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# EQUIVARIANT ALGEBRAIC VECTOR BUNDLES OVER ADJOINT REPRESENTATIONS

Dedicated to Professor Seiya Sasao on his 60th birthday

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## 0. Introduction

Let $G$ be a reductive complex algebraic group and let $B$, $F$ be $G$-modules over $\boldsymbol{C}$. Let $\mathrm{Vec}_G(B, F)$ denote the set of complex algebraic $G$-vector bundles over $B$ whose fiber at $0 \in B$ is $F$, and let $\mathrm{VEC}_G(B, F)$ denote the set of the $G$-isomorphism classes in $\mathrm{Vec}_G(B, F)$. The set $\mathrm{VEC}_G(B, F)$ has the trivial class represented by the product bundle $B \times F \rightarrow B$.

The solution of the Serre conjecture by QUILLEN [9] and SUSLIN [11] says that $\mathrm{VEC}_G(B, F)$ is trivial for any $B$ and $F$ when $G$ is trivial. In contrast to this SCHWARZ [10] discovered that $\mathrm{VEC}_G(B, F)$ is nontrivial for some $B$ and $F$ when $G$ belongs to a class of noncommutative groups that includes all classical groups (see also [5]); this depends upon an analysis of $\mathrm{VEC}_G(B, F)$ when the ring $\mathcal{O}(B)^G$ of invariants on $B$ is a polynomial ring in one variable. Subsequently KNOP [6] used the result of Schwarz for $G=SL_2$ to show that $\mathrm{VEC}_G(\mathfrak{g}, F)$ is nontrivial for many irreducible $G$-modules $F$ if $G$ is connected and noncommutative, where $\mathfrak{g}$ denotes the adjoint representation of $G$. Note that $\mathcal{O}(\mathfrak{g})^G$ is a polynomial ring in $n$ variables where $n$ is the rank of $G$. We refer the reader to [7] and [8] for further results, where $\mathrm{VEC}_G(B, F)$ is studied from a different point of view.

In this paper we closely look at the result of Schwarz on the $SL_2$ case together with the argument of Knop to prove

**Theorem A.** *If $G$ is semisimple, then $\mathrm{VEC}_G(\mathfrak{g}, F)$ is nontrivial for all but finitely many isomorphism classes of irreducible $G$-modules $F$.*

REMARK. If $G$ is commutative, then $\mathrm{VEC}_G(\mathfrak{g}, F)$ is trivial for any $G$-module $F$ because the action of $G$ on $\mathfrak{g}$ is trivial ([3, §2]).

Theorem A is a corollary of Theorem B stated below. Let $R_n$ be the

$SL_2$-module of homogeneous polynomials of degree $n$ in two variables. According to [10] $VEC_{SL_2}(R_2, R_m)$ forms an abelian group isomorphic to $\boldsymbol{C}^p$ where $p=[(m-1)^2/4]$. Suppose $G$ is connected and noncommutative. We fix a system $\Sigma$ of simple roots of $G$. Associted to a simple root $\alpha\in\Sigma$, Knop defined a map

$$\Phi^\alpha: VEC_G(\mathfrak{g}, F)\rightarrow VEC_{SL_2}(R_2, R_m)$$

where $m=\langle\chi, \alpha\rangle$ and $\chi$ is the highest weight of the irreducible $G$-module $F$. He proved that $\Phi^\alpha$ is surjective if $\chi$ is regular, i.e. unless $\chi$ is contained in a reflecting hyperplane $P_\beta$ for some $\beta\in\Sigma$.

DEFINITION. We call the $\alpha$-string $(\chi, \chi-\alpha, \cdots, \chi-m\alpha)$ of $\chi$ *singular* if it is contained in some $P_\beta$ and *regular* otherwise.

Clearly if $\chi$ is regular, then the $\alpha$-string of $\chi$ is regular for any $\alpha\in\Sigma$. But an $\alpha$-string happens to be regular even if $\chi$ is singular, e.g. if $G$ is semisimple and of rank two, then any dominant weight has a regular $\alpha$-string. Hence the following theorem extends the result of Knop mentioned above.

