xi_x\|$ at $t=0$ such that $\|\nabla_x \xi_x\|$ is uniformly bounded on $0 \leq t < T$. Q.E.D.

Lemma 3.3. *Let ξ be a solution of initial value problem* (3.3) *for closed curves. If $\|\nabla_x \xi_x\|$ is uniformly bounded on $0 \leq t < T$, then ξ is C^∞-ly uniformly bounded on $0 \leq t < T$. This estimation is independent of ε.*

Proof. We show that $\|\nabla_x{}^n \xi_x\|$ is uniformly bounded on $0 \leq t < T$ by induction. This holds for $n=1$. Suppose that it holds for n. From Lemma 3.1

and the assumption, we know that $\max|\nabla_x{}^{n-1}\xi_x|$ is bounded.

$$\frac{d}{dt}\|\nabla_x{}^{n+1}\xi_x\|^2 = 2\langle \nabla_x{}^{n+1}\xi_x, \nabla_t\nabla_x{}^{n+1}\xi_x\rangle$$

$$= 2\langle \nabla_x{}^{n+1}\xi_x, \sum_{i=0}^{n} \nabla_x{}^i(R(\xi_t,\xi_x)\nabla_x{}^{n-i}\xi_x) + \nabla_x{}^{n+1}\nabla_t\xi_x\rangle$$

$$= -2\varepsilon\|\nabla_x{}^{n+2}\xi_x\| + 2\langle \nabla_x{}^{n+1}\xi_x, \sum \nabla_x{}^i(R(\xi_t,\xi_x)\nabla_x{}^{n-i}\xi_x)\rangle$$

$$\leq C_1\|\nabla_x{}^{n+1}\xi_x\|\{1 + \|\nabla_x{}^n\xi_t\| + \||\xi_t|\nabla_x{}^n\xi_x|\|\}.$$

Here,

$$\||\xi_t|\nabla_x{}^n\xi_x|\| \leq C_2\max|\xi_t| \leq C_3(1 + \|\nabla_x{}^2\xi_x\|) \leq C_4(1 + \|\nabla_x{}^{n+1}\xi_x\|).$$

Therefore,

$$\frac{d}{dt}\|\nabla_x{}^{n+1}\xi_x\|^2 \leq C_5(1 + \|\nabla_x{}^{n+1}\xi_x\|^2).$$

Thus $\|\nabla_x{}^{n+1}\xi_x\|^2$ is estimated only by T, $\|\nabla_x\xi_x\|$ and the initial value. Q.E.D.

Proposition 3.4. *Initial value problem* (3.1) $\xi_t = J\nabla_x\xi_x$ *for closed curves in* M *has a short time solution.*

Proof. For a positive number ε, equation (3.3) becomes parabolic, hence has a C^∞ (ε-depending) short time solution. By Lemma 3.2 and 3.3, the solution is C^∞-ly bounded independently of ε. Therefore, there is a convergent subsequence when $\varepsilon \to 0$, and the limit satisfies equation (3.1). Note that, when we change time variable t to $-t$, the form of equation does not change. It means that we have also a solution for negative time. Q.E.D.

Theorem 3.5. *Let* M *be a Kähler manifold. Initial value problem* (3.1) $\xi_t = J\nabla_x\xi_x$ *for closed curves in* M *has a unique short time solution for any initial data.*

Proof. We have to show the uniqueness. Let ξ^o be the initial data. Taking a small tubular neighbourhood of ξ^o, we have an open set U of $\boldsymbol{R}^n$ and a local diffeomorphism φ from U into M such that the image of φ contains the image of ξ^o.

