

ON LAMBEK TORSION THEORIES

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According to Chase [3, Theorem 2.1], the argument of Morita [8, Theorem 1] yields that, for a left and right coherent ring, the injective envelope of the left regular module is flat if and only if this is the case with its opposite ring. In the present note, we will generalize this fact and provide conditions which are symmetrical for an arbitrary associative ring with identity.

Throughout R stands for an arbitrary associative ring with identity and all modules are unitary left or right R -modules. We denote by $\text{Mod } R$ (resp. $\text{Mod } R^{\text{op}}$) the category of all left (resp. right) R -modules and by $(\)^*$ both the R -dual functors. For a module X , we denote by $E(X)$ its injective envelope and by $\varepsilon_X: X \rightarrow X^{**}$ the usual evaluation map. For an $X \in \text{Mod } R$, we denote by $\tau(X)$ its torsion submodule with respect to the Lambek torsion theory on $\text{Mod } R$. Namely, $\tau(X)$ is the submodule of X such that (i) $\text{Hom}_R(\tau(X), E({}_R R)) = 0$ and (ii) $E({}_R R)$ cogenerates $X/\tau(X)$. For also an $M \in \text{Mod } R^{\text{op}}$, we denote by $\tau(M)$ its torsion submodule with respect to the Lambek torsion theory on $\text{Mod } R^{\text{op}}$.

We will prove the following

Theorem A. *The following are equivalent.*

- (a) $\tau(X) = \text{Ker } \varepsilon_X$ for every finitely presented $X \in \text{Mod } R$.
- (a)^{op} $\tau(M) = \text{Ker } \varepsilon_M$ for every finitely presented $M \in \text{Mod } R^{\text{op}}$.
- (b) f^{**} is monic for every monic $f: X \rightarrow Y$ in $\text{Mod } R$ with X finitely generated and Y finitely presented.
- (b)^{op} g^{**} is monic for every monic $g: M \rightarrow N$ in $\text{Mod } R^{\text{op}}$ with M finitely generated and N finitely presented.

Proposition B. *Let R be right coherent. Then the following are equivalent.*

- (a) $E({}_R R)$ is flat.
- (b) There is an $E \in \text{Mod } R$ which is faithful, injective and flat.
- (c) $\tau(X) = \text{Ker } \varepsilon_X$ for every finitely presented $X \in \text{Mod } R$.

Proposition C. *Let R be right noetherian. Then the following are equivalent.*

- (a) $E({}_R R)$ is flat.
- (b) Every finitely generated submodule of $E({}_R R)$ is torsionless.

In Proposition B, it always holds that (a) \Rightarrow (b) \Rightarrow (c). Thus Proposition B together with Theorem A yields a result of Morita [8, Theorem 1] that, if R is right coherent and $E(R_R)$ is flat, $E({}_R R)$ is flat. Also, since every finitely presented submodule of a flat module imbeds in a projective module, Proposition C generalizes the original statement of Morita [8, Theorem 1].

1. Preliminaries. In this section, we recall several basic facts which we need in later sections.

Lemma 1. *Let $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ be a finite presentation in $\text{Mod } R$ and put $M = \text{Cok}(P_0^* \rightarrow P_1^*)$. Then $\text{Ker } \varepsilon_X \simeq \text{Ext}_R^1(M, R)$ and $\text{Cok } \varepsilon_X \simeq \text{Ext}_R^2(M, R)$.*

Proof. See Auslander [1, Proposition 6.3].

REMARK. In the above lemma, we have a finite presentation $P_0^* \rightarrow P_1^* \rightarrow M \rightarrow 0$ in $\text{Mod } R^{\text{op}}$ with $X \simeq \text{Cok}(P_1^{**} \rightarrow P_0^{**})$, so $\text{Ker } \varepsilon_M \simeq \text{Ext}_R^1(X, R)$ and $\text{Cok } \varepsilon_M \simeq \text{Ext}_R^2(X, R)$.

Lemma 2. *Let $E \in \text{Mod } R$ be injective. Then, for a finitely presented $M \in \text{Mod } R^{\text{op}}$, there is a natural epimorphism*

$$\text{Tor}_1^R(M, E) \rightarrow \text{Hom}_R(\text{Ext}_R^1(M, R), E)$$

which is an isomorphism if R is right coherent.

Proof. See Cartan and Eilenberg [2, Chap. VI, Proposition 5.3]. Note for the last part that, if R is right coherent, every finitely presented $M \in \text{Mod } R^{\text{op}}$ admits a projective resolution whose terms are finitely generated.

