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# ON THE CLOSABLE PARTS OF PRE-DIRICHLET FORMS AND THE FINE SUPPORTS OF UNDERLYING MEASURES

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## 1. Introduction

Let $X$ be a locally compact separable metric space, $\mathcal{M}$ be the space of positive Radon measures on $X$ and let $\mathcal{M}'=\{\nu\in\mathcal{M}: supp[\nu]=X\}$. Fix $m\in\mathcal{M}'$ and a regular Dirichlet form $\mathcal{E}$ with domain $\mathcal{F}$ on $L^2(X; m)$, which possesses a nice core $\mathcal{C}$ as described in Section 3. Throughout the present paper, we assume that $\mathcal{E}$ is either irreducible or transient. Let $Cap(\cdot)$ be the 1-capacity associated with $\mathcal{E}$. A set $A$ is said to be $\mathcal{E}_1$-polar if $Cap(A)=0$. Define

$$\mathcal{M}_0=\{\nu\in\mathcal{M}: \nu \text{ charges no } \mathcal{E}_1\text{-polar set}\},$$
$$\mathcal{M}_{00}=\{\nu\in\mathcal{M}_0: Cap(X\backslash\tilde{S}_\nu)=0\},$$

where $\tilde{S}_\nu$ stands for the support of the positive continuous additive functional (abbreviated to PCAF) associated with $\nu\in\mathcal{M}_0$. $\tilde{S}_\nu$ is closed with respect to the fine topology for the associated Hunt process and we call it the fine support of $\nu$.

For $\mu\in\mathcal{M}$, we introduce the capacitary decomposition of $\mu_1$ with respect to $Cap(\cdot)$: a unique decomposition $\mu=\mu_0+\mu_1$, where $\mu_0\in\mathcal{M}_0$ and $\mu_1=\boldsymbol{I}_N\cdot\mu$ with an $\mathcal{E}_1$-polar set $N$. For details, see Section 2. This is a variant of the potential-theoretical decomposition of measures due to Blumenthal-Getoor [1, VI(3.6)].

In the present paper, we are interested in changing the underlying measure $m$ for another element of $\mathcal{M}'$ by keeping the pre-Dirichlet form $\mathcal{E}$ on $\mathcal{C}$ unchanged. We aim at showing the following necessary and sufficient condition for $\mu\in\mathcal{M}'$,

$$(\mathcal{E}, \mathcal{C}) \textit{ is closable on } L^2(X; \mu) \textit{ if and only if } \mu_0\in\mathcal{M}_{00}.$$

See Theorem 4.1, where the Hunt process associated with the closure is also specified by time changing with respect to $\mu_0$ and making points of $N$ traps.

The condition that $\nu\in\mathcal{M}_{00}$ is an indispensable requirement for the invariance of the pre-Dirichlet form under the random time change with respect to

$\nu\in\mathcal{M}_0\cap\mathcal{M}'$ (cf. [2,3]). It is easy to see that $\mathcal{M}_{00}\subset\mathcal{M}_0\cap\mathcal{M}'$ and that $\nu(X\backslash\tilde{S}_\nu)=0$ for any $\nu\in\mathcal{M}_0\cap\mathcal{M}'$. However the stronger condition that $Cap(X\backslash\tilde{S}_\mu)=0$ for $\nu\in\mathcal{M}_0\cap\mathcal{M}'$ is not alwasys satisfied (see Kuwae-Nakao [4] for a counter example) and is hard to be checked due to the involvement of the fine topology. What to be emphasized is that the above necessary and sufficient condition reduces the difficulty to the problem on closability, which is purely analytic and accordingly easier to be studied.

A nonnegative definite symmetrric bilinear form on a Hilbert space with dense domain is simply called a symmetric form. We say that a symmetric form $\mathcal{A}$ on $L^2(X;\mu)$ with domain $\mathcal{D}$ is a pre-Dirichlet form if it is Markovian. The closable part of a pre-Dirichlet from $\mathcal{A}$ with domain $\mathcal{D}$ on $L^2(X;\mu)$ is a pre-Dirichlet form $\mathcal{A}'$ with domain $\mathcal{D}$ such that (i) $(\mathcal{A}',\mathcal{D})$ is closable on $L^2(X;\mu)$, (ii) $\mathcal{A}'(u,u)\leq\mathcal{A}(u,u)$ for $u\in\mathcal{D}$ and (iii) if $\mathcal{B}$ is a closable pre-Dirichlet form with domain $\mathcal{D}$ on $L^2(X;\mu)$ and $\mathcal{B}(u,u)\leq\mathcal{A}(u,u)$ for $u\in\mathcal{D}$, then $\mathcal{B}(u,u)\leq\mathcal{A}'(u,u)$ for $u\in\mathcal{D}$. For the existence of closable parts, see Section 6. A key observation to establish the above necessary and sufficient condition is that the closure of the closable part of the pre-Dirichlet form $(\mathcal{E},\mathcal{C})$ on $L^2(X;\mu)$ is realized by the Dirichlet form $\mathcal{E}^{(\mu)}$ constructed in the following two steps: first time changing the Hunt process corresponding to $\mathcal{E}$ by the PCAF associated with $\mu_0$ and secondly making all points in $N$ traps. We note that $\mathcal{E}^{(\mu)}$ is a direct generalization of the time changed (regular) Dirichlet form recently formulated by Kuwae and Nakao [4] who treat the case that $\mu=\mu_0$. As another application of the observation, it will be seen in Corollary 4.2 that, whenever the original form $\mathcal{E}$ is transient and $(\mathcal{E},\mathcal{C})$ is closable on $L^2(X;\mu)$, where $\mu\in\mathcal{M}'$, the closure is transient if and only if $\mu\in\mathcal{M}_0$.