**Theorem B**. *Suppose $G$ is connected and noncommutative. Then*

(1) *$\Phi^\alpha$ is surjective if the $\alpha$-string of $\chi$ is regular,*

(2) *the image of $\Phi^\alpha$ contains a subspace of dimension $[m/2]([m/2]-1)/2$ if the $\alpha$-string of $\chi$ is singular.*

Theorem B implies that $VEC_G(\mathfrak{g}, F)$ is nontrivial provided $m\geq 4$. If $G$ is semisimple, then there are only finitely many irreducible $G$-modules $F$ such that $\langle\chi, \alpha\rangle\leq 3$ for all $\alpha\in\Sigma$. Therefore Theorem A follows from Theorem B.

## 1. The $SL_2$ case

In this section we translate the result of Schwarz on the $SL_2$ case into an explicit form. Let $G=SL_2$ and $T$ be its maximal torus consisting of diagonal matrices. Remember that $R_n$ is the $G$-module of homogeneous polynomials of degree $n$ in two variables, say $x$ and $y$. Since the $G$-orbit of $R_2^T=\{bxy|b\in\boldsymbol{C}\}$ is dense in $R_2$, the inclusion map $i: R_2^T\rightarrow R_2$ induces an injective homomorphism

$$i^*: \mathrm{Mor}(R_2, \mathrm{End}(R_m))^G\rightarrow \mathrm{Mor}(R_2^T, (\mathrm{End}(R_m))^T)^W$$

where $W$ denotes the Weyl group $N_G(T)/T$, which is of order two. Note that the one dimensional subspaces of $R_m$ spanned by $x^{m-n}y^n$ are mutually non-isomorphic $T$-modules.

**Lemma 1.1.** *Any element $\sigma\in\mathrm{Mor}(R_2^T, (\mathrm{End}(R_m))^T)^W$ is of the form*

$$(\sigma(bxy))(x^{m-n}y^n)=f_n(b)x^{m-n}y^n$$

*with polynomials $f_n(b)$ such that $f_n(-b)=f_{m-n}(b)$ for $n=0, 1, \cdots, m$.*

Proof. It follows from Schur's lemma that $\sigma$ is of the form

$$(\sigma(bxy))(x^{m-n}y^n)=f_n(b)x^{m-n}y^n$$

with polynomials $f_n(b)$ for any $n$. The element of $G$ mapping $x$ to $y$ and $y$ to $-x$ is a representative of the nontrivial element of $W$. It acts on $\mathrm{R}_2^{\mathrm{T}}$ as multiplication by $-1$ and on $(\mathrm{End}(\mathrm{R_m}))^{\mathrm{T}}$ by conjugation of the element of $\mathrm{End}(\mathrm{R_m})$ mapping $x^{m-n}y^n$ to $(-1)^n x^n y^{m-n}$. Hence it follows from the equivariance with respect to the action of $W$ that $f_n(-b)=f_{m-n}(b)$. This proves the lemma. □

It is well-known (and easy to prove) that $\mathcal{O}(\mathrm{R}_2)^{\mathrm{G}}$ is a polynomial ring $\boldsymbol{C}[\varDelta]$ where $\varDelta$ is the discriminant defined by $\varDelta(ax^2+bxy+cy^2)=b^2-4ac$. We note that $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R_m}))^{\mathrm{G}}$ is an algebra over $\mathcal{O}(\mathrm{R}_2)^{\mathrm{G}}=\boldsymbol{C}[\varDelta]$. The following lemma describes the algebra structure.

**Lemma 1.2.** $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R_m}))^{\mathrm{G}}=(\boldsymbol{C}[\varDelta])[\gamma]/\prod_{n=0}^{m}(\gamma-(m-2n)\sqrt{\varDelta})$ *where $\gamma$ is homogeneous of degree one with respect to the coordinates of* $\mathrm{R}_2$ *and expressed on* $\mathrm{R}_2^{\mathrm{T}}$ *as*

$$(*)\qquad (\gamma(bxy))(x^{m-n}y^n)=(m-2n)bx^{m-n}y^n.$$

REMARK. Since $(\gamma-(m-2k)\sqrt{\varDelta})(\gamma-(m-2(m-k))\sqrt{\varDelta})=\gamma^2-(m-2k)^2\varDelta$, the product $\prod_{n=0}^{m}(\gamma-(m-2n)\sqrt{\varDelta})$ is actually a polynomial of $\gamma$ and $\varDelta$.

Proof. This may be known, but for the sake of completeness we shall give the proof.

