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Z/kZ-FINITENESS FOR CERTAIN S1-SPACES

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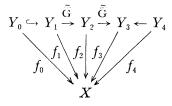
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Introduction

Let $G - \mathcal{FDCW}$ denote the category of G-spaces having the G-homotopy type of a finitely dominated G - CW complex for a compact Lie group G. Lück [8] has introduced a functor Wa^{c} from $G - \mathcal{FDCW}$ into the category of abelian groups and has realized the equivariant finiteness obstriction as the element $w^{c}(X)$ in $Wa^{c}(X)$. That is, a finitely dominated G - CW complex X is G-homotopy equivalent to a finite G - CW complex if and only if $w^{c}(X) = 0$. When G is the trivial group, there is an isomorphism from $Wa^{c}(X)$ to the reduced projective group $\tilde{K}_{0}(\mathbb{Z}[\pi_{1}(X)])$ which sends the element $w^{c}(X)$ to the Wall's finiteness obstruction ([14]).

Anderson [1] and Ehrlich [4] have studied a sufficient condition for $w^{(1)}(E) = 0$ for some fibration $E \to B$ with fiber S^1 . Munkholm, Pedresen [11], Lück [6, 7, 9] and others have studied the transfer map $\tilde{K}_0(\mathbb{Z}[\pi_1(B)]) \to \tilde{K}_0(\mathbb{Z}[\pi_1(E)])$. The purpose of this paper is to get a sufficient condition for $w^L(X) = 0$ for a S^1 -space X and a finite cyclic group L.

We call G-maps $f_0: Y_0 \rightarrow X$ and $f_0: Y_4 \rightarrow X$ equivalent if there exists a commutative diagram



such that (Y_1, Y_0) and (Y_3, Y_4) are relatively finite G-CW complexes, and $Y_1 \rightarrow Y_2$ and $Y_3 \rightarrow Y_2$ are G-homotopy equivalences. The group $Wa^c(X)$ consists of equivalence classes $[f: Y \rightarrow X]$ of the set of G-maps $f: Y \rightarrow X$ with Y finitely dominated and $w^c(X)$ is the equivalence class containing the identity 1_X of X. The additive structure on $Wa^c(X)$ is given by a disjoint sum:

$$[f: Y \to X] + [g: Z \to X] = [f \coprod g: Y \coprod Z \to X]$$

Let K be a closed subgroup of G. For a K-space X, we define $\operatorname{ind}_{K}^{G}X$ as the orbit space $G \times_{K} X$ of the product space $G \times X$ with respect to the K-action $k \cdot (g, x) = (gk^{-1}, kx)$. For a K-map $f: X \to Y$, we have an induced map $1 \times_{K} f: G \times_{K} X \to G \times_{K} Y$, denoted by $\operatorname{ind}_{K}^{G} f$. The induction functor $\operatorname{ind}_{K}^{G}$ induces a transformation $\operatorname{Ind}_{K}^{G}: (Wa^{K}, w^{K}) \to (Wa^{G}, w^{C})$. To consider G-maps as K-maps implies a transformation $\operatorname{Res}_{K}^{G}: (Wa^{G}, w^{C}) \to (Wa^{K}, w^{K})$.

Throughout this paper we denote by ν the restriction of the G-action of X to $G \times \{x_0\}$. If $G = S^1$ and X is connected, the order of the image of $H_1(\nu; \mathbb{Z})$ is independent of taking a point x_0 of X.

Our main results are as follows:

Theorem A. Let X be a connected S^1 -CW complex which has finitely many orbit types. Suppose that the S^1 -map $\nu: S^1 \to X$ defined as above induces a monomorphism $H_1(\nu; \mathbb{Z})$ between 1-dimensional homology groups. Then there exist a proper subgroup K of S^1 and a K-CW complex Y such that $S^1 \times_K Y$ is S^1 -homotopy equivalent to X.

Theorem B. Let $G=S^1$ and let X be a connected G-space. If the above defined G-map $\nu: G \to X$ induces an injective homomorphism $H_1(\nu; \mathbb{Z})$, then the restriction homomorphism $\operatorname{Res}^{G}_{H}(X): Wa^{G}(X) \to Wa^{H}(X)$ is trivial for any proper subgroup H of G.

