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ON UPPERBOUNDS OF VIRTUAL MORDELL-WEIL RANKS

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0. Introduction

0.0. Let $f: X \to C$ be a relatively minimal fibration of curves of genus $g \geq 1$ over a smooth projective curve C of genus b defined over an algebraically closed field k. Let $K = k(C)$ be the field of rational functions on C. In the theory of Mordell-Weil lattices due to Shioda (cf. [17], [18]) the following conditions are assumed:

(0.1) (i) *f admits a global section (O) as zero-section,*
(ii) *K/k-trace of the Jacobian J_F of the generic fibre F/K of f is trivial.*

Under these conditions the Mordell-Weil group $J(K)$ of K-rational points of J is finitely generated. The rank r of its free part is called the Mordell-Weil rank. We shall be concerned with characteristic zero case (in this case the second assumption in (0.1) is equivalent to $q(X) = b$). In [14, Theorem 1.3] an upperbound of r via the invariants of f is given. In particular, for the case of rational surfaces X it was shown in a joint paper ([15]) that $r \leq 4g+4$. Moreover the structure of fibrations with maximal rank $r = 4g+4$ and the structure of corresponding Mordell-Weil lattices are completely determined in [15] (a such fibration is obtained as a blowing up of a linear pencil of hyperelliptic curves on a Hirzebruch surface Σ_e with $0 \leq e \leq g$ $(g \geq 2)$).

In this note we consider a similar problem for locally non-trivial fibrations, not necessarily satisfying conditions (0.1). Let $\mathrm{NS}(X)$ be the Néron-Severi group of X. Then $\mathrm{NS}(X)/torsion$ admits the lattice structure with the intersection pairing. Hodge's index theorem asserts that its signature is $(1, \rho - 1)$, where $\rho := \mathrm{rank}\, \mathrm{NS}(X)$ is the Picard number of X.

DEFINITION 0.2 (cf. [11]). The virtual Mordell-Weil rank r of f is defined to be the rank of the essential sublattice of the Néron-Severi lattice (cf. [17], [18]), i.e.,

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as

$$(0.3)\qquad r=\rho-2-\sum_{t\in C}(n_t-1),$$

where n_t is the number of irreducible components of $X_t:=f^{-1}(t)$.

If f satisfies conditions (0.1) then this is nothing but the well-known formula for the Mordell-Weil rank r (*loc. cit*). This justifies our definition.

0.4. A natural question arising here is to give a best possible upperbound of virtual Mordell-Weil rank and we are interested in knowing when it becomes the real Mordell-Weil rank. In a similar way as in [14, Theorem 1.3] by using Xiao's inequality ([20]), one can have the following bound for a locally non-trivial fibration $f:X\to C$

$$(0.5)\qquad r\leq(6+4/g)d+2(q-b)+2g(b-1),$$

where $d=\deg(f_*\omega_{X/C})$, $q=q(X)$. Moreover we can show that the equality holds only if f is a hyperelliptic fibration, all fibres of f are irreducible and $q=b$.

In the non-hyperelliptic case with $f_*\omega_{X/C}$ semi-stable we have a sharper bound for r due to Konno's stronger version of the slope inequality ([5, Lemma 2.5]). In the light of new results of Konno (personal communication) we know that the case of equality implies that $\mathrm{Cliff}(f)=1$, i.e., genaral fibres of f are trigonal or plane quintic (see also his recent paper [6] where he treates the non-semistable case with $\mathrm{Cliff}(f)=1$). From the point of view of Mordell-Weil lattices the "computable" case $p_g=q=0$ is most interesting. In this case $r\leq 3g+6$ (also for the number of singular fibres $s:s\leq 7g+6$). We give two examples showing that bounds actually are sharp. It should be very interesting to get a complete description as in [15] for the maximal case.*

We can also have more precise structure theorem for the following pencils ($b=0$):

(I) *Pencils with* $\chi(\mathcal{O}_X)=1$. In this case the bound (0.5) can be read as $r\leq 4g+4+2q$. We remark that the equality $r=4g+4+2q$ leads us to the maximal case studied in [15] by using [11, Lemma 3.1.2]. Also $s\leq 8g+4$ and the equality $s=8g+4$ gives us Lefschetz pencils in the constructions of [15].