In the case that $\mathcal{E}$ is transient, it should be recalled that Röckner-Wielens [6] establishes a necessary and sufficient condition in order that $(\mathcal{E},\mathcal{C})$ is closable on $L^2(X;\mu)$ and the closure is transient. As will be seen in Remark after Corollary 4.3, their criterion is covered by ours.

The organization of this paper is as follows. In Section 2, we establish the capacitary decomposition. The next section is devoted to the construction of the Dirichlet form $\mathcal{E}^{(\mu)}$ on $L^2(X;\mu)$ by time changing and making suitable points traps. In Section 4, the Dirichlet form $\mathcal{E}^{(\mu)}$ will be identified with the closure of the closable part of the pre-Dirichlet form $(\mathcal{E},\mathcal{C})$ on $L^2(X;\mu)$. In addition, the necessary and sufficient conditions for closability and transience will be shown in the section. Two examples will be studied in Section 5 to illustrate our results. Section 6 will present general results on the closable parts of symmetric forms on a Hilbert space.

## 2. The capacitary decomposition of measures

In this section, we establish a decomposition of $\sigma$-finite measures with respect to a set function.

**Lemma 2.1.** *Let $(\Omega, \mathcal{B})$ be a measurable space and $\Phi(\cdot)$ be a countably subadditive nonnegative set function on it. For each $\sigma$-finite measure $\mu$ on $(\Omega, \mathcal{B})$, there exists a unique pair $(\mu_0, \mu_1)$ of measures on $(\Omega, \mathcal{B})$ such that*

(I) $\mu=\mu_0+\mu_1$,

(II) $\mu_0(A)=0$ *for any* $A\in\mathcal{B}$ *with* $\Phi(A)=0$,

(III) $\mu_1=\boldsymbol{I}_N\cdot\mu$ *for some* $N\in\mathcal{B}$ *with* $\Phi(N)=0$,

*where $\boldsymbol{I}_N$ is the indicator function of $N$.*

DEFINITION 2.1. *We call the above $\mu_0$ the smooth part of $\mu$ with respact to $\Phi$.*

Proof of Lemma 2.1. It is elementary to see the uniqueness of the decomposition. Thus, we only see the existence of such $\mu_0$ and $\mu_1$.

It is standard to extend the assertion to general $\sigma$-finite $\mu$'s once we have shown the assertion for a finite measure $\mu$. Hence we may assume that $\mu(\Omega)<\infty$. It then holds that

$$\alpha \equiv \sup\{\mu(A)\colon A\in\mathcal{B} \quad \text{and} \quad \Phi(A) = 0\} < \infty .$$

Take an increasing sequence $\{A_n\}\subset\mathcal{B}$ such that $\Phi(A_n)=0$ and $\lim_n \mu(A_n)=\alpha$. Set

$$A_\infty = \bigcup_{n=1}^{\infty} A_n .$$

Then, we obtain that $A_\infty\in\mathcal{B}$, $\Phi(A_\infty)=0$ and $\mu(A_\infty)=\alpha$. In particular, it holds that

$$\mu(A\backslash A_\infty) = 0 \qquad \text{for every} \quad A\in\mathcal{B} \quad \text{with} \quad \Phi(A) = 0. \tag{2.1}$$

We then define $\mu_0$, $\mu_1$ by

$$\mu_0 = \boldsymbol{I}_{X\backslash A_\infty}\cdot\mu , \quad \mu_1 = \boldsymbol{I}_{A_\infty}\cdot\mu .$$

Obviously, $(\mu_0, \mu_1)$ enjoys the properties (I) and (III) with $N=A_\infty$. Moreover, (2.1) implies that (II) is satisfied.

## 3. Definition of $\mathcal{E}^{(\mu)}$

As in Section 1, let $X$ be a locally compact separable metric space and $m\in\mathcal{M}'$. Consider a dense subalgebra $\mathcal{C}$ of $C_0(X)$ possessing the following two properties:

($\mathcal{C}$.1) For every compact set $K$ and relatively compact set $G$ with $K\subset G\subset X$, there is a $w\in\mathcal{C}$ such that $0\leq w\leq 1$ and $w$ takes value 1 on $K$ and 0 outside of $G$.

($\mathcal{C}$.2) For each $\varepsilon>0$, there is a real function $\beta_\varepsilon(t)$, $t\in(-\infty, \infty)$, such that $\beta_\varepsilon(t)=t$ on $[0,1]$, $-\varepsilon\leq\beta_\varepsilon(t)\leq 1+\varepsilon$ for any $t$, $0\leq\beta_\varepsilon(t)-\beta_\varepsilon(s)\leq t-s$ for $t>s$ and $\beta_\varepsilon(f)\in\mathcal{C}$ whenever $f\in\mathcal{C}$.