We rewrite equation (3.1) by

$$J\xi_t + \nabla_x\xi_x = 0, \tag{3.4}$$

and take its linearization by $\eta=\xi_s$:

$$\Phi(\eta):=\nabla_s\{J\xi_t+\nabla_x\xi_x\}=J\nabla_t\eta+\nabla_x{}^2\eta+R(\eta,\xi_x)\xi_x. \tag{3.5}$$

Since the coordinate expression of (3.4) is

$$J^i{}_j\xi^j_t+\xi^i_{xx}+\Gamma_j{}^i{}_k\xi^j_x\xi^k_x=0,$$

the coordinate expression of (3.5) is given by

$$\Phi(\eta)^i=J^i{}_j\eta^j_t+(\partial_kJ^i{}_j)\eta^k\xi^j_t+\eta^i_{xx}+2\Gamma_j{}^i{}_k\eta^j_x\xi^k_x+(\partial_l\Gamma_j{}^i{}_k)\eta^l\xi^j_x\xi^k_x.$$

Let ξ^i and $\bar{\xi}^i$ be coordinate expressions of two solutions of (3.1) with initial data ξ^o. By Lemma 3.2 and 3.3, taking small T, we know that ξ^i and $\bar{\xi}^i$ are bounded. Note that the difference $u^i(t,x):=\bar{\xi}^i(t,x)-\xi^i(t,x)$ can be regarded as a coordinate expression of a vector field u along ξ.

Using the differences $I^i{}_j(t,x):=J^i{}_j(\bar{\xi}(t,x))-J^i{}_j(\xi(t,x))$ and $T_j{}^i{}_k(t,x):=\Gamma_j{}^i{}_k(\bar{\xi}(t,x))-\Gamma_j{}^i{}_k(\xi(t,x))$, we have

$$(J^i{}_j(\xi)+I^i{}_j)(\xi^j_t+u^j_t)+(\xi^i_{xx}+u^i_{xx})+(\Gamma_j{}^i{}_k(\xi)+T_j{}^i{}_k)(\xi^j_x+u^j_x)(\xi^k_x+u^k_x)=0.$$

Therefore,

$$J^i{}_ju^j{}_t+u^i_{xx}+2\Gamma_j{}^i{}_k\xi^j_xu^k_x=O(I^j{}_k,T_k{}^j{}_l,u^j_xu^k_x),$$

where $O(*)$ means a sum of terms with a factor $*$. Thus we have

$$\Phi(u)=O(I^j{}_k,T_k{}^j{}_l,u^j,u^j_xu^k_x).$$

In particular,

$$|\Phi(u)|,|\nabla_x(\Phi(u))|\le C_1(|u|+|\nabla_xu|).$$

Now,

$$\begin{aligned}\frac{d}{dt}\|u\|^2&=2\langle u,\nabla_tu\rangle=2\langle Ju,J\nabla_tu\rangle=2\langle Ju,\Phi(u)-R(u,\xi_x)\xi_x-\nabla_x{}^2u\rangle\\&\le C_2\{\langle|u|,|\Phi(u)|\rangle+\|u\|^2\}+2\langle J\nabla_xu,\nabla_xu\rangle\\&\le C_3(\|u\|^2+\|\nabla_xu\|^2).\end{aligned}$$

Moreover,

$$\begin{aligned}\frac{d}{dt}\|\nabla_xu\|^2&=2\langle\nabla_xu,\nabla_t\nabla_xu\rangle=2\langle\nabla_xu,R(\xi_t,\xi_x)u+\nabla_x\nabla_tu\rangle\\&\le C_4(\|u\|^2+\|\nabla_xu\|^2)+2\langle J\nabla_xu,\nabla_x(J\nabla_tu)\rangle.\end{aligned}$$

Here,

$$\begin{aligned}\langle J\nabla_x u, \nabla_x(J\nabla_t u)\rangle &= \langle J\nabla_x u, \nabla_x\{\Phi(u) - R(u,\xi_x)\xi_x - \nabla_x{}^2 u\}\rangle \\ &\leq C_5\langle|\nabla_x u|, |\nabla_x(\Phi(u))| + |u| + |\nabla_x u|\rangle + \langle J\nabla_x{}^2 u, \nabla_x{}^2 u\rangle \\ &\leq C_6\{\|u\|^2 + \|\nabla_x u\|^2\}.\end{aligned}$$

Thus we have

$$\frac{d}{dt}\{\|u\|^2 + \|\nabla_x u\|^2\} \leq C_7\{\|u\|^2 + \|\nabla_x u\|^2\},$$

from which we can conclude that $u \equiv 0$. Q.E.D.

