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A GENERALIZATION OF A THEOREM OF MILNOR

Dedicated to Professor Seiya Sasao on his 60th birthday

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1. Introduction

We work in the smooth category with free actions by groups in the present paper. Let us recall Milnor's theorem:

Theorem 1.1 ([6; Corollary 12.13]). *Any h-cobordism W between lens spaces L and L' must be diffeomorphic to $L\times[0,1]$ if the dimension of L is greater than or equal to* 5.

Let $\boldsymbol{Z}_m$ be the cyclic group of order m. Then we see that Theorem 1.1 is put in another way as follows:

Theorem 1.2. *Let $S(V)$ and $S(V')$ be free linear $\boldsymbol{Z}_m$-spheres of dimension $2n-1\geqq 5$. Then any $\boldsymbol{Z}_m$-h-cobordism W between $S(V)$ and $S(V')$ must be $\boldsymbol{Z}_m$-diffeomorphic to $S(V)\times I$, where $I=[0,1]$.*

Let R be a ring with unit, G a finite group. Put $GL(R)=\varinjlim GL_n(R)$ and $E(R)=[GL(R),GL(R)]$ the commutator subgroup of $GL(R)$. Then $K_1(R)$ denotes the quotient group $GL(R)/E(R)$. Let $\boldsymbol{Z}$ be the ring of integers and $\boldsymbol{Q}$ the ring of rational munbers. Let $\boldsymbol{Z}[G]$ and $\boldsymbol{Q}[G]$ denote the group rings of G over $\boldsymbol{Z}$ and $\boldsymbol{Q}$. The Whitehead group of G is the quotient group

$$Wh(G)=K_1(\boldsymbol{Z}[G])/<\pm g{:}g\in G>.$$

The natural inclusion map $i{:}GL(\boldsymbol{Z}[G])\to GL(\boldsymbol{Q}[G])$ gives rise to a group homomorphism $i_*{:}K_1(\boldsymbol{Z}[G])\to K_1(\boldsymbol{Q}[G])$. Then $SK_1(\boldsymbol{Z}[G])$ is defined by setting

$$SK_1(\boldsymbol{Z}[G])=\ker[i_*{:}K_1(\boldsymbol{Z}[G])\to K_1(\boldsymbol{Q}[G])].$$

In [15], C.T.C. Wall showed that $SK_1(\boldsymbol{Z}[G])$ is isomorphic to the torsion subgroup of $Wh(G)$. We will apply the following algebraic result to extend Theorem 1.2.

Theorem A. *Let G be a finite group which can act linearly and freely on spheres. Then $SK_1(\mathbf{Z}[G])=0$ if and only if G is isomorphic to one of the following groups.*

(1) *A cyclic group.*
(2) *A group of type* I *in Appendix(a metacyclic group with certain condition).*
(3) *A quaternionic group $\mathbf{Q}(8t)$ with generators B,R and relations $B^{4t}=1$, $B^{2t}=R^2=(BR)^2$, where $t\geqq 1$.*
(4) *A group $\mathbf{Q}(8t,m_1,m_2)$ generated by A, B, R with relations $A^{m_1m_2}=B^{4t}=1$, $BAB^{-1}=A^{-1}$, $R^2=B^{2t}$, $RAR^{-1}=A^l$, $RBR^{-1}=B^{-1}$, where $m_1,m_2\geqq 1$, $m_1m_2>1$, $(m_1,m_2)=1$, $(2t,m_1m_2)=1$, $l\equiv -1(m_1)$, $l\equiv 1(m_2)$.*
(5) *The binary tetrahedral group $\mathbf{T}^*$.*
(6) *A generalized binary octahedral group $\mathbf{O}^*(48t)$ generated by B,P,Q,R with relations $B^{3t}=1$, $P^2=Q^2=(PQ)^2=R^2$, $BPB^{-1}=Q$, $BQB^{-1}=PQ$, $RPR^{-1}=QP$, $RQR^{-1}=Q^{-1}$, $RBR^{-1}=B^{-1}$, where t is odd.*
(7) *The binary icosahedral group $\mathbf{I}^*=SL(2,5)$.*
(8) *The group generated by $SL(2,5)$ and an element S, where $S^2=-1\in SL(2,5), SLS^{-1}=\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix} L \begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}^{-1}$ for $L\in SL(2,5)$.*

We obtain the following applications of Theorem A as generalizations of Theorem 1.2.