First we claim that $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R_m}))^{\mathrm{G}}$ is free and of rank $m+1$ as a $\boldsymbol{C}[\varDelta]$-module; more precisely, the degrees of the generators are $0, 1, 2, \cdots, m$. This can be seen as follows. By the self-duality of $\mathrm{R_m}$ and the Clebsch-Gordan formula ([4, p. 170]) we have

$$\mathrm{End}(\mathrm{R_m})\cong\mathrm{R_m}\otimes\mathrm{R_m}\cong\oplus_{k=0}^{m}\mathrm{R_{2k}}.$$

Hence $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R_m}))^{\mathrm{G}}\cong\oplus_{k=0}^{m}\mathrm{Mor}(\mathrm{R}_2, \mathrm{R_{2k}})^{\mathrm{G}}$. Here it is easy to see that $\mathrm{Mor}(\mathrm{R}_2, \mathrm{R_{2k}})^{\mathrm{G}}$ is free and of rank one as a $\boldsymbol{C}[\varDelta]$-module, in fact, the generator is given by the $k$th power map. This implies the claim.

Suppose $\gamma\in\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R_m}))^{\mathrm{G}}$ is homogeneous and of degree one. Then it follows from Lemma 1.1 that

$$(\gamma(bxy))(x^{m-n}y^n)=c_n bx^{m-n}y^n$$

with constants $c_n$ such that $c_n=-c_{m-n}$. Let $g\in G$ be the unipotent matrix with 1 in the upper right hand corner. Since $gx=x$ and $gy=x+y$ (hence $g(xy)=x^2$

$+xy$), it follows from equivariance that

$$\gamma(bx^2+bxy)=g\gamma(bxy)g^{-1}.$$

We view elements in $\mathrm{End}(\mathrm{R}_m)$ as matrices by taking a basis $\{x^m, x^{m-1}y, \cdots, y^m\}$ of $\mathrm{R}_m$. Since $\gamma$ is homogeneous and of degree one, the entries of the matrix $\gamma(ax^2+bxy+cy^2)$ are linear combinations of $a$, $b$ and $c$. The equivariance of $\gamma$ with respect to the action of $T$ implies that the $(i, j)$ entries of $\gamma(ax^2+bxy+cy^2)$ vanish whenever $|i-j|\geq 2$. (In fact, the diagonal entries are scalar multiples of $b$, the $(i, i+1)$ entries are those of $a$, and the $(i+1, i)$ entries are those of $c$.) In particular, the $(1, j)$ entries of $\gamma(bx^2+bxy)$ are zero for $j\geq 3$. The vanishing of the $(1, j)$ entries $(3\leq j\leq m+1)$ of the matrix at the right hand side of the identity above yields $m-1$ equations among the constants $c_n$. An elementary computation shows that

$$c_n=(1-n)c_0+nc_1.$$

This together with the relation $c_n=-c_{m-n}$ shows $c_n=(m-2n)c_0/m$.. The identities (*) are then obtained by setting $c_0=m$.

The identities (*) imply that $\gamma^j (0\leq j\leq m)$ are linearly independent over $\boldsymbol{C}[\Delta]$ when restricted to $\mathrm{R}_2^{\mathrm{T}}$. Since the $G$-orbit of $\mathrm{R}_2^{\mathrm{T}}$ is dense in $\mathrm{R}_2$, the $\gamma^j (0\leq j\leq m)$ are linearly independent over $\boldsymbol{C}[\Delta]$ as elements of $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R}_m))^{\mathrm{G}}$. Moreover the identities (*) show that the element is $\prod_{n=0}^{m}(\gamma-(m-2n)\sqrt{\Delta})$ is zero when restricted to $\mathrm{R}_2^{\mathrm{T}}$, and hence zero actually as an element of $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R}_m))^{\mathrm{G}}$. As claimed above $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R}_m))^{\mathrm{G}}$ is free and of rank $m+1$ as a $\boldsymbol{C}[\Delta]$-module. This shows that the identity $\prod_{n=0}^{m}(\gamma-(m-2n)\sqrt{\Delta})=0$ is the only relation in $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R}_m))^{\mathrm{G}}$. This completes the proof. □