4. A semilinear Schrödinger equation in a hermitian symmetric space

In a hermitian symmetric space, we can show the all-time existence of a solution of equation (3.1). We can prove it by a way similar to the case of S^2, but we give here a proof which uses results in the previous section. Therefore, we will give another proof for results in Section 1.

Lemma 4.1. *Let M be a locally hermitian symmetric space and ξ a solution of equation* (3.1) *for closed curves. Then the quantity*

$$\|\nabla_x\xi_x\|^2 + \frac{1}{4}\langle R(\xi_x, J\xi_x)\xi_x, J\xi_x\rangle \tag{4.1}$$

is constant in t.

Proof. We have

$$\frac{d}{dt}\|\nabla_x\xi_x\|^2 = 2\langle\nabla_x\xi_x, \nabla_t\nabla_x\xi_x\rangle = 2\langle\nabla_x\xi_x, R(\xi_t, \xi_x)\xi_x\rangle.$$

On the other hand,

$$\begin{aligned}\frac{d}{dt}\langle R(\xi_x, J\xi_x)\xi_x, J\xi_x\rangle &= 2\langle R(\nabla_t\xi_x, J\xi_x)\xi_x, J\xi_x\rangle + 2\langle R(\xi_x, J\nabla_t\xi_x)\xi_x, J\xi_x\rangle \\ &= 4\langle R(\nabla_x\xi_t, J\xi_x)\xi_x, J\xi_x\rangle \\ &= -4\langle R(\xi_t, J\nabla_x\xi_x)\xi_x, J\xi_x\rangle - 4\langle R(\xi_t, J\xi_x)\nabla_x\xi_x, J\xi_x\rangle \\ &\quad -4\langle R(\xi_t, J\xi_x)\xi_x, J\nabla_x\xi_x\rangle\end{aligned}$$

$$=4\langle R(\xi_t, J\xi_x)J\nabla_x\xi_x, \xi_x\rangle - 4\langle R(\xi_t, J\xi_x)\xi_x, J\nabla_x\xi_x\rangle$$
$$= -8\langle R(\xi_t, J\xi_x)\xi_x, J\nabla_x\xi_x\rangle = -8\langle R(J\nabla_x\xi_x, J\xi_x)\xi_x, \xi_t\rangle$$
$$= -8\langle R(\xi_t, \xi_x)\xi_x, \nabla_x\xi_x\rangle.$$

Thus,

$$\frac{d}{dt}\{\|\nabla_x\xi_x\|^2 + \frac{1}{4}\langle R(\xi_x, J\xi_x)\xi_x, J\xi_x\rangle\} = 0.$$

Q.E.D.

Theorem 4.2. *Let M be a complete locally hermitian symmetric space. Initial value problem* (3.1) $\xi_t = J\nabla_x\xi_x$ *for closed curves has a unique all time solution* $(-\infty < t < \infty)$ *for any initial value.*

Proof. Let ξ be a solution on $0 \le t < T$. By Lemma 4.1,

$$\|\nabla_x\xi_x\|^2 \le C_1(1 + \max|\xi_x|^2\|\xi_x\|^2) \le C_2(1 + \|\xi_x\|(\|\xi_x\| + \|\nabla_x\xi_x\|))$$
$$\le C_3(1 + \|\nabla_x\xi_x\|).$$

It means that $\|\nabla_x\xi_x\|$ is time-independently bounded. Therefore, by Lemma 3.3, ξ is uniformly C^∞-ly bounded, hence ξ can be extended beyond T.

Q.E.D.

Now, we compare this with the case of S^2. For this, we generalize the transformation defined in Proposition 1.2.