Let $\mathcal{E}$ with domain $\mathcal{F}$ be a Dirichlet form on $L^2(X; m)$ possessing $\mathcal{C}$ as its core: $\mathcal{C}$ is dense in the Hilbert space $(\mathcal{F}, \mathcal{E}_1)$, where $\mathcal{E}_1$ is defined by

$$\mathcal{E}_1(u, u) = \mathcal{E}(u, u)+(u, u)_m\,, \quad u\in\mathcal{F}\,,$$

$(\cdot, \cdot)_m$ being the inner product of $L^2(X;m)$. We denote by $\boldsymbol{M}=(X_t, P_x)$ the associated Hunt process. The associated 1-capacity $Cap(\cdot)$ is then a Choquet capacity given for compact $K$ by

$$Cap(K) = \inf\{\mathcal{E}_1(f, f): f\in\mathcal{C} \quad\text{and}\quad f\geq 1 \text{ on } K\}\,. \tag{3.1}$$

Let $\{T_t, t\geq 0\}$ be the semigroup on $L^2(X; m)$ associated with $\mathcal{E}$. A set $A$ is said to be $T_t$-invariant if $T_t(\boldsymbol{I}_A u)=\boldsymbol{I}_A\cdot T_t u$, $m$-a.e. for any $u\in L^2(X; m)$ and $t>0$. We say that $\mathcal{E}$ is irreducible if either $m(A)$ or $m(X\backslash A)$ vanishes whenever $A$ is $T_t$-invariant. $\mathcal{E}$ is said to be transient if there is a bounded $g\in L^1(X; m)$ with $g>0$ $m$-a.e. such that

$$\int_X |u|g dm\leq\sqrt{\mathcal{E}(u, u)} \qquad \text{for every} \quad u\in\mathcal{F}. \tag{3.2}$$

Throughout this and the next seation, we assume that

(A.1) $\mathcal{E}$ is either irreducible or transient.

Let $\mu\in\mathcal{M}'$. To define a Dirichlet form $\mathcal{E}^{(\mu)}$, we review briefly on time changed processes of $\boldsymbol{M}$. In what follows, we use $\mu_0$ to denote the smooth part of $\mu$ with respect to $Cap(\cdot)$ and assume that

(A.2) $\mu_0\neq 0$.

Since $\mu_0\in\mathcal{M}_0$, it admits a PCAF $A_t^0$ of $\boldsymbol{M}$ with Revuz measure $\mu_0$. Let $\tilde{S}\equiv\tilde{S}_{\mu_0}$ be the support of $A_t^0$:

$$\tilde{S} = \{x\in X\backslash N_0: P_x[A_t^0>0 \quad\text{for any } t>0] = 1\}\,,$$

$N_0$ being an exceptional set of $A_t^0$. It is known that, if we replace $N_0$ by an appropriate exceptional set, then

$$\tilde{S}\subset S \equiv supp[\mu_0] \quad\text{and}\quad \mu_0(S\backslash\tilde{S}) = 0\,. \tag{3.3}$$

See [2]. In the sequel, we assume that (3.3) is satisfied.

The $\mu_0$-killing Dirichlet space $(\mathcal{L}, \mathcal{K})$ on $L^2(X; m)$ is a regular Dirichlet space given by

$$\mathcal{K} = \mathcal{F}\cap L^2(X; \mu_0) \quad \textit{and} \quad \mathcal{L}(u, u) = \mathcal{E}(u, v)+(u, v)_{\mu_0}\,,$$

where $(\ ,\ )_{\mu_0}$ is the inner product of $L^2(X; \mu_0)$ and the equality $\mathcal{K}=\mathcal{F}\cap L^2(X; \mu_0)$

must be read as a statement about $m$-quasicontinuous versions of the elements of the Dirichlet spaces involved (cf. [4, 5]). It follows from Assumption (A.1) that $\mathcal{L}$ is transient (cf. [4]). The 0-capacity with respect to $\mathcal{L}$ is then well-defined, and a set is $\mathcal{L}_1$-polar if and only if it is of 0-capacity zero. Hence, in what follows, we simply say "$\mathcal{L}$-polar" and "$\mathcal{L}$-q.e." instead of "$\mathcal{L}_1$-polar" and "$\mathcal{L}_1$-q.e.". We denote by $(\mathcal{L}, \mathcal{K}_e)$ the extended Dirichlet space of $(\mathcal{L}, \mathcal{K})$ and by $\mathcal{P}$ the projeation of $\mathcal{K}_e$ onto the orghogonal complement of $\{u\in\mathcal{K}_e: u=0$ $\mathcal{L}$-q.e. on $\tilde{S}\}$.

The time changed process

$$\tilde{M} = (X_{\tau_t}, P_x)_{x\in\tilde{S}} \tag{3.4}$$

of $M$ by

$$\tau_t = \inf\{s>0 : A_s^0>t\}$$

then determines a strongly continuous symmetric resolvent $\{G_\alpha^0, \alpha>0\}$ on $L^2(S; \mu_0)$ and the associated Dirichlet space $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(S; \mu_0)$ is characterized by

$$\mathcal{F}^0 = \{u\in L^2(S; \mu_0): u = \mathcal{P}v|_S \ \mu_0\text{-a.e.} \quad \text{for some} \quad v\in\mathcal{K}_e\}, \tag{3.5}$$

$$\mathcal{E}^0(\mathcal{P}v|_S, \mathcal{P}v'|_S) = \mathcal{L}(\mathcal{P}v, \mathcal{P}v') - (\mathcal{P}v, \mathcal{P}v')_{\mu_0} \quad \text{for} \quad v, v'\in\mathcal{K}_e. \tag{3.6}$$

This has been shown in Kuwae and Nakao [4], where the Dirichlet space $(\mathcal{E}^0, \mathcal{F}^0)$ on $L^2(S; \mu_0)$ is proved to be regular. We can even specify the core as we shall see now.