REMARK. In case R is right coherent, $\text{Tor}_i^R(M, E) \simeq \text{Hom}_R(\text{Ext}_R^i(M, R), E)$ for all $i \geq 0$, all injective $E \in \text{Mod } R$ and all finitely presented $M \in \text{Mod } R^{\text{op}}$.

Lemma 3. *Let $E \in \text{Mod } R$. Then E is flat if and only if $\text{Tor}_1^R(M, E) = 0$ for all finitely presented $M \in \text{Mod } R^{\text{op}}$.*

Proof. The functor $\text{Tor}_1^R(-, E)$ commutes with direct limits and every module is isomorphic to a direct limit of finitely presented modules.

Lemma 4. *For an $X \in \text{Mod } R$, $\tau(X) = \text{Ker } \varepsilon_X$ if and only if $\text{Hom}_R(\text{Ker } \varepsilon_X, E({}_R R)) = 0$.*

Proof. $E({}_R R)$ cogenerates X^{**} and thus $\text{Im } \varepsilon_X$.

The next lemma is due essentially to Masaïke [6] (see also Sumioka [10, Theorem 2]).

Lemma 5. *The following are equivalent.*

- (a) $\tau(X) = \text{Ker } \varepsilon_X$ for every finitely generated $X \in \text{Mod } R$.
- (b) Every finitely generated submodule of $E({}_R R)$ is torsionless.

Proof. (a) \Rightarrow (b). By Lemma 4.

(b) \Rightarrow (a). Let $X \in \text{Mod } R$ be finitely generated. We claim $\text{Hom}_R(\text{Ker } \varepsilon_X, E({}_R R)) = 0$. Let $f: \text{Ker } \varepsilon_X \rightarrow E({}_R R)$. By the injectivity of $E({}_R R)$, f factors through the inclusion $j: \text{Ker } \varepsilon_X \rightarrow X$. Let $g: X \rightarrow E({}_R R)$ satisfy $f = g \circ j$. Then, since $\text{Im } g$ is torsionless, the injectivity of $E({}_R R)$ yields also that g factors through ε_X . Consequently f factors through $0 = \varepsilon_X \circ j$, namely $f = 0$.

Lemma 6. *Let $E \in \text{Mod } R$ be injective. Suppose every finitely generated submodule of E imbeds in a projective module. Then E is flat.*

Proof. Let $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a finite presentation in $\text{Mod } R^{\text{op}}$ and put $X = \text{Cok}(P_0^* \rightarrow P_1^*)$. There is a natural map

$$\delta_X: X^* \otimes_R E \rightarrow \text{Hom}_R(X, E)$$

such that $\delta_X(f \otimes e)(x) = f(x)e$ for $f \in X^*$, $e \in E$ and $x \in X$. Since by Auslander [1, Proposition 7.1] $\text{Tor}_1^R(M, E) \simeq \text{Cok } \delta_X$, it suffices to show that δ_X is surjective. Let $f: X \rightarrow E$. Since $\text{Im } f$ is finitely generated and imbeds in a projective module, the injectivity of E yields that f factors through a free module of finite rank, which implies $f \in \text{Im } \delta_X$.

REMARK. In the above lemma, if R is left noetherian, the converse holds.

2. Main results. In this section, we prove Theorem A and Propositions B and C stated in the introduction.

Proof of Theorem A. By symmetry, it suffices to prove the implications (a) $^{\text{op}} \Rightarrow$ (b) \Rightarrow (a).

(a) $^{\text{op}} \Rightarrow$ (b). Let $f: X \rightarrow Y$ be monic in $\text{Mod } R$ with X finitely generated and Y finitely presented. Note first that $Z = \text{Cok } f$ is finitely presented. Thus by Lemma 1 $\text{Ext}_R^1(Z, R) \simeq \text{Ker } \varepsilon_M$ for some finitely presented $M \in \text{Mod } R^{\text{op}}$. Hence by Lemma 4 $\text{Hom}_R(\text{Ext}_R^1(Z, R), E({}_R R)) = 0$. Since $\text{Cok } f^*$ imbeds in $\text{Ext}_R^1(Z, R)$, we get $\text{Ker } f^{**} \simeq (\text{Cok } f^*)^* = 0$.