(II) *Pencils with* $c_1^2(X)=-4(g-1)<0$. Here we have $r\leq 4g+4-8q$ (*resp.* $s\leq 8g+4-12q$). The maximal case is obtained as a blowing up of a pencil (*resp.* a Lefschetz pencil, except for $g=2q$) on a ruled surface $\Sigma_e^q\to E$, $g(E)=q$ $(-q\leq e\leq g-2q)$ whose general members are double coverings of curve E. In the maximal case the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined. The proof uses the fact that in this case X is double

* A full account of this situation is now in preparation ([12]).

covering of $E\times \boldsymbol{P}^1$ whose branch locus is a smooth irreducible curve of numerical type $(2g+2-4q, 2)$ (cf. [11, Theorem 3.1 and Lemma 3.1.2]).

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1. Bounds of virtual Mordell-Weil ranks

1.1. We use the following notation:

$X(Y,\gamma,B):=\mathrm{Spec}(\mathcal{O}_Y\oplus\mathcal{O}_Y(\gamma)^\vee)$: the double covering of a smooth surface Y with branch locus $B\sim 2\gamma$:

$f: X\to C$: a relatively minimal fibration, not locally trivial, of curves of genus $g\geq 1$;

C: a smooth projective curve with genus b.

S: the finite set of critical points on C, $s=$ the number of S.

$\omega_{X/C}$: the relative dualizing sheaf, $d:=\deg(f_*\omega_{X/C})$.

ω: the relative canonical class $K_{X/C}$.

$\lambda(f):=\omega^2/d$: the slope of f.

$\rho':=h^{1,1}-\rho$: the difference of the middle Hodge number $h^{1,1}$ and Picard number ρ of X. Note that ρ' is a non-negative number by virtue of Lefschet'z theorem on algebraic cycles.

$\chi(X_t):=$the topological Euler number of $X_t:=f^{-1}(t)$ for $t\in C$.

$e_t(X):=\chi(X_t)-(2-2g)$: the local Euler number over $t\in C$.

n_t: the number of irreducible components of X_t.

$g(\tilde{X}_t)$: the genus of the normalization of X_t.

Recall that the ground field k is the field of complex numbers $\boldsymbol{C}$.

Proposition 1.2. *Let $f: X\to C$ be a relatively minimal fibration as in* (1.1) *and r denote the virtual Mordell-Weil rank of f. Then we have*

$$r\leq\left(4+\frac{4}{g}\right)[\chi(\mathcal{O}_X)-(g-1)(b-1)]+2[\chi(\mathcal{O}_X)+q-1], \tag{1.2.1}$$

or equivalently,

$$r\leq\left(6+\frac{4}{g}\right)d+2(q-b)+2g(b-1). \tag{1.2.2}$$

Moreover the equality in the bounds above holds if and only if

1) *f is a hyperelliptic fibration with lowest slope $\lambda(f)=4-4/g$,*
2) *all fibers of f are irreducible,*
3) *$\rho'=0$.*

Proof (cf. [14, Theorem 1.3]). First in view of Xiao's inequality $\lambda(f)\geq 4-4/g$ ([20, Theorem 2]) one can put

$$\omega^2=\omega_f+(4-\frac{4}{g})d \tag{1.2.3}$$

with non-negative ω_f.

By an easy calculation using Leray's spectral sequence and Riemann-Roch we have

$$\chi(\mathcal{O}_X)=d+(g-1)(b-1). \tag{1.2.4}$$

Next since $\chi(X)=c_2(X)=2-2b_1+b_2$ and $b_1=2q$, where b_i is the i-th Betti number we infer from Noether's formula

$$h^{1,1}=b_2-2p_g=(4+\frac{4}{g})[\chi(\mathcal{O}_X)-(g-1)(b-1)]+2[\chi(\mathcal{O}_X)+q]-\omega_f.$$

Taking into account (0.3) one obtains

$$r=(4+\frac{4}{g})[\chi(\mathcal{O}_X)-(g-1)(b-1)]+2[\chi(\mathcal{O}_X)+q-1]-\omega_f-\sum_{t\in C}(n_t-1)-\rho', \tag{1.2.5}$$

or equivalently (in view of (1.2.4))

$$r=(6+\frac{4}{g})d+2(q-b)+2g(b-1)-\omega_f-\sum_{t\in C}(n_t-1)-\rho'. \tag{1.2.6}$$

Bounds (1.2.1)–(1.2.2) follow directly from (1.2.5)–(1.2.6). Moreover the equality holds if and only if

1) $\omega_f=0$, or equivalently, $\lambda(f)=4-4/g$,
2) $n_t=1$, $\forall t\in C$, i.e., all fibres of f are irreducible,
3) $\rho'=0$.