For a closed set $A\subset X$, let

$$C|_A = \{f|_A : f\in C\}.$$

Then we obtain

**Lemma 3.1.** *$\mathcal{E}^0$ possesses $C|_S$ as its core. In particular, if we set*

$$\mathcal{E}_0^{(\mu)}(f, g) = \mathcal{E}^0(f|_S, g|_S) \quad \text{for} \quad f, g\in C, \tag{3.7}$$

*then $(\mathcal{E}_0^{(\mu)}, C)$ is closable on $L^2(X; \mu)$.*

Lemma 3.1 enables us to define

DEFINITION 3.1. *For $\mu\in\mathcal{M}'$ with $\mu_0\neq 0$, we define $(\mathcal{E}^{(\mu)}, \mathcal{F}^{(\mu)})$ to be the closure of $(\mathcal{E}_0^{(\mu)}, C)$ on $L^2(X; \mu)$. If $\mu_0=0$, then we define $\mathcal{F}^{(\mu)}=L^2(X; \mu)$ and $\mathcal{E}^{(\mu)}=0$.*

Proof of Lemma 3.1. Notice that, for every $g\in\mathcal{K}_e$, $\mathcal{P}g=g$ $\mathcal{L}$-q.e. on $\tilde{S}$. Moreover, the definition of $\mathcal{L}$ implies that an $\mathcal{L}$-polar set is $\mathcal{E}_1$-polar. Since $\mu_0$ charges no $\mathcal{E}_1$-polar set, it then follows from (3.3) that

$$\mathcal{P}g = g \quad \mu_0\text{-a.e. on } S. \tag{3.8}$$

Since $\mathcal{C}\subset\mathcal{K}\subset\mathcal{K}_e$, we obtain that $\mathcal{C}|_S\subset\mathcal{F}^0$.

We next see that $(\mathcal{L}, \mathcal{K})$ possesses $\mathcal{C}$ as its core. Since $\mathcal{L}$ is regular, it suffices to show that, for each $f\in\mathcal{K}\cap C_0(X)$, there is a sequence $\{f_n\}\subset\mathcal{C}$ such that $f_n\to f$ in $\mathcal{K}$ with respeat to $\mathcal{L}_1$. To do this, take $\{g_n\}\subset\mathcal{C}$, $w\in\mathcal{C}$ and $\varepsilon>0$ satisfying that $g_n\to f$ in $(\mathcal{F}, \mathcal{E}_1)$, $0\le w\le 1$, $w=1$ on $supp[f]$ and $\sup_{x\in X}|f(x)|<\varepsilon^{-1}$. Define $h_n\in\mathcal{C}$ by

$$h_n = \varepsilon^{-1}\beta_\varepsilon(\varepsilon g_n)\cdot w\,.$$

Then, $h_{n_j}\to f$ $m$-a.e. for some subsequence $\{h_{n_j}\}$. Moreover, combined with [2, Theorem 1.4.2], the Markov property of $\mathcal{E}$ implies that

$$\sqrt{\mathcal{L}_1(h_n, h_n)}\le\varepsilon^{-1}\{\sqrt{\mathcal{E}(g_n, g_n)}+(1+\varepsilon)\,(\sqrt{\mathcal{E}(w, w)}+\sqrt{(w, w)_m+(w, w)_{\mu_0}})\}\,.$$

Hence the Cesàro mean of a subsequence of $\{h_n\}$ converges to $f$ in $(\mathcal{K}, \mathcal{L}_1)$ and $\mathcal{C}$ is dense in $(\mathcal{K}, \mathcal{L}_1)$.

The above observation implies that $\mathcal{C}$ is also dense in $(\mathcal{K}_e, \mathcal{L})$. To see that $\mathcal{C}|_S$ is dense in $(\mathcal{F}^0, \mathcal{E}^0)$, let $u\in\mathcal{F}^0$. Choose $v\in\mathcal{K}_e$ such that $u=\mathcal{P}v|_S$ $\mu_0$-a.e. on $S$. Let $\{v_n\}\subset\mathcal{C}$ be a sequence such that $v_n\to v$ in $(\mathcal{K}_e, \mathcal{L})$. Then, it holds that

$$\begin{aligned}\mathcal{E}_1^0(u-v_n|_S, u-v_n|_S) &= \mathcal{L}(\mathcal{P}(v-v_n), \mathcal{P}(v-v_n))\\ &\le \mathcal{L}(v-v_n, v-v_n)\\ &\to 0\,.\end{aligned}$$

The proof is complete.

## 4. Necessary and sufficient conditions

In this seation, we establish necessary and sufficient conditions for closability and transience. Our goal of this section will be

**Theorem 4.1.** *Consider the same Dirichlet form $\mathcal{E}$ as in Section 3. Let $\mu\in\mathcal{M}'$. Then, the following assertions hold:*
*(i) $(\mathcal{E}_0^{(\mu)}, \mathcal{C})$ is the closable part of $(\mathcal{E}, \mathcal{C})$ on $L^2(X; \mu)$.*
*(ii) $(\mathcal{E}, \mathcal{C})$ is closable on $L^2(X; \mu)$ if and only if $\mu_0\in\mathcal{M}_{00}$, where $\mu_0$ is the smooth part of $\mu$ with respect to $Cap(\cdot)$.*
*(iii) If $(\mathcal{E}, \mathcal{C})$ is closable on $L^2(X; \mu)$, then the closure is given by $(\mathcal{E}^{(\mu)}, \mathcal{F}^{(\mu)})$. Furthermore, if we denote by $\tilde{M}=(X_{\tau_t}, P_x)_{x\in\tilde{S}}$ the time changed process corresponding to $\mu_0$, then $(\mathcal{E}^{(\mu)}, \mathcal{F}^{(\mu)})$ is realized by a Hunt process $M^{(\mu)}=(X_t^{(\mu)}, P_x^{(\mu)})$ such that*

(a) *"the law of $X_\cdot^{(\mu)}$ under $P_x^{(\mu)}$"="the law of $X_{\tau_\cdot}$ under $P_x$" for $x\in X\setminus N$,*

(b) *$P_x^{(\mu)}[X_t=x$ for $t\ge 0]=1$ for $x\in N$,* *where* $N$ *is an* $\mathcal{E}_1$*-polar Borel set such that* $\mu_0=\boldsymbol{I}_{X\setminus N}\cdot\mu$ *and* $X\setminus N\subset\tilde{S}$.