This paper is organized as follows. Let $F \to E \xrightarrow{p} B$ be a *G*-fibration with fibre *F* ([15]). In [13], we have constructed a transfer $p^1: Wa^G(B) \to Wa^G(E)$. But this homomorphism does not always send $w^G(B)$ to $w^G(E)$. It is originated from that (Wa^G, w^G) is not a functionial additive invariant for $G - \mathcal{FDCW}$. In section 1 we study a *K*-*CW* structure on $p^{-1}(1K)$ for a *G*-*CW* complex *E* which has a *G*-map $E \to G/K$. In section 2 we show that if $G \times_K X$ has the *G*-homotopy type of a finite *G*-*CW* complex then *X* has the *K*-homotopy type of a finite *K*-*CW* complex. In section 3, we prove Theorem A in the case where *X* is free. We use the fact that $\pi_1(X/K)$ has the subgroup $\pi_1(S^1/K)$ as a direct summand for some closed subgroup *K* of *G*. The last section consists of the proof of the main theorems. The proof of Theorem A is obtained from applying the free case.

1. K-CW structure on X of a G-CW complex $G \times_{K} X$

Let G be a compact Lie group. We study a space $p^{-1}(\{pt\})$ for a G-map p from a G-space onto an orbit space G/K of G. We note that it has a canonical K-action.

Proposition 1.1. (cf.[12]) A G-map $p: E \rightarrow B$ is a G-fibration if and only if $p^{\kappa}: E^{\kappa} \rightarrow B^{\kappa}$ is a fibration for any closed subgroup K of G.

Proof. This follows essentially from Theorem 4.1 in [2].

Since $Y \to G \times_{\kappa} Y \to G/K$ is a G-fibration ([15, 13]), we have that $Y^{L} \to (G \times_{\kappa} Y)^{L} \to (G/K)^{L}$ is a fibration for any closed subgroup L of G.

We symbolize 1 as the identity element of G. For any G-map $p: X \rightarrow G/K$ it is a G-fibration with fibre $p^{-1}(1K)$. The following lemma is a key to show the main theorems. It implies that a G-map $p: X \rightarrow G/K$ with X a G-CW complex is G-homotopy equivalent to a G-fibration whose fibre is a K-CW complex.

Lemma 1.2. Let X be a G-CW complex which has a G-map $p: X \rightarrow G/K$. A K-CW complex can be constructed from the G-CW structure of X such that the K-space $V=p^{-1}(1K)$ is homotopy equivalent to it. In particular it is a finite K-CW complex if X is a finite G-CW complex.

Proof. Clearly we have $G \times_{\kappa} V$ and X are G-homeomorphic. Then we construct a K-CW complex W and a K-homotopy equivalence $W \to V$ by induction on the dimension of cells of X. By the existence of the G-map p, we obtain that L is subconjugate to K for any isotropy subgroup L of G in X. We can regard a 0-cell $G/L \times e^0$ of X as $G \times_{\kappa} K/aLa^{-1} \times e^0$ for $a \in G$ with $aLa^{-1} \leq K$. Suppose that $X = G \times_{\kappa} Y \cup_{\phi} G/L \times e^n$ for some K-CW complex Y. Let C be a connected component of $(G/K)^L$ which contains $p^L \circ \phi(1L \times e^n)$. Take $aK \in C$ and let $\psi: G/aLa^{-1} \to G/L$ be the canonical G-map. Then the pushout of

$$G|aLa^{-1} \times \dot{e}^n \hookrightarrow G|aLa^{-1} \times e^n$$

$$\downarrow \phi \circ (\psi \times 1)$$

$$G \times _{\mathbf{r}} Y$$

is G-homotopy equivalent to X. Then we can assume that $L \leq K$ and $p^L \circ \phi(1L \times \dot{e}^n)$ is contained in the connected component of 1K.

Since the map $p^{L} \circ \phi|_{1L \times i^n}$ is homotopic to a constant map, there is a map $\sigma: 1L \times e^n \to (G/K)^L$ such that σ coincides with $p^L \circ \phi$ over $1L \times e^n$ and $\sigma(0) = 1K$. We define a map $\tau: 1L \times e^n \times I \to (G/K)^L$ as $\tau(s, t) = \sigma((1-t)s)$. Since $(G \times_K Y)^L \to (G/K)^L$ is a fibration with fibre Y^L , there exists a homotopy $F: 1L \times e^n \times I \to (G \times_K Y)^L$ such that $F_0 = \phi|_{1L \times i^n}$ and $F_1(1L \times i^n) \subseteq Y^L$. This map can be canonically extended to a G-map Φ from $G/L \times i^n \times I$ to $G \times_K Y$. Let W be a K-CW complex obtained from the following pushout.