It remains to use Konno's result stating that fibrations with lowest slope $\lambda(f)=4-4/g$ are hyperelliptic ([5, Proposition 2.6]). □

REMARK 1.3. As a consequence of this proposition we obtain $q=b$ in the case of the equality of bounds above ([20, Corollary 1]). If we assume moreover that f admits a global section, then this is equivalent to the triviality of K/k-trace of the Jacobian J_F. So r becomes the real Mordell-Weil rank and it makes sense to study the structure of Mordell-Weil lattices in this case.

Corollary 1.4. *Let $f:X\to C$ be as in Proposition* 1.2. *Assume that r is* *maximal, i.e.,* $r=(6+4/g)d+2(q-b)+2g(b-1)$. *Then* X *is a double covering of a ruled surface* $\Sigma_e^b \to C$ *with smooth branch locus* B *and*

$$\omega^2=2(g-1)[2m-(g+1)e], \quad d=\frac{g}{2}[2m-(g+1)e], \tag{1.4.1}$$

where $B\equiv 2(g+1)C_0+2mF_0$ *with* C_0, F_0 *denoting the minimal section and a fibre on* Σ_e^b.

Proof. From Proposition 1.2 and Horikawa's theory it follows that X is the canonical resolution of a double cover of a ruled surface over C with simple singularities. It remains to use standard calculations with double coverings (see, *e.g.*, [4], or [1, V, 22], *cf.* also §3). The fact that B is smooth follows from the irreducibility of fibres of f. Indeed, if B were singular, a fibre of f could consist of extra curves arising from the resolution of singularities.

1.5. Consider the non-hyperelliptic case and assume that $f_*\omega_{X/C}$ is semi-stable then one can have a sharper bound thanks to Konno's stronger version of the slope inequality ([5, Lemma 2.5]). In particular if $p_g=q=0$ then it is easy to see that $f_*\omega_{X/\boldsymbol{P}^1}$ is semi-stable (*cf.* [11, A.4.4]). So we have $r\leq 3g+6$. We give here examples which show that this bound is sharp. Take a Lefschetz pencil of curves of degree m in the projective plane $\boldsymbol{P}^2$, considered in [19]. By blowing up m^2 distinct base points from $\boldsymbol{P}^2$ one obtains a smooth rational surface X with natural morphism $f:X\to \boldsymbol{P}^1$. The fact that f is non-hyperelliptic if $m>3$ is obvious. An easy computation shows that we have the following invariants:

1) $g=(m-1)(m-2)/2$,
2) $\omega^2=3m^2-12m+9$, $\omega_f=(m-3)^2$,
3) $r=m^2-1$, $s=3(m-1)^2$.

Thus the case of $m=4,5$ gives us the equality in the bound above. It should be very interesting to describe all such fibrations (see footnote in the Introduction).

2. Pencils with $\chi(\mathcal{O}_X)=1$

Proposition 2.1. *Let* $f:X\to \boldsymbol{P}^1$ *be a relatively minimal fibration of curves of genus* $g\geq 1$ *(having a section if* $g=1$*). Assume that* $\chi(\mathcal{O}_X)=1$. *Then we have*

$$r\leq 4g+4+2q. \tag{2.1.1}$$

Moreover the equality $r=4g+4+2q$ *implies that* X *is a rational surface (hence* $q=0$*) and* f *has a section. In particular,* r *gives actually the Mordell-Weil rank and if* $g\geq 2$ *we obtain the known constructions with Hirzebruch surfaces* Σ_e, $0\leq e\leq g$, *as in* [15].

Proof. In fact bound (2.1.1) can be easily followed from (1.2.1)–(1.2.2). In our special case we have $d=g$ and

$$\omega_f=\omega^2-4(g-1), \tag{2.1.2}$$

is a non-negative integer.