Example B. Let G be a finite group in Theorem A. Let X be a free G-homotopy sphere of dimension $2n-1\geqq 5$, and let $S(V)$ and $S(V')$ be free linear G-spheres of dimension $2n-1\geqq 5$. Then,

(1) Any G-h-cobordism W between X and itself must be G-diffeomorphic to $X\times I$.
(2) Any G-h-cobordism W between $S(V)$ and $S(V')$ must be G-diffeomorphic to $S(V)\times I$.

Example C. Let G be a finite group. Let X be a free G-homotopy sphere of dimension $4n+1\geqq 5$, and let $S(V)$ and $S(V')$ be free linear G-spheres of dimension $4n+1\geqq 5$. Then,

(1) Any G-h-cobordism W between X and itself must be G-diffeomorphic to $X\times I$.
(2) Any G-h-cobordism W between $S(V)$ and $S(V')$ must be G-diffeomorphic to $S(V)\times I$.

When G is a compact Lie group of positive dimension, a generalization of Theorem 1.2 is:

Theorem D. *Let G be a compact Lie group of positive dimension which can act freely on spheres. Let X^m and X'^m be free G-homotopy spheres of dimension m, and let $(W;X,X')$ be a G-h-cobordism of a free G-action.*

(1) *If $G = S^1$ and $m=2n-1\geqq 7$, then W must be S^1-diffeomorphic to $X\times I$.*
(2) *If $G= NS^1$ and $m=4n-1\geqq 7$, then W must be NS^1-diffeomorphic to $X\times I$ where NS^1 is the normalizer of S^1 in S^3.*
(3) *If $G=S^3$ and $m=4n-1\geqq 11$, then W must be S^3-diffeomorphic to $X\times I$.*

This paper is organized as follws: Section 2 presents the proof of Theorem A. In section 3 we prove Examples B and C, and state some results on G-h-cobordisms between G-homotopy spheres. We prove Theorem D in section 4. Appendix is devoted to quoting the table of the finite solvable groups which can act linearly and freely on odd dimensional spheres from [16].

2. Proof of Theorem A

First, let G be a finite solvable group which can act linearly and freely on spheres. As in [16; Theorem 6.1.11], there are 4 types for such kinds of groups. For the convenience of the readers, the table of these groups are cited in Appendix. We now recall the structure of $SK_1(\boldsymbol{Z}[G])$ of these groups G. We must prepare the following notations.

Let G_1, G_2, G_3 and G_4 denote the groups of type I, II, III and IV respectively mentioned in the table in Appendix. Let $(a_1,a_2,\cdots,a_\lambda)$ denote the greatest common divisor of integers $\{a_1,a_2,\cdots,a_\lambda\}$, and let m,n,r,l,k,u,v and d be the integers appeared in the definition of G_1, G_2, G_3 and G_4.For positive integers α,β,γ and δ, put

$$M_\beta=(r^\beta-1,m),$$
$$D(\alpha)=\{x\in \boldsymbol{N} \mid x \text{ is a divisor of } \alpha\},$$
$$D(\alpha,\beta)=\{x\in D(\alpha) \mid x \text{ can be divided by } \beta\},$$

$$D(\alpha)^{\delta}_{\gamma}=\{x\in D(\alpha)\mid x\gamma\equiv 0\ (\delta)\},$$
$$D(\alpha,\beta)^{\delta}_{\gamma}=\{x\in D(\alpha,\beta)\mid x\gamma\equiv 0(\delta)\}.$$