It follows immediately from Theorem 4.1 that

**Corollary 4.1.** *If* $\mu\in\mathcal{M}'$ *but* $\mu_0\notin\mathcal{M}'$, *then* $(\mathcal{E}, C)$ *is not closable on* $L^2(X;\mu)$.

In some special cases, we can show that $\mu_0\in\mathcal{M}_{00}$:

**Corollary 4.2.** *If* $m$ *is absolutely continuous with respect to* $\mu$, *then* $\mu_0\in\mathcal{M}_{00}$. *In particular,* $(\mathcal{E}, C)$ *is then closable on* $L^2(X;\mu)$ *and the closure is realized in the way as stated in Theorem* 4.1.

Proof. The second assertion is an immediate consequence of the first one and Theorem 4.1.

To see the first assertion, suppose that $\mu_0(A)=0$. Then, by Lemma 2.1, $\mu(A\setminus N)=0$, $N$ being an $\mathcal{E}_1$-polar set with $\mu_0=\boldsymbol{I}_{X\setminus N}\cdot\mu$. It follows from the assumption that $m(A\setminus N)=0$. Since $m\in\mathcal{M}_0$, this implies that $m(A)=0$. Thus, $m$ is absolutely continuous with respect to $\mu_0$.

By [2, Lemma 5.5.1], it holds that

$$E_x[e^{-R}]=1 \tag{4.1}$$

for $\mu_0$-a.e. $x\in X$, where $E_x$ stands for the expectation with respect to $P_x$ and

$$R=\inf\{t>0\colon A_t^0>0\}. \tag{4.2}$$

This imples that (4.1) holds for $m$-a.e. $x\in X$. By [2, Lemma 4.2.5], we see that (4.1) holds for $\mathcal{E}_1$-q.e. $x\in X$ and hence $\mu_0\in\mathcal{M}_{00}$.

Theorem 4.1 also yields the following criterion for transience.

**Corollary 4.3.** *Assume that the Dirichlet form* $(\mathcal{E},\mathcal{F})$ *on* $L^2(X;m)$ *is transient. Let* $\mu\in\mathcal{M}'$ *satisfy that* $\mu_0\in\mathcal{M}_{00}$. *Then the closure* $(\mathcal{E}^{(\mu)},\mathcal{F}^{(\mu)})$ *is transient if and only if* $\mu\in\mathcal{M}_0\cap\mathcal{M}'$.

It should be mentioned that $\mu\in\mathcal{M}_0$ if and only if $\mu=\mu_0$.

Proof. Suppose first that $\mathcal{E}^{(\mu)}$ is transient. Then the 0-capacity associated with $\mathcal{E}^{(\mu)}$ is well-defined and coincides with that of $\mathcal{E}$. If a set $A$ is $\mathcal{E}_1$-polar, then it is of zero capacity with respect to the 0-capacity and hence $\mathcal{E}_1^{(\mu)}$-polar. See [2]. Hence the Borel set $N$ obtained in Lemma 2.1 is $\mathcal{E}_1^{(\mu)}$-polar and $\mu=\mu_0$, since $\mu$ charges no $\mathcal{E}_1^{(\mu)}$-polar set.

Conversely, we assume that $\mu=\mu_0$. Define

$$\nu=\mu+m.$$

Then it is elementary to see that $\nu\in\mathcal{M}_0$. By virtue of Corollary 4.2, we can conclude that $\mathcal{E}^{(\nu)}$ is a time changed Dirichlet form of the transient Dirichlet form $\mathcal{E}$. Thus, by [2, Theorem 5.5.1], we have that $\mathcal{E}^{(\nu)}$ is transient. The transience property of $\mathcal{E}^{(\mu)}$ follows from this, because $\mu\leq\nu$.

REMARK. It is seen by Röckner-Wielens [6] that, if $\mathcal{E}$ is transient, then the following two conditions are equivalent for $\mu\in\mathcal{M}'$:

(a) $(\mathcal{E}, C)$ is closable on $L^2(X;\mu)$ and the closure is transient.

(b) A Borel measurable $\mathcal{E}_1$-quasicontinouos modification $\tilde{v}$ of $v\in\mathcal{F}_e$ vanishes $\mu$-a.e. if and only if $v=0$ $\mathcal{E}_1$-q.e.,

where $\mathcal{F}_e$ is the extended Dirichlet space of $\mathcal{F}$. The equivalence also follows from our results. Indeed, we first suppose that (a) is satisfied. Then, Theorem 4.1 and Corollary 4.3 imply that $\mu\in\mathcal{M}_0\cap\mathcal{M}'$. By Theorem 4.1 and [2, Theorem 5.5.2], we see that a set is $\mathcal{E}_1$-polar if and only if $\mathcal{E}_1^{(\mu)}$-polar. Thus, (b) holds.

Conversely, we assume that (b) is satisfied. Then, it is easy to see that $\mu\in\mathcal{M}_0$. Since $E_x[e^{-R}]$ in the proof of Corollary 4.2 is $\mathcal{E}_1$-quasicontinuous and equal to 1, $\mu_0$-a.e., the assumption implies that $E_x[e^{-R}]=1$, $\mathcal{E}_1$-q.e. Hence $\mu_0\in\mathcal{M}_{00}$. Applying Theorem 4.1 and Corollary 4.3, we see that (a) follows.