Next (1.2.5)–(1.2.6) can be rewritten as

$$r=4g+4+2q-\omega_f-\sum_{t\in S}(n_t-1)-\rho'. \tag{2.1.3}$$

Consequently $r\leq 4g+4+2q$ and the equality holds if and only if $\omega_f=0$, all singular fibres are irreducible, and $\rho'=0$. Since the elliptic case ($g=1$) is obvious we can assume $g\geq 2$. Then the condition $\omega_f=0$ implies that X is a ruled surface ([11, Theorem 3.1 and Lemma 3.1.2]), in particular $p_g=q=0$, and by the same token the rationality of X. (Note that the fact $p_g=q=0$ also follows from Remark 1.3). It remains to refer to [15, Theorem 4.1] for the rest of the Proposition. □

Lemma 2.2. (i) *For $t\in S$ one has*

$$e_t(X)>0 \tag{2.2.1}$$

except X_t is a non-singular elliptic curve with some multiplicity (*the case* $e_t(X)=0$).

(ii) *Moreover $e_t(X)=1$ if and only if either X_t is irreducible with at most one node as its singularity, or X_t is a curve with two smooth irreducible components C_1, C_2 meeting at one point transversally such that $g(C_1)+g(C_2)=g$.*

The first statement is nothing but Theorem 7 in [16, IV]. The proof of the second statement is purely technical and can be followed from the arguments in the proof of that Theorem and Lemma 4 (*loc. cit.*)

Corollary 2.3. *Under the assumptions of Proposition* 1.2 *we have*

$$s\leq(8+4/g)d. \tag{2.3.1}$$

In the case of equality we also have (1.4.1)

Proof. First note that

$$\chi(X)=c_2(X)=\sum_{t\in S}e_t(X)+4(g-1)(b-1) \tag{2.3.2}$$

(see, *e.g.*, [16, IV, §4] or [1, III, 11.4]). Furthermore from this, (1.2.3) and Noether's formula we have

$$\sum_{t\in S} e_t(X) = \left(8+\frac{4}{g}\right)d - \omega_f. \tag{2.3.3}$$

It remains to use (2.2.1) to get (2.3.1). The case of equality implies that $\omega_f=0$. The same arguments as in the proofs of Proposition 1.2 and Corollary 1.4 show that one has (1.4.1). □

Corollary 2.3.4. $e_t(X)\le 2(g-g(\tilde{X}_t))+2(n_t-1)$.

Proposition 2.4. *In the situation of Propositon* 2.1 *we have*

$$s\le 8g+4. \tag{2.4.1}$$

Furthermore every fibration with maximal number $s = 8g+4$ *is a rational hyperelliptic Lefschetz pencil with a section such that* $\omega^2=4(g-1)$.

Proof. Since $d=g$ it follows from (2.3.3) that

$$\sum_{t\in S} e_t(X)+\omega_f = 8g+4. \tag{2.4.2}$$

So (2.4.2) together with (2.2.1) implies the bound (2.4.1). Moreover $s=8g+4$ holds if and only if:

1) $\omega_f=0$,
2) $e_t(X)=1,\quad \forall t\in S$.

As in the proof of Proposition 2.1, $\omega_f=0$ implies that f is a rational hyperelliptic pencil. Furthermore since X is the canonical resolution of a double cover of $\boldsymbol{P}^1\times\boldsymbol{P}^1$ with simple singularities, singular fibres with two smooth irreducible components in the second statement of Lemma 2.2 can not occur. Thus we have a Lefschetz pencil. □

Corollary 2.4.3. *Let* $f: X\to \boldsymbol{P}^1$ *be as in Proposition* 2.1. *If* $p_g=q>0$ *then* $r\le 4g+2q$ (*resp.* $s\le 8g$) *and the equality is possible only in case* $p_g=q=1$.

Proof. 1) From [20, Corollary 1] and the assumption $q>0$ it follows that $\lambda(f)\ge 4$. Furthermore $\lambda(f)=4$ implies $q=1$ ([20, Theorem 3]). It remains to use (2.1.3) and (2.4.2). □

REMARK 2.4.4. For the detailed construction of Lefschetz pencils in Proposition

2.4 we refer to [15]. Note that those fibrations are irregular in the sense of [11, §3] if $g \geq 2$.

3. Pencils with $c_1^2(X) = -4(g-1) < 0$

3.1. In this section we consider the class of pencils with $c_1^2(X) = -4(g-1) < 0$. First recall some facts from the theory of double covering of surfaces. Let B be an even reduced effective divisor on a smooth surface Y. Consider the double covering $X(Y,\gamma,B)$ with branch locus B and γ such that $B \sim 2\gamma$ (*cf.* 1.1). Let $X_{CR} = X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 = X(Y,\gamma,B)$ be the canonical resolution of X_0. Putting $Y_0 = Y$, $B_0 = B$ then

1) each X_i is a double covering of Y_i with branch locus B_i,
2) Y_i is a blowing up of Y_{i-1} at a singular point of B_i with multiplicity m_i, $i \leq n-1$,
3) B_n is non-singular.