Denote by $M_k^m$ (resp., $N_k^m$) the linear space consisting of homomogeneous elements of degree $k$ in $i^*$ $\mathrm{Mor}(\mathrm{R}_2, \mathrm{End}(\mathrm{R}_m))^{\mathrm{G}}$ (resp., $\mathrm{Mor}(\mathrm{R}_2^{\mathrm{T}}, (\mathrm{End}(\mathrm{R}_m))^{\mathrm{T}})^{\mathrm{W}}$) and set $M^m=\prod_{k\geq 1} M_k^m$, $N^m=\prod_{k\geq 1} N_k^m$. An elementary calculation together with Lemmas 1.1 and 1.2 shows that

$$\dim N_k^m=\begin{cases}(m+1)/2, & \text{if } m \text{ is odd}\\ (m+1+(-1)^k)/2, & \text{if } m \text{ is even,}\end{cases}$$

$$\dim M_k^m=\begin{cases}[k/2]+1, & \text{if } k\leq m-2\\ \dim N_k^m, & \text{if } k\geq m-1,\end{cases}$$

and

$$\dim N^m/M^m=[(m-1)^2/4].$$

Remember that $\Delta: \mathrm{R}_2\to \boldsymbol{C}$ is an invariant polynomial. It is known that any element of $\mathrm{Vec}_{\mathrm{SL}_2}(\mathrm{R}_2, \mathrm{R}_m)$ is trivial over $\Delta^{-1}(\boldsymbol{C}-(0))$ ([5, VII.2.6]). Moreover, given $E\in \mathrm{Vec}_{\mathrm{SL}_2}(\mathrm{R}_2, \mathrm{R}_m)$, there is a finite subset $S$ of $\boldsymbol{C}-(0)$ such that $E$ is trivial over $\Delta^{-1}(\boldsymbol{C}-S)$ ([3, 6.2]). Hence one can find a transition function $\psi_E$ of $E$ in $\mathrm{Mor}(\Delta^{-1}(\mathbf{C}-(\mathrm{S}\cup(0))), \mathrm{End}(\mathrm{R_m}))^{\mathrm{G}}$. The choice of $\psi_E$ is not unique and one can always arrange $\psi_E$ such that the restriction $\psi_E|\mathrm{R}_2^{\mathrm{T}}$ is defined at 0 with value the identity. By the $T$-equivariance $\psi_E|\mathrm{R}_2^{\mathrm{T}}$ is a diagonal matrix with rational functions as entries with respect to the basis $\{x^m, x^{m-1}y, \cdots, y^m\}$ of $\mathrm{R_m}$. We expand those rational functions into formal power series of the coordinate $b$ of $\mathrm{R}_2^{\mathrm{T}}$. This gives a correspondence

$$\psi: \mathrm{Vec}_{\mathrm{SL}_2}(\mathrm{R}_2, \mathrm{R_m}) \to 1+N^m$$

defined by $\psi(E)=\psi_E|\mathrm{R}_2^{\mathrm{T}}$. A general result of Schwarz [10] or Kraft-Schwarz [5, VII.3.4] applied to the $\mathrm{SL}_2$ case implies

**Theorem 1.3** ([5], [10]). *The map $\psi$ induces a bijection*

$$\Psi: \mathrm{VEC}_{\mathrm{SL}_2}(\mathrm{R}_2, \mathrm{R}_m) \cong 1+N^m/M^m.$$

## 2. The map $\Phi^\alpha$

In this section $G$ is connected and noncommutative. We recall the definition of the map $\Phi^\alpha: \mathrm{VEC_G}(\mathfrak{g}, \mathrm{F}) \to \mathrm{VEC}_{\mathrm{SL}_2}(\mathrm{R}_2, \mathrm{R_m})$ mentioned in the introduction. Let $T$ be a maximal torus of $G$. Denote the Lie algebra of $T$ by $\mathfrak{t}$. Let $L$ be the subgroup of $G$ generated by $T$ and the root groups $U_\alpha$ and $U_{-\alpha}$ (see [2, 26.3]). Let $L'$ be the commutator subgroup of $L$ and $Z$ be the identity component of the center of $L$. Then $L'$ is isomorphic to $\mathrm{SL}_2$ or $\mathrm{SO}_3$, the subgroup $Z$ is a codimension one torus in $T$ and $L=ZL'$. We choose and fix an element $\xi_0 \in \mathfrak{t}$ whose centralizer is exactly $L$. This is equivalent to saying that $\xi_0 \in P_\alpha$ but $\xi_0 \notin P_\beta$ for any $\beta \neq \alpha \in \Sigma$. Denote by $\mathfrak{a}$ the affine space $\xi_0 + \mathrm{Lie}\, L' \subset \mathfrak{g}$, which is $L$ invariant. The action of $Z$ on $\mathfrak{a}$ is trivial and $\mathfrak{a}$ is isomorphic to $\mathrm{R}_2$ as $L'$-varieties because Lie $L'$ and $\mathrm{R}_2$ are isomorphic representations.