If d is an even integer, we put $d'=d/2$, and put

$$\begin{aligned} t(2)=\#\{(\alpha,\beta)\mid \beta\in D(v)^{v}_{k-1},\alpha\in D(M_{2^u\beta}),\\ (\alpha+aM_{2^u\beta})(l-1,r^{n/4}-1)\equiv 0(m)\\ \text{for some integer } a \text{ with } 0\leqq a<m/M_{2^u\beta}\}\\ -\#\bigcup_{\substack{0\leq b<d\\ \lambda=0,1}} D(m)^{m}_{(l-1,r^{n/4}-1,l^{\lambda}r^{b}+1)},\end{aligned}$$

$$\begin{aligned} t'(2)=\#\{(\alpha,\beta)\mid \beta\in D(v)^{v}_{k-1},\alpha\in D(M_{2^u\beta}),\\ (\alpha+aM_{2^u\beta})(l-1,r^{n/4}-1)\equiv 0(m) \text{ or}\\ (\alpha+aM_{2^u\beta})(lr^{d'}-1,r^{n/4}-1)\equiv 0(m)\\ \text{for some integer } a \text{ with } 0\leqq a<m/M_{2^u\beta}\}\\ -\#\bigcup_{\substack{0\leq b<d\\ \lambda=0,1}} (D(m)^{m}_{(l-1,r^{n/4}-1,l^{\lambda}r^{b}+1)}\bigcup D(m)^{m}_{(lr^{d'}-1,r^{n/4}-1,l^{\lambda}r^{b}+1)}),\end{aligned}$$

$$t(3)=\sum_{\beta\in D(n,3)}\#D(M_{\beta})-1,$$

$$t(4)=\sum_{\beta\in D(n,3)}\#D(M_{\beta})-\sum_{\beta\in D(n,3)^{n}_{k+1}}\#D(M_{\beta})^{m}_{l+1}.$$

Then we have:

Theorem 2.1 ([12; Theorem]). *Let G_1, G_2, G_3 and G_4 denote the groups of type* I, II, III *and* IV *respectively.*

(1) $SK_1(\boldsymbol{Z}[G_1])=0.$

(2) $SK_1(\boldsymbol{Z}[G_2])\cong \boldsymbol{Z}_2^{t(2)}$ *if d is an odd integer,*
$SK_1(\boldsymbol{Z}[G_2])\cong \boldsymbol{Z}_2^{t'(2)}$ *if d is an even integer.*

(3) $SK_1(\boldsymbol{Z}[G_3])\cong \boldsymbol{Z}_2^{t(3)}.$

(4) $SK_1(\boldsymbol{Z}[G_4])\cong \boldsymbol{Z}_2^{t(4)}.$

By Theorem 2.1, we get (1) and (2) of Theorem A. Let G_2^1 be a group G_2 such that d is odd. At first, we determine the group G_2^1 satisfying $SK_1(\boldsymbol{Z}[G_2^1])=0$. Put

$$\mathscr{T}_{+}=\{(\alpha,\beta)\mid \beta\in D(v)^{v}_{k-1},\alpha\in D(M_{2^u\beta}),$$

$(\alpha+aM_{2^u\beta})(l-1,r^{n/4}-1)\equiv 0(m)$
for some integer a with $0\leqq a<m/M_{2^u\beta}\}$,

and

$$\mathscr{T}_-=\{(\alpha,v)\mid \alpha\in \bigcup_{\substack{0\leq b<d\\ \lambda=0,1}} D(m)^m_{(l-1,r^{n/4}-1,l^\lambda r^b+1)}\}.$$

By [12; §3], $t(2)$ the 2-rank of $SK_1(\boldsymbol{Z}[G_2^1])$ is calculated by

$$t(2)=\sharp\mathscr{T}_+-\sharp\mathscr{T}_-.$$

It is easy to see that $\mathscr{T}_-$ is a subset of $\mathscr{T}_+$. Suppose that $t(2)=0$. Then it is necessary that $D(v)^v_{k-1}=\{v\}$. In fact, if there exists an element β of $D(v)^v_{k-1}$ which is different from v, we see that the ordered pair of numbers $(M_{2^u\beta},\beta)$ is in $\mathscr{T}_+$, but is not in $\mathscr{T}_-$. Hence, if β in $D(v)$ satisfies $\beta(k-1)\equiv 0\ (v)$, it must be equal to v. Thus we have $(k-1,v)=1$. Since $k^2\equiv 1\ (n)$ and $k\equiv -1\ (2^u)$, it holds that $k\equiv -1\ (n)$. Since d is a divisor of $k-1$ and d is odd , by [12; Observation 3.1] $(k-1,v)$ is divisible by d. Hence we have $d=1$, thereby $r\equiv 1\ (m)$. By using $(n(r-1),m)=1$, we get $m=1$, that is, A is equal to the identity element of G_2^1. Thus if $SK_1(\boldsymbol{Z}[G_2^1])=0$, G_2^1 must be isomorphic to a group of order $2n$ which is generated by the elements of the form B and R, and which has relations:

$$B^n=1,\ R^2=B^{n/2},\ RBR^{-1}=B^{-1},$$

where n is a number of the form $2^u v$ for some $u\geqq 2$, $(v,2)=1$, $v\geqq 1$. Conversely, we can easily check that SK_1 for this group vanishes. By putting $t=n/4$, we have (3) of Theorem A.

Let G_2^0 be a group G_2 such that d is even. Next, we determine the group G_2^0 satisfying $SK_1(\boldsymbol{Z}[G_2^0])=0$. Since d is even, we have $m>1$. Put

$\mathscr{T}'_+=\{(\alpha,\beta)\mid \beta\in D(v)^v_{k-1},\ \alpha\in D(M_{2^u\beta}),$
$(\alpha+aM_{2^u\beta})(l-1,\ r^{n/4}-1)\equiv 0(m)$ or
$(\alpha+aM_{2^u\beta})(lr^{d'}-1,\ r^{n/4}-1)\equiv 0(m)$
for some integer a with $0\leqq a<m/M_{2^u\beta}\}$,

and

$$\mathscr{T}'_-=\{(\alpha,v)\mid \alpha\in \bigcup_{\substack{0\leq b<d\\ \lambda=0,1}} (D(m)^m_{(l-1,r^{n/4}-1,l^\lambda r^b+1)}\bigcup D(m)^m_{(lr^{d'}-1,r^{n/4}-1,l^\lambda r^b+1)})\}.$$

By [12; §3], $t'(2)$ the 2-rank of $SK_1(\boldsymbol{Z}[G_2^0])$ is calculated by

$$t'(2)=\#\mathscr{T}'_+-\#\mathscr{T}'_- .$$

It is easy to see that $\mathscr{T}'_-$ is a subset of $\mathscr{T}'_+$. Then by the same argument as before, we have $D(v)^v_{k-1}=\{v\}$ and $(k-1,v)=1$. Since $k^2\equiv 1$ (n) and $k+1\equiv 0$ (2^u), it holds that $k\equiv -1$ (n). Since d is even, by [12; Observation 3.1], $d'=d/2$ is a divisor of $(k-1,v)$. Hence, we have $d=2$, thereby $r\not\equiv 1$ (m) and $r^2\equiv 1$ (m). Now we claim that $r\equiv -1$ (m). In fact, since $(r+1)(r-1)\equiv 0$ (m) and $(r-1,m)=1$, it holds that $r+1\equiv 0$ (m) or $m=1$. However, it must hold $r\equiv -1$ (m) because $m>1$. Therefore, we have

$$(lr^{d'}-1,\ r^{n/4}-1)=(l+1,(-1)^{n/4}-1).$$

Thus, for $\#\mathscr{T}'_+=\#\mathscr{T}'_-$, it is necessary that

$$\begin{aligned}&\#\{\alpha\in D(m)\,|\,\alpha(l-1,\ (-1)^{n/4}-1)\equiv 0\ (m)\\ &\qquad\qquad \text{or } \alpha(l+1,\ (-1)^{n/4}-1)\equiv 0\ (m)\}\\ &=\#\bigcup_{\substack{b=0,1\\ \lambda=0,1}}(D(m)^m_{(l-1,(-1)^{n/4}-1,(-1)^b l^\lambda+1)}\bigcup D(m)^m_{(l+1,(-1)^{n/4}-1,(-1)^b l^\lambda+1)}).\end{aligned}$$