Recall that X_0 has at most rational double points as its singularities if and only if all m_i are less than 4 ([4, Lemma 5]).

Lemma 3.2. *In the notation above we have*

$$\chi(\mathcal{O}_{X_{CR}}) = 2\chi(\mathcal{O}_Y) + \frac{1}{2}\gamma(K_Y + \gamma) - \frac{1}{2}\sum_i \left[\frac{m_i}{2}\right]\left(\left[\frac{m_i}{2}\right] - 1\right), \tag{3.2.1}$$

$$c_1^2(X_{CR}) = 2(K_Y + \gamma)^2 - 2\sum_i \left(\left[\frac{m_i}{2}\right] - 1\right)^2. \tag{3.2.2}$$

Proof. See [4, Lemma 6].

Theorem 3.3. *Let $f: X \to \boldsymbol{P}^1$ be a relatively minimal fibbration with $g \geq 2$. Assume that*

1) $c_1^2(X) = -4(g-1)$,
2) $\chi(\mathcal{O}_X) \geq 3 - g$.

Then X is a ruled surface defined as a family $\pi: X \to E$, $g(E) = q$. The morphisms f and π define a 2-to-1 map from X to $Y = E \times \boldsymbol{P}^1$ with a branch locus B. The second projection of Y induces f and B is of numerical type $(2g+2-4q, 2)$. Moreover X is the canonical resolution of $X(Y,\gamma,B)$ with rational double singularities. In particular, if $q = 0$ then X is a rational surface and f is hyperelliptic.

Proof. From [11, Theorem 3.1 and Lemma 3.1.2] we have known that X is a ruled surface $\pi: X \to E$, $g(E) = q$. The morphisms f and π define a 2-to-1 map from X to Y. So it can be easily seen that the branch locus has the desired numerical type. It remains to show that X is the canonical resolution of $X(Y,\gamma,B)$ with rational double singularities. Indeed arguments similar to those in [8, §2] show that the natural morphism $h: X_{CR} \to X$ is a contraction of all (-1)-curves on X_{CR}. Now calculating $\chi(\mathcal{O}_{X_{CR}})$, $c_1^2(X_{CR})$ by (3.2.1)–(3.2.2) we obtain:

$$\chi(\mathcal{O}_{X_{CR}})=1-q-\frac{1}{2}\sum_i [\frac{m_i}{2}]([\frac{m_i}{2}]-1),$$

$$c_1^2(X_{CR})=-4(g-1)-2\sum_i([\frac{m_i}{2}]-1)^2.$$

Therefore

$$\sum_i [\frac{m_i}{2}]([\frac{m_i}{2}]-1)=0,$$

$$\sum_i([\frac{m_i}{2}]-1)^2=0.$$

That is $X=X_{CR}$ and all m_i are less than 4 as desired. □

Corollary 3.3.1. *Under the assumptions of Theorem* 3.3 *let m be the number of critical points of* π. *We have*

$$(m=4g+4-8q) \Leftrightarrow (B \textit{ is a smooth irreducible curve})$$

Theorem 3.4. *Let* $f: X \to P^1$ *be as in Theorem* 3.3. *Then we have*

$$r \leq 4g+4-8q \quad \text{and} \quad s \leq 8g+4-12q. \tag{3.4.1}$$

Moreover the equality $r=4g+4-8q$ *implies that it is obtained by blowing up from a pencil on a ruled surface* $\pi': \Sigma_e^q \to E$, $g(E)=q$ $(-q \leq e \leq g-2q)$ *whose general members are double coverings of curve E. Furthermore in the case* $2q \neq g$ *the equality* $s=8g+4-12q$ *implies the same conclusions with a Lefschetz pencil. In the maximal cases the structure of the essential sublattice in the Néron-Severi lattice is uniquely determined.*

Proof. 1) The first bound in (3.4.1) is obvious since $\rho=b_2=4g+6-8q$ and by using (0.3). The second one follows immediately from Lemma 2 and

$$\sum_{t \in S} e_t(X)=8g+4-12q \tag{3.4.2}$$

(cf. (2.4.2)). As a consequence we obtain $g+1 \geq 2q$.