Given $E \in \mathrm{Vec_G}(\mathfrak{g}, F)$, we restrict it to $\mathfrak{a}$. Since $\mathfrak{a}$ is fixed under the action of $Z$, the restricted bundle $E|\mathfrak{a}$ decomposes into eigenbundles according to the weights of $F$ viewed as a $Z$-module. Let $(E|\mathfrak{a})^\chi$ denote the eigenbundle of $E|\mathfrak{a}$ corresponding to the highest weight $\chi$ restricted to $Z$. Since $Z$ commutes with $L'$, $(E|\mathfrak{a})^\chi$ is an $L'$-vector bundle. The correspondence $E \to (E|\mathfrak{a})^\chi$ induces the desired map $\Phi^\alpha$.

## 3. Proof of Theorem B

Let $\Delta \in \boldsymbol{C}[\mathfrak{g}]^G$ be the discriminant and put $\mathfrak{g}_0 = \Delta^{-1}(\boldsymbol{C}-(0))$. For a finite subset $S \subset \boldsymbol{C}-(0)$ we set $\mathfrak{g}_S = \Delta^{-1}(\boldsymbol{C}-S)$. Similarly we set $\mathfrak{t}_0 = \mathfrak{t} \cap \mathfrak{g}_0$, $\mathfrak{t}_S = \mathfrak{t} \cap \mathfrak{g}_S$. Since $\mathfrak{g}_0$ is the set of regular semisimple elements, we have

$$\mathfrak{g}_0 = G \times^{N_G(T)} \mathfrak{t}_0. \tag{3.1}$$

We construct a $G$-vector bundle over $\mathfrak{g}$ by glueing the product $G$-vector bundles $\mathfrak{g}_0 \times F \to \mathfrak{g}_0$ and $\mathfrak{g}_S \times F \to \mathfrak{g}_S$ over $\mathfrak{g}_{S_0} = \mathfrak{g}_0 \cap \mathfrak{g}_S$ using a transition function, where $S_0 = S \cup (0)$. The transition function is a $G$-equivariant morphism

$$\varphi:\ \mathfrak{g}_{S_0} \to \mathrm{GL(F)}$$

where $G$ acts on $\mathrm{GL(F)}$ by conjugation. It follows from (3.1) that the restriction map

$$\mathrm{Mor}(\mathfrak{g}_{S_0},\ \mathrm{GL(F)})^{\mathrm{G}} \to \mathrm{Mor}(\mathfrak{t}_{S_0},\ \mathrm{GL(F)}^{\mathrm{T}})^{\mathrm{W}}$$

is bijective, where $W$ is the Weyl group $N_G(T)/T$. Thus we are led to study $W$-equivariant morphisms from $\mathfrak{t}_{S_0}$ to $\mathrm{GL(F)}^T$.

Decompose

$$F = \oplus_{\eta \in \chi(T)} M(\eta) \quad \text{as } T\text{-modules}$$

where $\chi(T)$ denotes the set of characters of $T$ and $M(\eta)$ is a (not necessarily one dimensional) $T$-module with character $\eta$. It follows from Schur's lemma that

$$GL(F)^T = \prod_{\eta \in \chi(T)} GL(M(\eta)).$$

Hence an element of $\mathrm{Mor}(\mathfrak{t}_{S_0},\ \mathrm{GL(F)}^{\mathrm{T}})^{\mathrm{W}}$ is given by a family of morphisms

$$\varphi_\eta:\ \mathfrak{t}_{S_0} \to \mathrm{GL(M(\eta))}$$

satisfying

$$\varphi_{w\eta}(\xi) = \bar{w} \circ \varphi_\eta(w^{-1}\xi) \circ \bar{w}^{-1} \quad \text{for all } w \in W \text{ and } \xi \in \mathfrak{t}_{S_0} \tag{3.2}$$

where $\bar{w} \in N_G(T)$ is a representative of $w$. The action of $\bar{w}$ induces an isomorphism from $M(\eta)$ to $M(w\eta)$ as $T$-modules.