However, we can easily check that this formula always holds. Thus, if $SK_1(\boldsymbol{Z}[G_2^0])=0$, G_2^0 must be isomorphic to a group of order $2n$ which is generated by the elements of the form A, B and R, and which has relations:

$$\begin{aligned}&A^m=B^n=1,\ BAB^{-1}=A^{-1},\\ &R^2=B^{n/2},\ RAR^{-1}=A^l,\ RBR^{-1}=B^{-1},\end{aligned}$$

where m,n and l satisfy the following conditions:

$$\begin{aligned}&m>1,\ (n,m)=1,\ l^2\equiv 1\ (m),\\ &n=2^u v(u\geqq 2,\ (v,2)=1,\ v\geqq 1).\end{aligned}$$

Conversely, we can easily check that SK_1 for this group vanishes. Now, we put $t=n/4$. Since $l^2\equiv 1(m)$, there exist two integers m_1 and m_2 such that $m=m_1m_2$, $(m_1,m_2)=1$, $l\equiv -1(m_1)$, and $l\equiv 1(m_2)$. Conversely if we write $m=m_1m_2$ where $(m_1,m_2)=1$, there exists an integer l uniquely modulo m such that $l\equiv -1(m_1)$ and $l\equiv 1(m_2)$. We denote this group by $\mathbf{Q}(8t,m_1,m_2)$ (This notation is based on [11]). Thus we get (4) of

Theorem A.

Next, we determine the group G_3 satisfying $SK_1(\boldsymbol{Z}[G_3])=0$. Assume that

$$t(3)=\sum_{\beta\in D(n,3)} \sharp D(M_\beta)-1=0.$$

Since $\sharp D(M_\beta)\geqq 1$ for every $\beta\in D(n,3)$, it is necessary that $\sharp D(n,3)=1$. Hence, n must be 3, thereby d is 1 or 3. However, if $d=3$, n/d is not divisible by 3. Hence d must be equal to 1, thereby $r\equiv 1(m)$. By using $(n(r-1),m)=1$, we have $m=1$, that is, A is equal to the identity element of G_3. Thus, if $SK_1(\boldsymbol{Z}[G_3])=0$, G_3 must be isomorphic to a group of order 24 which is generated by the elements of the form B,P and Q, and which has relations:

$$B^3=1,\ P^2=Q^2=(PQ)^2,\ BPB^{-1}=Q,\ BQB^{-1}=PQ.$$

This group is the binary tetrahedral group $\mathbf{T}^*$. Conversely, we can easily see that $SK_1(\boldsymbol{Z}[\mathbf{T}^*])=0$. This proves (5) of Theorem A.

Next, we determine the group G_4 satisfying $SK_1(\boldsymbol{Z}[G_4])=0$. Suppose that

$$t(4)=\sum_{\beta\in D(n,3)} \sharp D(M_\beta)-\sum_{\beta\in D(n,3)^n_{k+1}} \sharp D(M_\beta)^m_{l+1}=0.$$

Then it is necessary that $D(n,3)=D(n,3)^n_{k+1}$. In fact, if there exists an element β_0 of $D(n,3)-D(n,3)^n_{k+1}$, since $\sharp D(M_{\beta_0})\geqq 1$, we have $t(4)\neq 0$. Hence, for every element β in $D(n,3)$, it must hold that $\beta(k+1)\equiv 0$ (n). In particular, we have $3(k+1)\equiv 0(n)$. Thus k must satisfy $k+1\equiv 0(n/3)$. We claim that $k\equiv -1(n)$. In fact, if k is congruent to $n/3-1$ or $2n/3-1$ modulo n, the conditions $k+1\equiv 0(3)$ and $n\equiv 0$ (3) imply $n\equiv 0$ (9), but it is a contradiction to the condition $k^2\equiv 1(n)$. Therefore we have

$$r^{k-1}\equiv r^{n-2}\equiv r^n\equiv 1\ (m),$$

which implies d is a divisor of $(n-2,n)$. Since a group G_4 has odd n, we have $d=1$. By the same argument as above, we have $m=1$, that is, A is equal to the identity element of G_4. Thus if $SK_1(\boldsymbol{Z}[G_4])=0$, G_4 must be a group of order $16n$ which is generated by the elements of the form B, P, Q and R, and which has relations:

$$B^n=1,\ P^2=Q^2=(PQ)^2=R^2,$$
$$BPB^{-1}=Q,\ BQB^{-1}=PQ,\ RPR^{-1}=QP,$$
$$RQR^{-1}=Q^{-1},\ RBR^{-1}=B^{-1},$$

where n is divisible by 3, but is not divisible by 2. This group is the generalized binary octahedral group $\mathbf{O}^*(48t)$. Conversely, we can easily check that $SK_1(\boldsymbol{Z}[\mathbf{O}^*(48t)])=0$. This proves (6) of Theorem A.