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$$K|L imes e^n \hookrightarrow K|L imes e^n \ igcup_1 \ Y$$

By the property of pushout, we get a K-map $k: W \rightarrow V$.

Since the G-map $\operatorname{ind}_{K}^{G}k$ is a G-homotopy equivalence, we have the K-map k is a K-homotopy equivalence.

Theorem 1.3. Let $f: Y \to X$ be a K-map between K-CW complexes and $(V, G \times_{\kappa} Y)$ be a G-CW pair. If there is a G-map $g: V \to G \times_{\kappa} X$ which is an extension of the G-map $1 \times_{\kappa} f$, then there exists a K-map $k: W \to X$ unique up to K-homotopy equivalence which fullfills the following conditions.

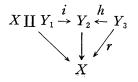
- (1) (W, Y) is a K-CW pair.
- (2) There is a G-homotopy equivalence $h: (G \times_{\kappa} W, G \times_{\kappa} Y) \rightarrow (V, G \times_{\kappa} Y)$ such that $\operatorname{ind}_{\kappa}^{c} k$ and $g \circ h$ are G-homotopic.
- (3) The number of the relative cells of (W, Y) equals that of $(V, G \times_{\kappa} Y)$.

2. Induction homomorphism

Let $D^{c}(X)$ be the set of equivalence classes of the set of G-maps $f: Y \to X$ where Y has the G-homotopy type of a G-CW complex. Here the equivalence relation is defined as in introduction. For a G-map $f: Y \to X$, we denote by $[f: Y \to X]$ its represented element of $D^{c}(X)$. The additive structure on $D^{c}(X)$ is given as the one of $Wa^{c}(X)$. A G-map from a finite G-CW complex to X represents the zero element of $D^{c}(X)$. Then $D^{c}(X)$ is a semigroup and we obtain a map $Wa^{c}(X) \to D^{c}(X)$ which preserves the abelian structures.

Lemma 2.1. The element of $D^{G}(X)$ represented by the identity map of X is invertible if and only if X is finitely dominated.

Proof. The "if" part is trivial and then we show the "only if" part. There is the commutative diagram



such that $(Y_2, X \coprod Y_1)$ is a relatively finite *G*-*CW* complex, Y_3 is a finite *G*-*CW* complex, and *h* is a *G*-homotopy equivalence. Let $h^{-1}: Y_2 \rightarrow Y_3$ be the *G*-homotopy inverse of *h*. Then *r* is a domination with section $h^{-1} \circ i$.

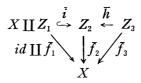
Proposition 2.2. Let K be a closed subgroup of G. A K-space X is a finitely dominated K-space if and only if $G \times_{\kappa} X$ is a finitely dominated G-space.

Proof. Suppose $G \times_{\kappa} X$ is dominated by a finite G-CW complex Y_3 . There is a commutative diagram such that f_3 is a domination with section $h^{-1}ij$.

$$\begin{array}{c} G \times_{\kappa} X \stackrel{j}{\hookrightarrow} (G \times_{\kappa} X) \coprod Y_{1} \stackrel{i}{\hookrightarrow} \stackrel{h}{Y_{2}} \stackrel{h}{\leftarrow} Y_{3} \\ id \coprod f_{1} & \downarrow f_{2} / f_{3} \\ G \times_{\kappa} X \end{array}$$

We let $Z_1 = (pf_1)^{-1}(1K)$, $Z_2 = (pf_2)^{-1}(1K)$, and $Z_3 = (pf_3)^{-1}(1K)$ for short, where $p: G \times_K X \to X$ is the canonical projection. We have G-homeomorphisms h_I $(l=1, \dots, 3)$ such that the following diagram is commutative.

By taking the Z_l 's, the G-maps f_l induce K-maps $\overline{f}_l: Z_l \to X$ $(l=1, \dots, 3)$. Since \overline{h} is a K-homotopy equivalence and the diagram



commutes, Z_3 dominates X. By Lemma 1.2, Z_3 has the K-homotopy type of a finite K-CW complex. This completes the proof.