(b) \Rightarrow (a). Let $Y \in \text{Mod } R$ be finitely presented. We claim $\text{Hom}_R(\text{Ker } \varepsilon_Y, E({}_R R)) = 0$. It suffices to show that $X^* = 0$ for every finitely generated submodule X of $\text{Ker } \varepsilon_Y$. Let X be a finitely generated submodule of $\text{Ker } \varepsilon_Y$ and let $f: X \rightarrow Y$ denote the inclusion. Note that $f^* \circ \varepsilon_Y^* = (\varepsilon_Y \circ f)^* = 0$. Thus, since ε_Y^* is epic, we get $f^* = 0$. Now, since f^{**} is monic, we get $X^{**} = 0$ and thus $X^* = 0$.

Proof of Proposition B. (a) \Rightarrow (b). Obvious.

(b) \Rightarrow (c). Let $P_1 \rightarrow P_0 \rightarrow X \rightarrow 0$ be a finite presentation in $\text{Mod } R$ and put $M = \text{Cok}(P_0^* \rightarrow P_1^*)$. Since by Lemma 1 $\text{Ker } \varepsilon_X \simeq \text{Ext}_R^1(M, R)$ with M finitely presented, by Lemma 2 we have $\text{Hom}_R(\text{Ker } \varepsilon_X, E) = 0$. Since the functor $\text{Hom}_R(\text{Ker } \varepsilon_X, -)$ commutes with direct products, and since $E({}_R R)$ imbeds in a direct product of copies of E , we conclude $\text{Hom}_R(\text{Ker } \varepsilon_X, E({}_R R)) = 0$. Hence by Lemma 4 $\tau(X) = \text{Ker } \varepsilon_X$.

(c) \Rightarrow (a). Let $P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a finite presentation in $\text{Mod } R^{\text{op}}$ and put $X = \text{Cok}(P_0^* \rightarrow P_1^*)$. Then X is finitely presented and by Lemma 1 $\text{Ker } \varepsilon_X \simeq \text{Ext}_R^1(M, R)$. Thus by Lemmas 2 and 4 $\text{Tor}_1^R(M, E({}_R R)) \simeq \text{Hom}_R(\text{Ext}_R^1(M, R), E({}_R R)) = 0$. Hence by Lemma 3 $E({}_R R)$ is flat.

Proof of Proposition C. (a) \Rightarrow (b). By Theorem A, Proposition B and Lemma 4, every finitely presented submodule of $E({}_R R)$ is torsionless. On the other hand, every finitely generated right module is finitely presented.

(b) \Rightarrow (a). By Theorem A, Proposition B and Lemma 5.

REMARK. Let us consider the following conditions:

(a) Every finitely generated submodule of $E({}_R R)$ imbeds in a projective module.

(b) $E({}_R R)$ is flat.

(c) Every finitely generated submodule of $E({}_R R)$ is torsionless.

(d) $\tau(X) = \text{Ker } \varepsilon_X$ for every finitely presented $X \in \text{Mod } R$.

Then we have shown that (a) \Rightarrow (b) \Rightarrow (d) and that (a) \Rightarrow (c) \Rightarrow (d).

We have shown that, if R is right coherent, (d) \Rightarrow (b) so (c) \Rightarrow (b). On the other hand, we know from Sumioka [10] that, if R satisfies the descending chain condition on annihilator left ideals, (c) \Rightarrow (a) so (c) \Rightarrow (b). It seems that there is no direct implication between these two results. Note however that they have the same effect on right noetherian rings R .

We have shown that the condition (d) is symmetrical. There is another symmetrical condition given by Masaïke [7, Theorem 2]. Namely, the condition (c) together with the descending chain condition on annihilator left ideals is symmetrical. Although these two results have the same effect on left and right noetherian rings R , we do not know whether there is any direct implication between them.

3. Remarks. Throughout this section R is assumed to be left and right coherent. We have seen that $E({}_R R)$ is flat if and only if $E(R_R)$ is. Unfortunately, this cannot be extended to higher weak dimensions. Namely, $\text{weak dim } E({}_R R) = 1$ does not necessarily imply $\text{weak dim } E(R_R) < \infty$.

We denote by $\text{mod } R$ (resp. $\text{mod } R^{\text{op}}$) the category of all coherent left (resp. right) R -modules. Note that an R -module is coherent if and only if it is finitely presented, and that every finitely generated submodule of a coherent R -module

is coherent (see e.g. Popescu [9]).