The proof of Theorem 4.1 will be broken into a series of lemmas.

**Lemma 4.1.** *Let $m_i\in\mathcal{M}'$ and $\mathcal{E}^{(m_i)}$ be a Dirichlet form on $L^2(X;m_i)$ possessing $C$ as a core, $i=1,2$. Assume that there is a $C\geq 0$ such that*

$$\mathcal{E}^{(m_2)}(f,f)\leq C\mathcal{E}^{(m_1)}(f,f) \qquad \text{for } f\in C. \tag{4.3}$$

*Then, for every $\mathcal{E}_1^{(m_1)}$-polar set $N'$, it holds that*

$$G_\alpha^{(m_2)}(\boldsymbol{I}_{N'}u)=\frac{1}{\alpha}\boldsymbol{I}_{N'}u \quad m_2\text{-a.e.} \qquad \text{for every} \quad u\in L^2(X;m_2), \tag{4.4}$$

*where $\{G_\alpha^{(m_2)},\ \alpha>0\}$ is the resolvent associated with $\mathcal{E}^{(m_2)}$.*

Proof. It suffices to show the equality in (4.4) with $N'=K$ and $u=\boldsymbol{I}_K$ for every $\mathcal{E}_1^{(m_1)}$-polar compact set $K$.

Let $G$ be an arbitrary relatively compact open set with $K\subset G$. Choose a $w\in C$ such that $0\leq w\leq 1$, and $w=1$ on $K$, and $=0$ outside of $G$. By virtue of (3.1), there is a sequence $\{g_n\}\subset C$ such that $g_n\geq 1$ on $K$ and $\lim_n \mathcal{E}_1^{(m_1)}(g_n,g_n)=0$. Define

$$\tilde{g}_n=\beta_{1/n}(g_n)\cdot w.$$

Then, $\tilde{g}_{n_j}\to 0$ $m_1$-a.e. for some subsequence $\{\tilde{g}_{n_j}\}$. Moreover, by [2, Theorem 1.4.2] and the Markov property of $\mathcal{E}$, we obtain the inequality

$$\sqrt{\mathcal{E}_1^{(m_1)}(\tilde{g}_n,\tilde{g}_n)}\leq\sqrt{\mathcal{E}_1^{(m_1)}(g_n,g_n)}+2\Big(1+\frac{1}{n}\Big)\sqrt{\mathcal{E}_1^{(m_1)}(w,w)}.$$

Thus, the Cesàro mean $h_n$ of a subsequence of $\{\tilde{g}_n\}$ converges to 0 with respect to the norm $\sqrt{\mathcal{E}_1^{(m_1)}(\cdot,\cdot)}$.

It then follows from (4.3) that

$$\sup_n \mathcal{E}_1^{(m_2)}(h_n, h_n) \leq C \sup_n \mathcal{E}_1^{(m_1)}(h_n, h_n) + 4m_2(G) < \infty .$$

Thus, the Cesàro mean $\tilde{h}_n$ of a subsequence of $\{h_n\}$ converges to an $\mathcal{E}_1^{(m_2)}$-quasi-continuous $h \in \mathcal{F}^{(m_2)}$, the domain of $\mathcal{E}^{(m_2)}$, with respect to the norm $\sqrt{\mathcal{E}_1^{(m_2)}(\cdot,\cdot)}$. The inequality (4.3), moreover, implies that

$$\mathcal{E}^{(m_2)}(h, h) = 0 . \tag{4.5}$$

Let us define a quadratic form $\psi_K$ on $\mathcal{F}^{(m_2)}$ by

$$\psi_K(v) = \mathcal{E}^{(m_2)}(v, v) + \alpha\left(v - \frac{I_K}{\alpha}, v - \frac{I_K}{\alpha}\right)_{m_2}, \quad v \in \mathcal{F}^{(m_2)} .$$

Then, it is known [2, p.23] that

$$\psi_K(G_\alpha^{(m_2)} I_K) = \inf \{\psi_K(v) : v \in \mathcal{F}^{(m_2)}\} .$$

Substituting $v = h/\alpha$, it follows from (4.5) that

$$\psi_K(G_\alpha^{(m_2)} I_K) \leq (h - I_K, h - I_K)_{m_2}/\alpha .$$

Note that $h=1$ $\mathcal{E}_1^{(m_2)}$-q.e. on $K$, and $=0$ $\mathcal{E}_1^{(m_2)}$-q.e. outside of $G$. Hence, letting $G \downarrow K$, we obtain that

$$\psi_K(G_\alpha^{(m_2)} I_K) = 0$$

and

$$G_\alpha^{(m_2)} I_K = \frac{1}{\alpha} I_K, \qquad m_2\text{-a.e.}$$

The proof is complete.

**Lemma 4.2.** *Let $(\mathcal{A}, \mathcal{C})$ be a closable pre-Dirichlet form on $L^2(X; \mu)$ such that $\mathcal{A}(u, u) \leq \mathcal{E}(u, u)$ for $u \in \mathcal{C}$. Then, $(\mathcal{A}, \mathcal{C})$ is also closable on $L^2(S; \mu_0)$.*

Proof. Let $\{R_\alpha, \alpha > 0\}$ be the revolvent associated with the closure $(\bar{\mathcal{A}}, \bar{\mathcal{C}})$ of $(\mathcal{A}, \mathcal{C})$ on $L^2(X; \mu)$. Applying Lemma 4.1, we have that

$$\alpha(uI_{X \setminus N} - \alpha R_\alpha(uI_{X \setminus N}), uI_{X \setminus N})_\mu = \alpha(u - \alpha R_\alpha u, u)_\mu \to \bar{\mathcal{A}}(u, u), \quad \text{as } \alpha \to \infty \tag{4.6}$$

for any $u \in \mathcal{C}$. Since $\mu((X \setminus S) \setminus N) = 0$, this implies that

$$uI_{S \setminus N} \in \bar{\mathcal{C}}, \quad \bar{\mathcal{A}}(u, u) = \bar{\mathcal{A}}(uI_{S \setminus N}, uI_{S \setminus N}) . \tag{4.7}$$