2) Now assume $r=4g+4-8q$. Then all singular fibres are irreducible. Since by Theorem 3.3, X is the canonical resolution of $X(Y,\gamma,B)$ with rational double singularities it follows that B is a smooth irreducible curve, so that $m=4g+4-8q$. Let us denote by $\{E_i^{\pm}\}_{i=1}^{4g+4-8q}$ the irreducible components of corresponding singular fibres of π. Obviously $E_i^{\pm}$s are sections of f. After a succession of blowings down (each time one of $E_i^{\pm}$ which we denote simply by E_i) we obtain a standard ruled surface $\pi':\Sigma_e^q \to E$. Surface Σ_e^q has degree $-e$ and a section C_0 such that $C_0^2=-e$. For invariant e we know that $e\geq -q$ ([9, 7]).

Thus we have a birational morphism $\varphi: X\to \Sigma_e^q$. Setting $F'=\varphi(F)$ the image of a smooth general fibre we may assume that F' is also smooth and birational to F. An easy computation shows that

$$F'\equiv 2C_0+aF_0,$$

where $a=g+1+e-2q$ and $F_0\simeq \boldsymbol{P}^1$ is a fibre of π'. It means that we obtain a linear pencil of a linear system $|2C_0+\bar{a}F_0|$ on Σ_e^q with $\deg \bar{a}=a$ and X is obtained as a blowing up of the base points of this linear pencil. We have to consider two cases.

$(+)$ If $e\geq 0$ then from [2, V, 2.20] it follows that $a\geq 2e$, or equivalently, $e\leq g+1-2q$. Assume that $e=g+1-2q$ then $F'\cdot C_0=0$. Let C_0' be the proper transform of C_0 by φ, one can see that $F\cdot C_0'=0$ and $C_0'^2=C_0^2=-e=2q-g-1$. In view of the irreducibility of fibres of f it is possible only if $e=2q-g-1=0$. Hence π is a smooth fibering, that is, $X=\Sigma_e^q$. On one hand C_0 is fibre of f by the above. On the other hand C_0 is a section of π. This contradicts the fact that $F\cdot F_0=2$. We have proved $e\leq g-2q$.

$(++)$ If $e<0$ then from [2, V, 2.21] we have known that $a\geq e$, or equivalently, $g+1\geq 2q$.

3) The Néron-Severi group NS(X) in the maximal case is as follows.

$$\mathrm{NS}(X)\simeq \boldsymbol{Z}\cdot C_0\oplus \boldsymbol{Z}\cdot F_0\oplus(\oplus_{i=1}^{4g+4-8q}\boldsymbol{Z}\cdot E_i) \tag{3.4.3}$$

where we denote total transforms of C_0, F_0 under φ by the same letters. We have a relation

$$F\sim 2C_0+\bar{a}F_0-\sum_{i=1}^{4g+4-8q}E_i. \tag{3.4.4}$$

4) One can show easily the assertions for s with adding the Lefschetz property to the pencinls. In fact since $e_t(X)=1$ and $g\neq 2q$ we see that a singular fibre with two smooth irreducible componnents C_1, C_2 with $g(C_1)+g(C_2)=g$ (cf. Lemma 2.2) does not appear. Since X is the canonical resoluton of $X(Y,\gamma,B)$, one obtains $g(C_1)=g(C_2)=q$, that is impossible by the assumption $g\neq 2q$. Thus arguing as above we get decomposition (3.4.3). Therefore in the case $s=8g+4-12q$ (even without the condition $g\neq 2q$) the structure of the essential sublattice in the

Néron-Severi lattice is uniquely determined. □

Corollary 3.4.5. *Let $f: X \to \boldsymbol{P}^1$ be as in Theorem 3.3. Assume that all singular fibres are irreducible, then*

$$s \geq 8 + \frac{4q-4}{g+1-2q}.$$

In particular we have $s \geq 7$ and if either $q \geq 1$, or $q=0$ and $g \geq 4$, then $s \geq 8$.

Proof. By virtue of the Riemann-Hurwitz formula one sees that $g(\tilde{X}_t) \geq 2q-1$ (Note that since S is not empty we get another proof of estimate $g+1>2q$). The corollary now follows easily from (3.4.2) and Corollary 2.3.4. □

3.5. The maximal case with rational base $q=0$ leads us to known constructions with Hirzebruch surfaces Σ_e. As a rule for constructing examples with maximal numbers s, r we need the very ampleness of linear system $|2C_0 + \bar{a}F_0|$ on Σ_e^q (cf. [15]). In general one can construct certain examples with maximal numbers $r=4g+4-8q$, $s=8g+4-12q$ under some conditions with respect to e.