Let ξ be a solution of (3.1). We attach to it a frame field $\{e_i\}$, and seek conditions for the differential u of its development. In bellow, we use the fact that the curvature tensor R of M is hermitian and parallel.

From $\nabla_x e_i \equiv 0$,

$$\partial_x(e_j, \nabla_t e_i) = (e_j, \nabla_x\nabla_t e_i) = (e_j, R(\xi_x, \xi_t)e_i + \nabla_t\nabla_x e_i)$$
$$= (e_j, R(\xi_x, J\nabla_x\xi_x)e_i) = (e_j, R(e_k, Je_l)e_i)u^k u^l_x$$
$$= \frac{1}{2}\partial_x\{(e_j, R(e_k, Je_l)e_i)u^k u^l\}.$$

Here, we used the fact that $(e_j, R(e_k, Je_l)e_i)$ is symmetric with respect to k, l and is constant with respect to x. Using freedom of $\{e_i\}$ for t-direction, we may put

$$\nabla_t e_i = \frac{1}{2}u^k u^j R(e_k, Je_j)e_i = \frac{1}{2}R(\xi_x, J\xi_x)e_i.$$

Then,

$$u^i_t e_i = \nabla_t(u^i e_i) - u^i \nabla_t e_i = \nabla_t \xi_x - \frac{1}{2} u^i u^k u^j R(e_k, Je_j)e_i$$

$$= \nabla_x \xi_t - \frac{1}{2} u^i u^k u^j R(e_k, Je_j)e_i$$

$$= J\nabla_x{}^2 \xi_x - \frac{1}{2} u^i u^k u^j R(e_k, Je_j)e_i$$

$$= u^i_{xx} Je_i - \frac{1}{2} u^i u^k u^j R(e_k, Je_j)e_i.$$

For the quasi-periodicity condition of u, we replace $\nabla_t e_i = -(1/2)|\xi_x|^2 Je_i$ in the case of S^2 to $\nabla_t e_i = (1/2)R(\xi_x, J\xi_x)e_i$. We can extend Lemma 1.2 as follows.

Proposition 4.3. *Let M be a locally hermitian symmetric space. Let $\xi^o(x)$ be a curve in M with frame field $F^o = \{e^o_i(x)\}$, and $u^o(x)$ the differential of its development.*

1) *Let $\xi(t,x)$ be a solution of initial value problem* (3.1) *with initial data ξ^o. We extend $e^o_i(x)$ to $e_i(t,x)$ along ξ by the ODE: $\nabla_t e_i = (1/2)R(\xi_x, J\xi_x)e_i$. Then, for each t_0, $F = \{e_i(t_0, x)\}$ is a frame field along $\xi(t_0, x)$. And, the family $u(t,x)$ of the differential of the development of $\xi(t,x)$ is a solution of initial value problem:*

$$u_t = Ju_{xx} - \frac{1}{2} R(u, Ju)u. \tag{4.2}$$

Moreover, if ξ is a family of closed curves, then the correction of period of u is constant.

2) *Conversely, let $u(t,x)$ be a solution of* (4.2) *with initial data u^o. We extend $\{\xi^o, e^o_i\}$ to $\{\xi(t,x), e_i(t,x)\}$ by the system of ODEs: $\xi_t = Ju^i_x e_i$, $\nabla_t e_i = (1/2)R(u, Ju)e_i$. Then $\xi(t,x)$ is a solution of initial value problem* (3.1). *Moreover, if ξ^o is closed and if u is quasi-periodic with constant correction, then ξ is a family of closed curves.*

We also can construct a vortex filament type equation. This generalization is based on the identification $(\boldsymbol{R}^3, * \times *) = (\mathfrak{so}(3), [*, *])$. Let M be a hermitian symmetric space G/K, where G is the isometry group of M and K is the isotropy group. We use standard decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $\mathfrak{g}$ (resp. $\mathfrak{k}$) is the Lie algebra of G (resp. K), and the vector space $\mathfrak{m}$ is canonically identified with the tangent space of M at the origin.