Let $\Phi: D^{G}(G \times_{\kappa} X) \to D^{K}(X)$ be a homomorphism induced by a mapping assigning $k: W \to X$, described as in Theorem 1.3, to any G-map $g: V \to G \times_{\kappa} X$. It is an inverse isomorphism of a homomorphism $D^{K}(X) \to D^{G}(G \times_{\kappa} X)$ induced by $\operatorname{ind}_{K}^{G}$. Since $G \times_{\kappa} W \cong_{G} V$, it follows from Proposition 2.2 that $\Phi(Wa^{G}(G \times_{\kappa} X)) \subset Wa^{K}(X)$. Then we have: T. Sumi

Theorem 2.3. Let K be any closed subgroup of G and let X be a K-space. The induction homomorphism $\operatorname{Ind}_{K}^{c}(X)$: $Wa^{\kappa}(X) \rightarrow Wa^{c}(G \times_{\kappa} X)$ is an isomorphism. In particular $G \times_{\kappa} X$ has the G-homotopy type of a finite G-CW complex if and only if X has the K-homotopy type of a finite K-CW complex.

3. Free S¹-spaces

In this section we study Theorem A for free S^1 -spaces. We denote a group S^1 by G and let X be a connected free G-CW complex such that $H_1(\nu; \mathbb{Z})$ is injective. If the projection $X \to X/G$ is a principal G-bundle and the fundamental group of X is abelian, Anderson [1] has shown that the universal cover of X is $\pi_1(X/G)$ -homeomorphic to the product of the universal cover of X/G and the real space \mathbb{R} with some $\pi_1(X/G)$ -action on \mathbb{R} . We show that for some $K \leq G$ and some CW complex V, the G-space X/K is G-homotopy equivalent to $G/K \times_{\{1K\}} V = G/K \times V$.

Lemma 3.1. Let X be a connected free G-space such that the G-map ν : $G \rightarrow X$ induces a monomorphism $H_1(\nu; \mathbb{Z})$. There is a finite subgroup K of G such that $\pi_1(X|K)$ isomorphic to $\pi_1(G|K) \oplus \pi_1(X|G)$.

Proof. For any $K \leq G$, we have a short exact sequence:

$$1 \rightarrow \pi_1(G/K) \rightarrow \pi_1(X/K) \rightarrow \pi_1(X/G) \rightarrow 1$$

We construct a splitting $\pi_1(X/K) \rightarrow \pi_1(G/K)$ for some K < G. By the assumption, there is an epimorphism $\mu: \pi_1(X) \rightarrow \mathbb{Z}$ such that the following diagram commutes:

$$\pi_1(G) \xrightarrow{\nu} \pi_1(X)$$

$$n \qquad \qquad \downarrow \mu$$

$$Z$$

Here *n* is multiplication by $n \ge 0$. Let *K* be a subgroup of *G* with order *n*.

$$\pi_{1}(G) \xrightarrow{\nu} \pi_{1}(X) \xrightarrow{\mu} Z$$

$$\downarrow n \qquad \qquad \downarrow p \qquad \qquad \stackrel{\pi}{\longrightarrow} \pi_{1}(X/K)$$

$$\downarrow n \qquad \qquad \downarrow p \qquad \qquad \stackrel{\pi}{\longrightarrow} \pi_{1}(X/K)$$

$$\downarrow \partial \qquad \qquad \downarrow \partial$$

$$K = = K$$

By a chasing method, the equation $\bar{\nu}(m)p(y) = \bar{\nu}(m')p(y')$ implies that there is

 $z \in \pi_1(G)$ satisfying m = m' + nz and $y = \nu(z^{-1})y'$. Then we have

$$m + \mu(y) = m' + nz + \mu(\nu(z^{-1})) + \mu(y') = m' + \mu(y').$$

For any $x = \overline{\nu}(m)p(y) \in \pi_1(X/K)$ we define as $\overline{\mu}(x) = m + \mu(y)$. Then the map $\overline{\mu}: \pi_1(X/K) \to \mathbb{Z}$ is a homomorphism with $\overline{\mu} \circ p = \mu$, since the image of $\overline{\nu}$ is a subgroup of the center of $\pi_1(X/K)$. Since both $\mu \circ \nu$ and *n* are multiplication by *n*, we have $\overline{\mu} \circ \overline{\nu} = 1$ and $\overline{\mu}$ is the required splitting.

Proposition 3.2. Let X be as in Theorem A. If X is free, then there are a proper subgroup K of G and a CW complex V such that $G|K \times V$ and X|K are G-homotopy equivalent.