Next, we consider the case that G is non-solvable.

Lemma 2.2 ([16; 6.3.1 Theorem]). *Let G be a finite non-solvable group. If G has a fixed point free representation, then G is one of the following two types.*

TYPE V. *$G=K\times SL(2,5)$ where K is a solvable fixed point free group of type* I *in Appendix and order prime to* 30.

TYPE VI. *$G=<G_5,S>$ where $G_5=K\times SL(2,5)$ is a normal subgroup of index* 2 *and type* V, $S^2=-1\in SL(2,5)$, $SLS^{-1}=\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}L\begin{pmatrix}0 & -1\\ 2 & 0\end{pmatrix}^{-1}$ *for $L\in SL(2,5)$, and S normalizes K.*

Let G be a finite group of type V or VI. For an odd prime p, since p-Sylow subgroups of G are cyclic, $SK_1(\boldsymbol{Z}[G])_{(p)}=0$. Hence by [7;Theorem 3], $SK_1(\boldsymbol{Z}[G])$ is generated by induciton from 2-elementary subgroups of G, that is, $SK_1(\boldsymbol{Z}[G])=0$ if and only if G has not a subgroup which is isomorphic to $\Gamma\times S_2$ where Γ is a cyclic group of order prime to 2 and S_2 is a 2-group. In these cases, $SK_1(\boldsymbol{Z}[G])=0$ if and only if G has not a subgroup of the form $\Gamma\times\mathbf{Q}_8$ (see [5]). Hence K must be $\{1\}$ which proves (7) and (8) of Theorem A.

3. G-h-cobordisms between G-homotopy spheres

Let $Wh(G)$ be the Whitehead group of G, $L_m^s(G)$ and $L_m^h(G)$ the Wall groups (for the Wall groups, see [2], [14]). $\boldsymbol{Z}[G]$ is the integral group ring with involution $-$ defined by $\overline{\Sigma a_g g}=\Sigma a_g g^{-1}$ where $a_g\in\boldsymbol{Z}$ and $g\in G$. For a matrix (x_{ij}) with coefficients in $\boldsymbol{Z}[G]$, $\overline{(x_{ij})}$ is defined by $(\overline{x_{ji}})$. Then $Wh(G)$ has the induced involution also denoted by $-$. We define a subgroup $\tilde{A}_m(G)$ of $Wh(G)$ by

$$\tilde{A}_m(G)=\{\tau\in Wh(G)\,|\,\bar{\tau}=(-1)^m\tau\},$$

and put

$$A_m(G)=\tilde{A}_m(G)/\{\tau+(-1)^m\bar{\tau}\mid\tau\in Wh(G)\}.$$

Let $c{:}A_{2n+1}(G)\to L^s_{2n}(G)$ be the map in the Rothenberg exact sequence

$$\cdots\to A_{2n+1}(G)\xrightarrow{c}L^s_{2n}(G)\xrightarrow{d}L^h_{2n}(G)\to\cdots,$$

and $\tilde{c}$: $\tilde{A}_{2n+1}(G)\to L^s_{2n}(G)$ the map determing c (for this exact sequence, see [8; Proposition 4.1]).