Proposition D. *The following are equivalent.*

- (a) $\text{weak dim } E({}_R R) \leq 1$.
- (b) $\varepsilon_X: X \rightarrow X^{**}$ is an essential monomorphism for every torsionless $X \in \text{mod } R$.
- (c) f^{**} is monic for every monic $f: X \rightarrow Y$ in $\text{mod } R$ with Y torsionless.
- (d) $\text{Ext}_R^1(-, R)^*$ vanishes on $\text{mod } R$.

Proof. (a) \Rightarrow (b). Let $X \in \text{mod } R$. By Lemmas 1 and 2 we have $\text{Hom}_R(\text{Cok } \varepsilon_X, E({}_R R)) = 0$. Thus $\text{Cok } \varepsilon_X$ does not contain a non-zero torsionless submodule. On the other hand, every non-zero submodule of X^{**} is torsionless. Hence $\text{Im } \varepsilon_X$ is large in X^{**} .

(b) \Rightarrow (c). Let $f: X \rightarrow Y$ be monic in $\text{mod } R$ with Y torsionless. Since $f^{**} \circ \varepsilon_X = \varepsilon_Y \circ f$ is monic, so is f^{**} .

(c) \Rightarrow (d). Let $0 \rightarrow Y \rightarrow P \rightarrow X \rightarrow 0$ be exact in $\text{mod } R$ with P projective. Then $\text{Ext}_R^1(X, R)^* \simeq \text{Ker}(Y^{**} \rightarrow P^{**}) = 0$.

(d) \Rightarrow (a). Let $\dots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0$ be a projective resolution in $\text{mod } R^{\text{op}}$. We claim $\text{Tor}_2^R(M, E({}_R R)) \simeq \text{Hom}_R(\text{Ext}_R^2(M, R), E({}_R R)) = 0$. It suffices to show that $X^* = 0$ for every finitely generated submodule X of $\text{Ext}_R^2(M, R)$. Put $N = \text{Cok}(P_2 \rightarrow P_1)$ and $Y = \text{Cok}(P_1^* \rightarrow P_2^*)$. By Lemma 1 $\text{Ext}_R^2(M, R) \simeq \text{Ext}_R^1(N, R) \simeq \text{Ker } \varepsilon_Y$. Also, since N is torsionless, $\text{Ext}_R^1(Y, R) \simeq \text{Ker } \varepsilon_N = 0$. Let X be a finitely generated submodule of $\text{Ext}_R^2(M, R)$ and let $f: X \rightarrow Y$ be an imbedding which factors through $\text{Ker } \varepsilon_Y$. Then, as in the proof of (b) \Rightarrow (a) in Theorem A, we get $f^* = 0$. Hence, applying $(\)^*$ to an exact sequence $0 \rightarrow X \rightarrow Y \rightarrow \text{Cok } f \rightarrow 0$, we get $X^* \simeq \text{Ext}_R^1(\text{Cok } f, R)$. Therefore $X^{**} \simeq \text{Ext}_R^1(\text{Cok } f, R)^* = 0$, which implies $X^* = 0$.

EXAMPLE. Let R be a subalgebra of $(K)_6$, the 6×6 matrix algebra over a field K , with the basis elements

$$\begin{aligned} e_1 &= e_{11} + e_{22} + e_{33}, & e_2 &= e_{44} + e_{55}, & e_3 &= e_{66}, \\ \alpha &= e_{15} + e_{25}, & \beta &= e_{42} + e_{53}, & \gamma &= e_{46} + e_{56}, \\ \alpha\beta &= e_{13} + e_{23}, & \alpha\gamma &= e_{16} + e_{26}, & \beta\alpha &= e_{45}, \\ & & \beta\alpha\beta &= e_{43} & \text{and } \beta\alpha\gamma &= e_{46}, \end{aligned}$$

where e_{ij} are matrix units. Then it is not difficult to check that $\text{proj dim } E({}_R R) = 1$ and that $\text{proj dim } E(R_R) = \infty$.

REMARK. The above example together with Proposition D shows that the vanishing of $\text{Ext}_R^1(-, R)^*$ on $\text{mod } R$ is not symmetrical. On the other hand, we know from [4] that the vanishing of $\text{Ext}_R^2(-, R)^*$ on $\text{mod } R$ is symmetrical.