Let $\{f_n\} \subset \mathcal{C}$ be a sequence converging to 0 in $L^2(S; \mu_0)$ such that $\mathcal{A}(f_n - f_m, f_n -$

$f_m)\to 0$ as $n, m\to\infty$. Then, by (4.7), $\{f_n \boldsymbol{I}_{S\setminus N}\}$ is an $\bar{\mathcal{A}}$-Cauchy sequence and converges to 0 in $L^2(X; \mu)$. Combined with the closedness of $\bar{\mathcal{A}}$, (4.7) also implies that

$$\mathcal{A}(f_n, f_n) = \bar{\mathcal{A}}(f_n \boldsymbol{I}_{S\setminus N}, f_n \boldsymbol{I}_{S\setminus N})\to 0 .$$

Thus, $(\mathcal{A}, \mathcal{C})$ is closable on $L^2(S; \mu_0)$.

**Lemma 4.3.** *$(\mathcal{E}_0^{(\mu)}, \mathcal{C})$ is the closable part of $(\mathcal{E}, \mathcal{C})$ on $L^2(X; \mu)$.*

Proof. Let $(\mathcal{A}, \mathcal{C})$ is a closable pre-Dirichlet form on $L^2(X; \mu)$ with $\mathcal{A}(u, u)\leq \mathcal{E}(u, u)$ for $u\in\mathcal{C}$. Fix an arbitrary $f\in\mathcal{C}$. By definition, it holds that

$$\mathcal{E}^{(\mu)}(f, f) = \mathcal{L}(\mathcal{P}f, \mathcal{P}f)-(f, f)_{\mu_0} . \tag{4.8}$$

Since $\mathcal{C}$ is dense in $\mathcal{K}_e$, there is a sequence $\{f_n\}\subset\mathcal{C}$ such that

$$\mathcal{L}(f_n-\mathcal{P}f, f_n-\mathcal{P}f)\to 0 .$$

The definition of $\mathcal{L}$ and (3.8) then imply that

(4.9) $\{f_n\}$ is an $\mathcal{E}$-Cauchy sequence and $f_n\to f$ in $L^2(X; \mu_0)$.

Combining with (4.8), we see that

$$\mathcal{E}^{(\mu)}(f, f) = \lim_n \mathcal{E}(f_n, f_n) . \tag{4.10}$$

It follows from (4.9) that (i) $\mathcal{A}((f_n-f)-(f_m-f), (f_n-f)-(f_m-f))\to 0$ and (ii) $f_n-f\to 0$ in $L^2(X; \mu_0)$. By Lemma 4.2, we have that $\mathcal{A}(f_n-f, f_n-f)\to 0$. It therefore holds that

$$\mathcal{A}(f, f) = \lim_n \mathcal{A}(f_n, f_n)\leq\lim_n \mathcal{E}(f_n, f_n) = \mathcal{E}^{(\mu)}(f, f) ,$$

which completes the proof.

**Lemma 4.4.** *$(\mathcal{E}, \mathcal{C})$ is closable on $L^2(X; \mu)$ if and only if $\mu_0\in\mathcal{M}_{00}$. Furthermore, if it is closable, then $\mathcal{E}^{(\mu)}$ is its closure.*

Proof. We first assume that $\mu_0\in\mathcal{M}_{00}(\subset\mathcal{M}_0\cap\mathcal{M}')$. Since a set is $\mathcal{L}$-polar if and only if $\mathcal{E}_1$-polar (cf. [4]), the orthogonal projeation $\mathcal{P}$ of $\mathcal{K}_e$ into itself is then the identity map. Thus, we have that $\mathcal{E}^{(\mu)}=\mathcal{E}$ on $\mathcal{C}\times\mathcal{C}$, which implies the closability of $(\mathcal{E}, \mathcal{C})$ on $L^2(X; \mu)$ and that $\mathcal{E}^{(\mu)}$ is the closure of $(\mathcal{E}, \mathcal{C})$ on $L^2(X; \mu)$.

We next assume that $(\mathcal{E}, \mathcal{C})$ is closable. By Theorem 2.1 and Lemma 4.3, we have that, for any $f\in\mathcal{C}$,

$$\begin{aligned}\mathcal{L}(f, f) &= \mathcal{E}(f, f)+(f, f)_{\mu_0}\\ &\leq \mathcal{E}^{(\mu)}(f, f)+(f, f)_{\mu_0}\\ &= \mathcal{L}(\mathcal{P}f, \mathcal{P}f) .\end{aligned}$$

This implies that $f=\mathcal{P}f$ $\mathcal{L}$-q.e., for $f\in\mathcal{C}$. Hence, by virtue of the equivalence of $\mathcal{L}$-polarity and $\mathcal{E}_1$-polarity (cf.[4]), we have that

$$u\in\mathcal{K} \text{ vanishes } \mathcal{E}_1\text{-q.e. on } X \text{ if so on } \tilde{S}. \tag{4.11}$$