Proposition 3.1. *Let G be a finite group such that $SK_1(\boldsymbol{Z}[G])=0$. Then the following hold*:

(1) *If X is a free G-homotopy sphere of dimension $2n-1\geqq 5$, any G-h-cobordism W between X and itself must be G-diffeomorphic to $X\times I$.*

(2) *If $S(V)$ and $S(V')$ are free linear G-spheres of dimension $2n-1\geqq 5$, any G-h-cobordism W' between $S(V)$ and $S(V')$ must be G-diffeomorphic to $S(V)\times I$.*

Proof. (1) In the case $|G|\leqq 2$, since it holds that $Wh(G)=0$, the conclusion follows from the s-cobordism theorem. Our proof will be done under $|G|\geqq 3$. Let W be a G-h-cobordism between X and itself, with dim $W=2n\geqq 6$. To distinguish the inclusions of X to W, we put $\partial W=X\amalg X'$, where X' is a copy of X. Let $i{:}X\to W$ and $i'{:}X'\to W$ be the natural inclusion maps. Let r be a G-homotopy inverse of i. Since the order of G is greater than or equal to 3 and G acts freely on a homotopy sphere X with $\dim X\geqq 5$, any G-self-homotopy equivalence of X is G-homotopic to the identity map. Hence, we have

$$\tau(r\circ i')=\tau(id)=0.$$

On the other hand,

$$\begin{aligned}\tau(r\circ i')&=\tau(r)+r_*\tau(i')\\&=-r_*\tau(i)+r_*\tau(i')\\&=r_*(\tau(i')-\tau(i)).\end{aligned}$$

Thus we have $\tau(i')=\tau(i)$, that is,

$$\tau(W,X)=\tau(W,X').$$

By the duality theorem ([6; p. 394]), we also get

$$\tau(W,X')=-\overline{\tau(W,X)}.$$

Hence by these formulae, we see that $\tau=-\bar{\tau}$, that is, τ is an element of $\tilde{A}_{2n+1}(G)$.

Since G has periodic cohomolgy, $\tilde{A}_{2n+1}(G)$ is isomorphic to $SK_1(\boldsymbol{Z}[G])$ by [9; Theorem 3]. Hence we have $\tilde{A}_{2n+1}(G)=0$, thereby $\tau=0$.

(2) Let C be a cyclic subgroup of G. By Theorem 1.2, $\mathrm{res}_c V=\mathrm{res}_c V'$ as real C-modules. Thus $V=V'$ as real G-modules, and then $S(V')$ is G-diffeomorphic to $S(V)$. Since $SK_1(\boldsymbol{Z}[G])=0$, the conclusion now follows from (1) of this proposition. □

Proof of Examples. Example B follows from Theorem A and Proposition 3.1 immediately. By [10], if a finite group G whose 2-Sylow subgroups are quaternionic acts freely on spheres, its dimension must be $4n-1(n\in\boldsymbol{N})$. Hence, if a finite group G can act freely on spheres of dimension $4n+1$, the 2-Sylow subgroups of G are cyclic. Thus G must be of Type I in Appendix, thereby $SK_1(\boldsymbol{Z}[G])=0$, which proves Example C.

In [13], we studied G-h-cobordisms between G-homotopy spheres and obtained the following results:

Theorem 3.2 ([13; Theorem A]).*Let G be a finite group, and X a free G-homotopy sphere of dimension $2n-1\geqq 5$. Then the following* (1) *and* (2) *are equivalent.*

(1) *Any G-h-cobordism W between X and itself must be G-diffeomorphic to $X\times I$.*
(2) $\ker\tilde{c}$ *is trivial.*

Corollary 3.3 ([13; Corollary B]). *Suppose* $\ker\tilde{c}=0$. *Let $S(V)$ and $S(V')$ be free linear G-spheres of dimension $2n-1\geqq 5$. Then a G-h-cobordism W between $S(V)$ and $S(V')$ must be G-diffeomorphic to $S(V)\times I$.*