There is an element Z of the center of $\mathfrak{k}$ such that $\mathrm{ad}_Z|_{\mathfrak{m}} \cong J$, and M is locally isomorphic to the orbit $\mathrm{Ad}_G Z \subset \mathfrak{g}$. We assume that M and the orbit are isomorphic, and identify them. Then, a curve ξ in M is regarded as a curve in $\mathfrak{g}$, and we have $J\nabla_x\xi_x = [\xi, \xi_{xx}]$. Thus we have the following

Proposition 4.4. *Consider a PDE for a curve γ in $\mathfrak{g}$*

$$\gamma_t = [\gamma_x, \gamma_{xx}] \quad (\gamma_x \in M). \tag{4.3}$$

There is a one-to-one correspondence between solutions of (4.3) *and solutions of* (3.1) *by putting $\xi = \gamma_x$.*

Now, we give exact solutions of equation (3.1) and explicitly describe them. Let $\xi(x)$ be a curve in $M \subset \mathfrak{g}$ such that $\xi(0) = Z \in M$. We attach a frame field $\{e_i\}$ to ξ. Then the differential u of the development of ξ can be viewed as a curve in $\mathfrak{m}$ by

$$u(x) := u^i(x)e_i(0).$$

Conversely, for a given u, the curve ξ can be reconstructed as follows. Let $g(x)$ be a curve in G satisfying the ODE: $g^{-1}g' = Ju$, $g(0) = 1_G$. Then, we can represent ξ and e_i as $\xi = \mathrm{Ad}_g Z$ and $e_i = \mathrm{Ad}_g e_i(0)$. In fact,

$$\nabla_x(\mathrm{Ad}_g\, e_i(0)) = ((\mathrm{Ad}_g\, e_i(0))')^T = (\mathrm{Ad}_g[g^{-1}g', e_i(0)])^T = \mathrm{Ad}_g[Ju, e_i(0)]_{\mathfrak{m}} = 0,$$
$$(\mathrm{Ad}_g\, Z)' = \mathrm{Ad}_g[g^{-1}g', Z] = \mathrm{Ad}_g\,[Ju, Z] = \mathrm{Ad}_g\, u = u^i \mathrm{Ad}_g\, e_i(0),$$

where we denote by $*^T$ the tangential component to M in $\mathfrak{g}$ and by $*_{\mathfrak{m}}$ the $\mathfrak{m}$ component in $\mathfrak{g}$.

Suppose that $\xi(x)$ satisfies the ODE:

$$J\nabla_x\xi_x = L(\xi) + a\xi_x, \tag{4.4}$$

where L is a Killing vector field of M and a is a real constant. Then, the family $\xi(t,x) := (\exp tL)(\xi(x + at))$ is a solution of equation (3.1). In fact,

$$\begin{aligned}\xi_t(t,x) &= (\exp tL)_*\{L(\xi(x+at)) + a\xi_x(x+at)\}\\ &= (\exp tL)_*(J\nabla_x\xi_x(x+at))\\ &= J\nabla_x\xi_x(t,x).\end{aligned}$$

ODE (4.4) in $M \subset \mathfrak{g}$ is given by

$$[\xi', \xi''] = [X, \xi] + a\xi', \tag{4.5}$$

where X is an element of $\mathfrak{g}$ which generates L, i.e., $\mathrm{ad}_X|_M = L$. Using g, we rewrite this equation to an equation for u. From

$$\xi' = \mathrm{Ad}_g\, u,$$

$$\xi''=\mathrm{Ad}_g([Ju,u]+u'),$$

we have

$$[\xi,\xi'']=\mathrm{Ad}_g[Z,[Ju,u]+u']=\mathrm{Ad}_g Ju',$$
$$[X,\xi]=[X,\mathrm{Ad}_g Z]=\mathrm{Ad}_g[\mathrm{Ad}_g^{-1}X,Z]=-\mathrm{Ad}_g J(\mathrm{Ad}_g^{-1}X)_{\mathfrak{m}},$$
$$a\xi'=a\,\mathrm{Ad}_g u.$$