Proposition E. *The following are equivalent.*

- (a) $()^{**}: \text{mod } R \rightarrow \text{mod } R$ is left exact.
- (a)^{op} $()^{**}: \text{mod } R^{\text{op}} \rightarrow \text{mod } R^{\text{op}}$ is left exact.
- (b) $()^{**}: \text{mod } R \rightarrow \text{mod } R$ is mono-preserving and $\text{Ext}_R^1(\text{Ext}_R^1(-, R), R)$ vanishes on $\text{mod } R$.
- (b)^{op} $()^{**}: \text{mod } R^{\text{op}} \rightarrow \text{mod } R^{\text{op}}$ is mono-preserving and $\text{Ext}_R^1(\text{Ext}_R^1(-, R), R)$ vanishes on $\text{mod } R^{\text{op}}$.
- (c) Both $E({}_R R)$ and $E(E({}_R R)/R)$ are flat.
- (c)^{op} Both $E(R_R)$ and $E(E(R_R)/R)$ are flat.

Proof. We know from Theorem A and Proposition B that $()^{**}: \text{mod } R \rightarrow \text{mod } R$ is mono-preserving if and only if so is $()^{**}: \text{mod } R^{\text{op}} \rightarrow \text{mod } R^{\text{op}}$, and that $()^{**}: \text{mod } R \rightarrow \text{mod } R$ is mono-preserving if and only if $E({}_R R)$ is flat.

(a) \Leftrightarrow (a)^{op}. By [5, Proposition 3.4].

(a)^{op} \Rightarrow (b). Let $X \in \text{mod } R$. Since by Lemma 2 $\text{Hom}_R(\text{Ext}_R^1(X, R), E(R_R)) \simeq \text{Tor}_1^R(E(R_R), X) = 0$, $\text{Ext}_R^1(X, R)^* = 0$ so by [5, Lemma 3.3] $\text{Ext}_R^1(\text{Ext}_R^1(X, R), R) = 0$.

(b)^{op} \Rightarrow (c). Let $M \in \text{mod } R^{\text{op}}$. We claim $\text{Tor}_1^R(M, E(E({}_R R)/R)) \simeq \text{Hom}_R(\text{Ext}_R^1(M, R), E(E({}_R R)/R)) = 0$. It suffices to show that $\text{Hom}_R(X, E(E({}_R R)/R)) = 0$ for every finitely generated submodule X of $\text{Ext}_R^1(M, R)$. Let X be a finitely generated submodule of $\text{Ext}_R^1(M, R)$. Since by Lemma 2 $\text{Hom}_R(\text{Ext}_R^1(M, R), E({}_R R)) \simeq \text{Tor}_1^R(M, E({}_R R)) = 0$, $\text{Hom}_R(X, E({}_R R)) = 0$. Thus $\text{Hom}_R(X, E(E({}_R R)/R)) \simeq \text{Ext}_R^1(X, R)$. Also $X^* = 0$ so by Lemma 1 $X \simeq \text{Ext}_R^1(N, R)$ for some $N \in \text{mod } R^{\text{op}}$. Hence $\text{Ext}_R^1(X, R) \simeq \text{Ext}_R^1(\text{Ext}_R^1(N, R), R) = 0$. Therefore $\text{Hom}_R(X, E(E({}_R R)/R)) = 0$.

(c) \Rightarrow (a). By [5, Proposition 3.5].

The next proposition is due essentially to Sumioka [10, Theorem 3].

Proposition F. *Let R be left and right noetherian. Suppose the maximal right quotient ring Q of R is a left quotient ring of R . Then $\text{weak dim } E({}_R R) \leq 1$ implies $E({}_R R)$ is flat.*

Proof. Note first that by Masaike [6, Proposition 2] every finitely generated submodule of Q_R is torsionless. Suppose $\text{weak dim } E({}_R R) \leq 1$. Let $M \in \text{mod } R^{\text{op}}$. We claim $\tau(M) = \text{Ker } \varepsilon_M$. By Lemma 1 $\text{Ker } \varepsilon_M \simeq \text{Ext}_R^1(X, R)$ for some $X \in \text{mod } R$. Thus by Proposition D $(\text{Ker } \varepsilon_M)^* \simeq \text{Ext}_R^1(X, R)^* = 0$. Hence by Sumioka [10, Proposition 3] $\text{Hom}_R(\text{Ker } \varepsilon_M, E(R_R)) = 0$, so by Lemma 4 $\tau(M) = \text{Ker } \varepsilon_M$. The assertion now follows from Lemma 5 and Proposition C.

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Added in proof. Recently, the author was informed by S. Takashima that, in Proposition B, “coherent” can be replaced by “ τ -coherent”, and that, in Proposition C, “noetherian” can be replaced by “ τ -noetherian”. Accordingly, the questions in Remark of Section 2 have been settled.

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