Note that linear system $|2E+\bar{a}\boldsymbol{P}^1|$ on $\boldsymbol{P}^1 \times E$ with $\deg \bar{a} = a = g+1-2q$ is very ample if $a \geq 2q+1$. This gives an example with $e=0$. In fact one can prove the following proposition.

Proposition 3.5.1. *If $4q \leq g$, then for $-q \leq e \leq \frac{(g+1)}{2} - 2q$ linear systems $|2C_0 + \bar{a}F_0|$ with $\deg \bar{a} = a = g+1+e-2q$ are very ample on Σ_e^q.*

Proof. Denote by $\mathscr{L} = \mathcal{O}(2C_0 + \bar{a}F_0)$ and consider any two (possibly coinciding) points P_1, P_2 on Σ_e^q being contained in fibres F_1, F_2 respectively. As is well known, to prove the very ampleness of $\mathscr{L}$ it suffices to show:

$$(*) \qquad H^1(m_i \otimes \mathscr{L}) = H^1(m_1 m_2 \otimes \mathscr{L}) = 0,$$

where m_i is the ideal sheaf of P_i.

On the other hand due to two exact sequences

$$0 \to \mathscr{L}(-F_i) \to m_i \otimes \mathscr{L} \to m_i \otimes \mathscr{L} / \mathscr{L}(-F_i) \to 0$$

$$0 \to \mathscr{L}(-F_1 - F_2) \to m_1 m_2 \otimes \mathscr{L} \to m_1 m_2 \otimes \mathscr{L} / \mathscr{L}(-F_1 - F_2) \to 0$$

one can see easily that the vanishing statement $(*)$ follows from the following two statements:

1) the corresponding to $(*)$ vanishing statement for $\mathscr{L}$ restricted on fibres of π': it is easy since $\deg \mathscr{L}_{|F_i} = 2$ and $F_i \simeq \boldsymbol{P}^1$ so that $\mathscr{L}_{|F_i}$ is very ample on F_i,

2) $H^1(\mathscr{L}(-\bar{b}F_0))=0$ for any effective divisor $\bar{b}$ on E with $\deg \bar{b}\leq 2$.

For the second statement by virtue of the Kodaira-Ramanujam vanishing theorem ([13]) it suffices to verify the numerical positivity of divisor $D\equiv 2C_0+(a-2)F_0-K_{\Sigma_e^q}$. Here are standard calculations using [2, V, 2.20–2.21].

(i) Since $g\geq 4q$ by the assumption we have $D^2=8(g+1-4q)>0$.

(ii) Case $e\geq 0$: let $C'\equiv bC_0+cF_0$ be an irreducible curve ($b>0$, $c\geq be$) then from the condition for e we have

$$D\cdot C_0=g+1-4q-2e\geq 0,$$

$$D\cdot C'=-4be+b(g+1+2e-4q)+4c\geq -4be+b(g+1+2e-4q)+4be>0.$$

(iii) Case $e<0$: for an irreducible curve $C'\equiv bC_0+cF_0$ we have

(+) either $b=1$, $c\geq 0$, so that

$$D\cdot C'=-4e+(g+1+2e-4q)+4c>-4e+2e=-2e>0,$$

(+) or $b\geq 2$, $2c\geq be$, and here

$$D\cdot C'=-4be+b(g+1+2e-4q)+4c>-4be+2be+2be=0.$$

This completes the proof. □

REMARK 3.5.2. If $q=1$ then using [2, V, exer.2.12] it is easy to verify that the linear systems above are very ample if $-1\leq e\leq g-6$.

REMARK 3.5.3. In fact the method in the proof of Proposition 3.5.1 enables us to establish the very ampleness of the divisor $a_0C_0+\bar{a}F_0$ on Σ_e^q with

$$a_0>0,\qquad a\geq \max\{\frac{a_0e+1}{2}+2q,\ (a_0+1)e+2q\}.$$

This is not sharp in general. The only case where it gives sharp estimates is $e=q=0$ (cf. [2, V, 2.18]).