We define $\varphi_\eta \equiv 1$ unless $\eta$ is in the $W$-orbit of the $\alpha$-string of $\chi$, i.e. unless $\eta = w(\chi - n\alpha)$ for some $w \in W$ and $0 \leq n \leq m$. If $\eta$ is in the $W$-orbit of the $\alpha$-string of $\chi$, then dim $M(\eta) = 1$ ([1, p. 125, Exercise 1]); so $\mathrm{GL(M(\eta))} = \boldsymbol{C}^*$. Hence $\varphi_\eta$ is a rational function on $\mathfrak{t}$ which has neither zero nor a pole on $\mathfrak{t}_{S_0}$. Moreover in this case (3.2) reduces to

$$\varphi_{w\eta}(\xi) = \varphi_\eta(w^{-1}\xi) \quad \text{for all } w \in W \text{ and } \xi \in \mathfrak{t}_{S_0}. \tag{3.3}$$

In order to choose a family $\{\varphi_\eta\}$ which satisfies (3.3), It suffices to choose a subfamily $\{\varphi_\eta | \eta$ is in the $\alpha$-string of $\chi\}$ which satisfies (3.3) whenever $\eta$ and $w\eta$ are in the $\alpha$-string of $\chi$. We note that the reflection $s_\alpha$ relative to the reflecting hyperplane $P_\alpha$ reflects the $\alpha$-string of $\chi$, i.e. $s_\alpha(\chi - n\alpha) = \chi - (m-n)\alpha$ for any $n$

**Lemma 3.4.** (1) *If $w(\chi - k\alpha) = \chi - l\alpha$ for some $0 \leq k,\ l \leq m$, then $k = l$ or $k = m - l$.*

(2) *If $w(\chi - k\alpha) = \chi - k\alpha$ and $\chi - k\alpha$ is regular, then $w$ is the identity.*

Proof. (1) First we recall the following general fact ([1, 10.3]). Let $\lambda$, $\mu$ be elements in the closure $\bar{C}$ of the Weyl chamber relative to the simple root system $\Sigma$. If $w\lambda=\mu$ for some $w\in W$, then $\lambda=\mu$.

Suppose $\chi-k\alpha$ and $\chi-l\alpha$ are both in $\bar{C}$. Then it follows from the above fact that $k=l$. Suppose $\chi-k\alpha$ is in $\bar{C}$ but $\chi-l\alpha$ is not in $\bar{C}$. Then $s_\alpha(\chi-l\alpha)=\chi-(m-l)\alpha$ is in $\bar{C}$. Since $s_\alpha w(\chi-k\alpha)=\chi-(m-l)\alpha$, we are in the same situation as above, hence $k=m-l$. The remaining two cases can be treated in the same way.

(2) The isotropy subgroup of $W$ at a regular element in $\mathfrak{t}$ is trivial ([1, 10.3]). This implies (2). $\square$

We denote $\varphi_{\chi-n\alpha}$ by $\varphi_n$. We shall find a family $\{\varphi_n(0\leq n\leq m)\}$ satisfying (3.3). Let $\delta$ be the product of positive roots. It is well known that

$$\delta(s_\beta\xi)=-\delta(\xi) \quad \text{for any } \beta\in\Sigma \tag{3.5}$$

([1, 10.2]). We take a family of polynomials $\{p_n(0\leq n\leq m)|p_n(0)=1\}$ in one variable such that

$$p_0\equiv p_m\equiv 1 \quad \text{and} \quad p_n(-\delta)=p_{m-n}(\delta) \quad \text{for any } n \tag{3.6}$$

and define

$$\varphi_n(\xi)=p_n(\delta(\xi)). \tag{3.7}$$

Suppose the $\alpha$-string of $\chi$ is regular. Since $\varphi_0\equiv\varphi_m\equiv 1$, it follows from (3.3) and Lemma 3.4 that the identity $\varphi_{m-n}(\xi)=\varphi_n(s_\alpha\xi)$ for each $n$ is the only condition which the family $\{\varphi_n\}$ must satisfy. But it is satisfied by (3.5), (3.6) and (3.7).