Proof. Let K be a subgroup of G such that $\pi_1(X/K) \cong \pi_1(G/K) \oplus \pi_1(X/G)$. We denote by $p: V \to X/K$ the covering space corresponding to $\pi_1(X/G) \le \pi_1(X/K)$. The G-map $G/K \times V \to X/K$, sending (gK, v) to $g \cdot p(v)$, induces an isomorphism of homotopy groups. By a Whitehead theorem of the equivariant version [10], it is a G-homotopy equivalence.

By Lemma 1.2 there is a K-CW complex Y such that $G \times_{\kappa} Y$ and X is G-homotopy equivalent.

REMARK. Let Y be a K-space obtained from the G-homotopy pullback of the G-map p through the covering map $V \rightarrow X/K$. Then the G-map $G \times_{\kappa} Y \rightarrow X$ induced by the given K-map $Y \rightarrow X$ is a G-homotopy equivalence.

4. Proof of Theorems A and B

In this section, we also denote S^1 by G.

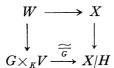
Proposition 4.1. Let X be as in Theorem A and let H be a finite subgroup of G. Then $H_1(p \circ v; \mathbf{Z})$ is monic for the projection $p: X \to X/H$.

Proof. As the *H*-action on *X* comes from a *G*-action by restriction, *H* acts trivially on $H_1(X; \mathbb{Z})$. Applying Theorem 2.4 [3, p. 120], we obtain that the projection induces an isomorphicm $H_1(p; \mathbb{Q}): H_1(X; \mathbb{Q}) \rightarrow H_1(X/H; \mathbb{Q})$. Then $H_1(p; \mathbb{Z})$ is injective on any free abelian subgroup of rank one in $H_1(X; \mathbb{Z})$.

We note that $\pi_1(\nu)$ is monic does not imply that $\pi_1(p \circ \nu)$ is injective.

Proof of Theorem A. Let H be a cyclic subgroup of which order is a common multiple of order of all isotropy subgroups in X. Clearly X/H is a connected free G/H-CW complex. By the argument in the previous section, there are a free K/H-CW complex V and a G/H-homotopy equivalence $h: G/H \times_{K/H} V$ $\rightarrow X/H$. We see canonically V as a K-CW complex. Then h induces a Ghomotopy equivalence $h': G \times_K V \rightarrow X/H$. The G-space W obtained from a T. Sumi

G-homotopy pullback of h' through the projection $X \rightarrow X/H$ is G-homotopy equivalent to X.



On the other hand, by Lemma 1.2, there is a K-CW complex Y such that $G \times_{\kappa} Y$ and W are G-homotopy equivalent. This completes the proof.

To prove Theorem B we may show the following:

Proposition 4.3. Let X be a connected finitely dominated G-space which fulfills that $H_1(v; \mathbb{Z})$ is injective. Then the K-space X has the K-homotopy type of a finite K-CW complex for any finite subgroup K of G.

Proof. X has the G-homotopy type of a G-CW complex with finitely many orbit types [5, Theorem 1.4]. By Theorem A and Proposition 2.2, there is a finitely dominated L-space Y such that $X \simeq_G G \times_L Y$. Since $G \times_L Y \rightarrow G/L$ is a K-fibration, we have the result. (See Theorem 3.6 [13].)

Theorem 4.4. Let G be any compact Lie group and let K be a subgroup of G. Let X be a finitely dominated G-space with X/G connected. If the rank of the image of $H_1(W_GK \rightarrow X/K; \mathbb{Z})$ is not zero, then $\operatorname{Res}_{K}^{G}(X): Wa^{G}(X) \rightarrow Wa^{K}(X)$ is a zero map.

Proof. Let T be a maximal torus of W_cK . Since $H_1(T \to W_cK; \mathbb{Z})$ is epic, there is a proper subgroup \overline{C} of W_cK such that \overline{C} is isomorphic to S^1 and $H_1(\overline{C} \to X/K; \mathbb{Z})$ is injective. Then there is a \overline{C} -map $f: X/K \to \overline{C}/L$ for some finite subgroup L of \overline{C} . Let C (resp. L) be the preimage of \overline{C} (resp. L) under the projection $N_cK \to W_cK$. Clearly $h: \overline{C}/L \cong C/L$. Since K is a normal subgroup of C, the projection $p: X \to X/K$ is a C-map. Then $h \circ f \circ p: X \to C/L$ is an equivariant C-fibration. The K-space C/L has a trivial K-action and its Euler characteristic is zero. If we apply Theorem 2.6 [13] to the equivariant Kfibration $h \circ f \circ p$, we conclude the proof.

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