Therefore, we get

$$u'=-(\mathrm{Ad}_g^{-1}X)_{\mathfrak{m}}-aJu.$$

We want to eliminate $(\mathrm{Ad}_g^{-1}X)_{\mathfrak{m}}$. From $(\mathrm{Ad}_g^{-1}X)'=-[Ju,\mathrm{Ad}_g^{-1}X]$, we have

$$(\mathrm{Ad}_g^{-1}X)'_{\mathfrak{m}}=-[Ju,(\mathrm{Ad}_g^{-1}X)_{\mathfrak{k}}],$$
$$(\mathrm{Ad}_g^{-1}X)'_{\mathfrak{k}}=-[Ju,(\mathrm{Ad}_g^{-1}X)_{\mathfrak{m}}]=-[Ju,-u'-aJu]=[Ju,u']=\frac{1}{2}[Ju,u]'.$$

Thus $(\mathrm{Ad}_g^{-1}X)_{\mathfrak{k}}-(1/2)[Ju,u]$ is a constant, which we denote by $A\in\mathfrak{k}$. The constant A is given by $X_k-(1/2)[JV,V]$, where $V=u(0)=\xi'(0)$. Using A, we have

$$u''=-(\mathrm{Ad}_g^{-1}X)'_{\mathfrak{m}}-aJu'=[Ju,A]+\frac{1}{2}[Ju,[Ju,u]]-aJu',$$

or,

$$u''+\frac{1}{2}[[Ju,u],Ju]=-[A,Ju]-aJu', \tag{4.6}$$

with initial data $u(0)=V$, $u'(0)=-X_{\mathfrak{m}}-aJV$. We can easily verify that $u(t,x)$: $=\mathrm{Ad}_{\exp tA}u(x+at)$ satisfies equation (4.2) via the formula of the curvature tensor: $R(v_1,v_2)v_3=-[[v_1,v_2],v_3]$.

We can reverse this procedure and conclude as follows.

Proposition 4.5. *Let $\xi(x)$ be a solution of equation* (4.5) *with initial data $\xi(0)=Z$, $\xi'(0)=V(\in\mathfrak{m})$. Then, $\xi(t,x):=\mathrm{Ad}_{\exp tX}(\xi(x+at))$ is a solution of equation* (3.1). *Let $u(x)$ be a solution of equation* (4.6) *with initial data $u(0)=V(\in\mathfrak{m})$, $u'(0)=W(\in\mathfrak{m})$. Then, $u(t,x):=\mathrm{Ad}_{\exp tA}u(x+at)$ is a solution of equation* (4.2). *By the procedure in* Proposition 4.3, *these solutions correspond to one another with relations $A=X_{\mathfrak{k}}-(1/2)[JV,V]$ and $W=-X_{\mathfrak{m}}-aJV$.*

REMARK 4.6. All irreducible hermitian symmetric spaces are classified into four classical types and two exceptional types. Classical types are (AIII) $SU(p+q)/S(U_p \times U_q)$, (DIII) $SO(2n)/U(n)$, (BDI) $SO(n+2)/SO(n) \times SO(2)$ and (CI) $Sp(n)/U(n)$. Their corresponding nonlinear Schrödinger equations are expressed as follows, where c is a real number.

Type	$\mathfrak{m}$	Equation
AIII	$\{p \times q$ matrices$\}$	$u_t = \sqrt{-1}(u_{xx} + cu{}^t\bar{u}u)$
DIII	$\mathfrak{so}(n, \boldsymbol{C})$	$u_t = \sqrt{-1}(u_{xx} + cu{}^t\bar{u}u)$
BDI	$\boldsymbol{C}^n$	$u_t = \sqrt{-1}(u_{xx} + c(2\|u\|^2 u - {}^t uu\bar{u}))$
CI	$\{$symmetric n-matrices$\}$	$u_t = \sqrt{-1}(u_{xx} + cu\bar{u}u)$

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