Theorem 3.2 is shown by using surgery theory. Corollary 3.3 is an immediate consequence of Theorem 3.2. Since by [9; Theorem 3] $SK_1(\boldsymbol{Z}[G])\cong\tilde{A}_{2n+1}(G)$ for a periodic group G, Proposition 3.1 is a special case of Theorem 3.2 and Corollary 3.3. Moreover, as in [13], there exists a finite group G such that $SK_1(\boldsymbol{Z}[G])\neq 0$ and $\ker\tilde{c}=0$. For example, let p be an odd prime, q a prime such that $q\geqq 5$. Let G be $\mathbf{Q}_8\times\boldsymbol{Z}_p$, $\mathbf{T}^*\times\boldsymbol{Z}_q$, or $\mathbf{O}^*\times\boldsymbol{Z}_q$, where $\mathbf{Q}_8$, $\mathbf{T}^*$, and $\mathbf{O}^*$ denote the quaternionic group, the binary tetrahedral group, and the binary octahedral group respectively. Then we see that $SK_1(\boldsymbol{Z}[G])\cong \boldsymbol{Z}_2$ and any G-h-cobordism W between a free G-homotopy sphere X of dimension $4n-1\geqq 7$ and itself must be G-diffeomorphic to $X\times I$, because $\ker\tilde{c}=0$.

4. Proof of Theorem D

Let G be a compact Lie group of positive dimension which can act freely on a sphere. Then by [3; p. 153, Theorem 8.5], G must be isomorphic to S^1, S^3 or NS^1 the normalizer of S^1 in S^3. If G is S^1, the dimension of a sphere on which G acts freely is $2n-1(n\geqq 1)$. If G is NS^1 or S^3, it is $4n-1(n\geqq 1)$ because G has a subgroup which is isomorphic to $\mathbf{Q}_8$. Now we recall the equivariant Whitehead group which is defined by S. Illman. By [4; Corollary 2,8],

$$Wh_{S^1}(X^m)\cong Wh(1)=0 \quad \text{where} \quad m=2n-1\geqq 7,$$
$$Wh_{NS^1}(X^m)\cong Wh(\boldsymbol{Z}_2)=0 \quad \text{where} \quad m=4n-1\geqq 7,$$
$$Wh_{S^3}(X^m)\cong Wh(1)=0 \quad \text{where} \quad m=4n-1\geqq 11.$$

Thus $(W;X,X')$ is a G-s-cobordism in the sense of [1]. The conclusion now follows from the conditions about the dimension of the homotopy sphere by using[1; Theorem 1].

5. Appendix

Let G be a finite solvable group. Then G has a fixed point free complex representation if and only if G is of type I, II, III, IV below, with the additional condition: if d is the order of r in the multiplicative group of residues modulo m, of integers prime to m, then n/d is divisible by every prime divisor of d.

TYPE I. A group of order mn that is generated by the elements of the from A and B, and that has relations:

$$A^m=B^n=1,\ BAB^{-1}=A^r,$$

where m,n and r satisfy the following conditions:

$$m\geq 1,n\geq 1,(n(r-1),m)=1,r^n\equiv 1(m).$$

TYPE II. A group of order $2mn$ that is generated by the elements of the form A, B and R, and that has relations:

$$R^2=B^{n/2},\ RAR^{-1}=A^l,\ RBR^{-1}=B^k$$

in addition to the relations in I, where m, n, r, l and k satisfy the following conditions:

$$l^2 \equiv r^{k-1} \equiv 1(m),\ k \equiv -1(2^u),$$
$$n = 2^u v(u \geqq 2, (v,2) = 1),\ k^2 \equiv 1(n)$$

in addition to the conditions in I.

Type III. A group of order $8mn$ that is generated by the elements of the form A,B,P and Q, and that has relations:

$$P^2 = Q^2 = (PQ)^2,\ AP = PA,\ AQ = QA,$$
$$BPB^{-1} = Q,\ BQB^{-1} = PQ$$

in addition to the relations in I, where m, n and r satisfy the following conditions:

$$n \equiv 1(2),\ n \equiv 0(3)$$

in addition to the conditions in I.

Type IV. A group of order $16mn$ that is generated by the elements of the form A,B,P,Q and R, and that has relations:

$$R^2 = P^2,\ RPR^{-1} = QP,\ RQR^{-1} = Q^{-1},$$
$$RAR^{-1} = A^l,\ RBR^{-1} = B^k$$

in addition to the relations in III, where m, n, r, k and l satisfy the following conditions:

$$l^2 \equiv r^{k-1} \equiv 1(m),\ k \equiv -1(3),\ k^2 \equiv 1(n)$$

in addition to the conditions in III.

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