3.6. From the very ampleness of $|2C_0+\bar{a}F_0|$ in a similar way as in [15] one can find a Lefschetz pencil of this linear system such that $4g+4-8q$ distinct base points $p_1,\cdots,p_{4g+4-8q}$ do not lie on C_0 and any two of them are not on the same fibre of π'. By blowing up the base points we obtain the desired fibration $f:X\to \boldsymbol{P}^1$ with maximal numbers r and s. Indeed if we denote by $\{E_i\}_{i=1}^{4g+4-8q}$ the exceptional curves dominating points $\{p_i\}_{i=1}^{4g+4-8q}$ respectively then for the Néron-Severi group $\mathrm{NS}(X)$ we have decomposition (3.4.3) with relation (3.4.4). Therefore $\rho=4g+6-8q$, and by (0.3) $r=4g+4-8q$. Furthermore it is easy to see that $c_1^2(X)=-4(g-1)$. So (3.4.2) implies $s=8g+4-12q$.

REMARK 3.6.1. Examples with $q=1$ give us the best upper bound of the self-intersection of curves on irrational ruled surfaces (cf. [3]).

Note Added in Revision. A part of results presented here remains valid in positive characteristic (at least, $\neq 2$) due to Moriwaki's version of the Cornalba-Harris-Xiao inequality in any characteristic (cf. [10]). We will come back to this theme in the other occasion.

References

[1] W. Barth, C. Peters, A. Van de Ven: Compact complex surfaces, Springer-Verlag, 1984.

[2] R. Hartshorne: Algebraic Geometry, Springer-Verlag, 1977.

[3] R. Hartshorne: *Curves with high self-intersection on algebraic surfaces*, Publ. Math. IHES, **36** (1969), 111–125.

[4] E. Horikawa: *On deformations of quintic surfaces*, Invent. Math., **31** (1975), 43–85.

[5] K. Konno: *Non-hyperelliptic fibrations of small genus and certain irregular canonical surfaces*, Annali, Scuola Normale, Serie IV **XX, 4** (1993), 575–595.

[6] K. Konno: *A lower bound of the slope of trigonal fibrations*, Internat. J. Math., **7** (1996), 19–27.

[7] M. Maruyama: On classification of ruled surfaces, Kyoto Univ.,Lectures in Math. 3, Kinokuniya, Tokyo, 1970.

[8] S. Matsusaka: *Some numerical invariants of hyperelliptic fibrations*, J. Math. Kyoto Univ., **30**–1 (1990), 33–57.

[9] M. Nagata: *On self-intersection number of a section on a ruled surface*, Nagoya Math. J., **37** (1970), 191–196.

[10] A. Moriwaki: *Bogomolov conjecture over function fields for stable curves with only irreducible fibers*, to appear in Comp. Math..

[11] K.V. Nguyen: *On Beauville's conjecture and related topics*, J. of Math. of Kyoto Univ., **35**–2 (1995), 275–298.

[12] K.V. Nguyen and M.-H. Saito: *On Mordell-Weil lattices of non-hyperelliptic fibrations on surfaces with* $p_g = q = 0$, Manuscript 1995.

[13] C.P. Ramanujam: *Remarks on the Kodaira vanishing theorem*, J. of Indian Math. Soc., **36** (1972), 41–51, **38** (1974), 121–124.

[14] M.-H. Saito: *On upperbounds of Mordell-Weil ranks of higher genus fibrations*, Proc. of the Workshop "Hodge Theory and Algebraic Geometry", Hokkaido Univ., June 20–24, 1994, Juten Lect. Notes N^0 10, 26–35.

[15] M.-H. Saito and K.-I. Sakakibara: *On Mordell-Weil lattices of higher genus fibrations on rational surfaces*, J. of Math. of Kyoto Univ., **34**–4 (1994), 859–871.

[16] I.R. Shafarevich et al.: *Algebraic surfaces*, Proc. Steklov Inst. Math., **75** (1965), transl. by Amer. Math. Soc. (1967).

[17] T. Shioda: *On the Mordell-Weil lattices*, Comment. Math. Univ. St. Pauli, **39** (1990), 211–240.

[18] T. Shioda: *Mordell-Weil lattices for higher genus fibration*, Proc. Japan Acad. **68A** (1992), 247–250.

[19] T. Shioda: *Generalization of a Theorem of Manin-Shafarevich*, Proc. Japan Acad. **69A** (1993), 10–12.

[20] G. Xiao: *Fibered algebraic surfaces with low slope*, Math. Ann., **276** (1987), 449–466.

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