Suppose the $\alpha$-string of $\chi$ is singular. Then we require one more condition on the family $\{p_n\}$ that they be all even functions. Since $\delta(w\xi)^2=\delta(\xi)^2$ for any $w\in W$ by (3.5), it follows from Lemma 3.4, (3.6) and (3.7) that (3.3) is satisfied.

Let $[E_p]$ denote the isomorphism class of the $G$-vector bundle $E_p\in\mathrm{Vec}_G(\mathfrak{g}, F)$ defined by a family of polynomials $\{p_n\}$ satisfying the conditions mentioned above. We shall observe $\Phi^\alpha([E_p])$. As discussed in §1 elements in $\mathrm{VEC}_{\mathrm{SL}_2}(\mathrm{R}_2, \mathrm{R}_m)$ are detected by their transition functions restricted to $\mathrm{R}_2^T$. By definition $\Phi^\alpha([E_p])=[(E_p|\mathfrak{a})^\chi]$ and $\mathfrak{a}$ is the affine space $\xi_0+\mathrm{Lie}\,L'$ which is isomorphic to $\mathrm{R}_2$ as $L'$-varieties. Then $\mathrm{R}_2^T$ corresponds to $\mathfrak{t}\cap\mathfrak{a}=\{\xi_0+bh_\alpha|b\in \boldsymbol{C}\}$ where $h_\alpha\in\mathfrak{t}\cap\mathrm{Lie}\,L'$ with $\alpha(h_\alpha)=1$. Thus $\Phi^\alpha([E_p])$ corresponds to the family $\{p_n(\delta(\xi_0+bh_\alpha))\}$ through the map $\Psi$ in Theorem 1.3. Remember that $\xi_0$ is chosen in such a way that $\xi_0\in P_\alpha$ but $\xi_0\notin P_\beta$ for any $\beta\neq\alpha\in\Sigma$. Since $\delta$ is the product of positive roots, $\delta(\xi_0+bh_\alpha)$ is a polynomial of $b$ with zero constant term and nonzero degree one term.

In case the $\alpha$-string of $\chi$ is regular, the condition we imposed on $\{p_n\}$ is only (3.6). Then it is not difficult to see that the composition $\Psi\circ\Phi^\alpha$ is surjective, hence $\Phi^\alpha$ is surjective as $\Psi$ is bijective.

In case the $\alpha$-string of $\chi$ is singular, the conditions we imposed on $\{p_n\}$ are (3.6) and that $p_n$ are even functions. Then it is also not difficult to see that $\Psi\circ\Phi^\alpha$ contains the image of even degree elements of $1+N^m$ in $1+N^m/M^m$. An elementary calculation shows that the image is a subspace of dimension $[m/2]([m/2]-1)/2$. This completes the proof of Theorem B.

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## References


[1] J.E. Humphreys: *Introduction to Lie algebras and representation theory,* GTM 9 (fifth edition), Springer-Verlag. 1972

[2] J.E. Humphreys: *Linear algebric groups,* GTM **21** (third edition), Springer-Verlag.

[3] H. Kraft: *G-vector bundles and the linearization problem,* in Group actions and invariant theory, CMS **10**, 111-123.

[4] H. Kraft: *Geometrische methoden in der invariantentheorie,* Aspekte der Mathematik D1, Vieweg, 1984.

[5] H. Kraft and G. Schwarz: *Reductive group actions with one dimensional quotient,* Publ. Math. IHES **76** (1992), 1-97.

[6] F. Knop: *Nichtlinearisierbare Operationen halbeinfacher Gruppen auf affinen Räumen,* Invent. Math. **105** (1991), 217-220.

[7] M. Masuda and T. Petrie: *Equivariant algebraic vector bundles over representations of reductive groups: Theory,* Proc. Nat. Acad. Sci. USA **88** (1991), 9061-9064.

[8] M. Masuda, L. Moser-Jauslin and T. Petrie: *Equivariant algebraic vector bundles over representations of reductive groups: Applications,* Proc. Nat. Acad. Sci. USA **88** (1991), 9065-9066.

[9] D. Quillen: *Projective modules over polynomial rings,* Invent. Math. **36** (1976), 167-171.

[10] G. Schwarz: *Exotic algebraic group actions,* C.R. Acad. Sci. Paris, Sér. I **309** (1989), 89-94.

[11] A. Suslin: *Projective modules over a polynomial ring,* Soviet Math. Doklady **17** (1976), 1160-1174.



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