For $\alpha>0$ and bounded $f\in L^2(X; m)$ with $f\geq 0$ $m$-a.e., we define

$$\tilde{R}_\alpha f(x)=E_x[\int_0^{\tilde{\sigma}} e^{-\alpha t}f(X_t)dt],$$

where $E_x$ stands for the expeatation with respect to $P_x$ and $\tilde{\sigma}=\inf\{t>0: X_t\in\tilde{S}\}$. Then, $\tilde{R}_\alpha f\in\mathcal{F}$ and vanishes $\mathcal{E}_1$-q.e. on $\tilde{S}$. For any $g\in\mathcal{C}$, $g\cdot\tilde{R}_\alpha f\in\mathcal{K}$ and vanishes $\mathcal{E}_1$-q.e. on $\tilde{S}$. By (4.11), $g\cdot\tilde{R}_\alpha f=0$ $\mathcal{E}_1$-q.e. on $X$. Letting $g\uparrow 1$, we obtain that $\tilde{R}_\alpha f$ vanishes $\mathcal{E}_1$-q.e. on $X$. Hence

$$P_x[\tilde{\sigma}=0]=1 \qquad \mathcal{E}_1\text{-q.e. } x\in X.$$

Since $\tilde{S}$ is finely closed with respect to $\boldsymbol{M}$, this implies that $Cap(X\backslash\tilde{S})=0$.

**Lemma 4.5.** *For $\mu\in\mathcal{M}'$, let $\{G_\alpha^0, \alpha>0\}$ (resp. $\{G_\alpha^{(\mu)}, \alpha>0\}$) be the resolvent on $L^2(S; \mu_0)$ (resp. $L^2(X; \mu)$) associated with $\mathcal{E}^0$(resp. $\mathcal{E}^{(\mu)}$). Then*

$$G_\alpha^0 u=G_\alpha^{(\mu)}u \quad \mu_0\text{-a.e. } \textit{for any bounded} \quad u\in L^2(X; \mu).$$

Proof. Let $u\in L^2(X; \mu)$ be bounded. Since

$$\begin{aligned}\mathcal{E}^{(\mu)}(f,f)&=\mathcal{L}(\mathcal{P}f, \mathcal{P}f)-(f,f)_{\mu_0}\\ &\leq\mathcal{L}(f,f)-(f,f)_{\mu_0}=\mathcal{E}(f,f), \quad f\in\mathcal{C},\end{aligned}$$

applying Lemma 4.1, we obtain that

$$G^{(\mu)}(\boldsymbol{I}_N u)=\frac{1}{\alpha}\boldsymbol{I}_N u \qquad \mu\text{-a.e.} \tag{4.12}$$

Hence it holds that

$$\mathcal{E}^{(\mu)}(G^{(\mu)} u, v)=(u-\alpha G_\alpha^{(\mu)}u, v)_\mu=(u-\alpha G_\alpha^{(\mu)} u, v)_{\mu_0}, \qquad v\in\mathcal{C}. \tag{4.13}$$

On the other hand, if a sequence $\{f_n\}\subset\mathcal{C}$ converges to $G_\alpha^{(\mu)}u$ in $\mathcal{F}^{(\mu)}$, then $\{f_n|_S\}$ is a Cauchy sequence in $\mathcal{F}^0$ and converges to $G_\alpha^{(\mu)}u$ in $L^2(S; \mu_0)$, because $\mathcal{E}^{(\mu)}(f,f)=\mathcal{E}^0(f|_S, f|_S)$ for $f\in\mathcal{C}$. Hence we see that $G_\alpha^{(\mu)}u\in\mathcal{F}^0$ and

$$\mathcal{E}^0(G_\alpha^{(\mu)}u, v|_S)=\mathcal{E}^{(\mu)}(G_\alpha^{(\mu)}u, v), \qquad v\in\mathcal{C}.$$

Combining this with (4.13), we obtain the desired conclusion.

**Lemma 4.6.** *In the case that $\mu_0\in\mathcal{M}_{00}$, $\mathcal{E}^{(\mu)}$ is realized in the way described in Theorem* 4.1.

Proof. This is a consequence of Lemmas 4.1 and 4.5.

## 5. Examples

EXAMPLE 5.1. Let $X=\boldsymbol{R}^d$ $(d\geq 2)$, $m=\lambda$, the Lebesgue measure on $\boldsymbol{R}^d$, and $\mathcal{C}=C_0^\infty(\boldsymbol{R}^d)$. Consider the Dirichlet integral

$$\mathcal{E}(u, v)=\int_{\boldsymbol{R}^d}\sum_{i=1}^d \frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_i}(x)d\lambda(x)$$

and $\mu\in\mathcal{M}'$ given by

$$\mu=\boldsymbol{I}_{R^d\setminus B}\cdot\lambda+\sum_{p_n\in B}2^{-n}\delta_{p_n},$$

where $B$ is the unit open ball in $\boldsymbol{R}^d$, $\delta_x$ stands for the Dirac measure concentrated at $x$ and $\{p_n\}$ is the set of all rational points in $\boldsymbol{R}^d$. It is elementary to see that

$$\mu_0=\boldsymbol{I}_{\boldsymbol{R}^d\setminus B}\cdot\lambda \quad\text{and}\quad S=\tilde{S}=\boldsymbol{R}^d\setminus B\,.$$

In view of Corollary 4.1, $(\mathcal{E}, \mathcal{C})$ is not closable on $L^2(\boldsymbol{R}^d; \mu)$. By (3.8) and the definition of $\mathcal{L}$, the closable part $\mathcal{E}_0^{(\mu)}$ is given by

$$\mathcal{E}_0^{(\mu)}(f,f)=\int_B\sum_{i=1}^d\left(\frac{\partial\mathcal{P}f}{\partial x_i}(x)\right)^2 d\lambda(x)+\int_{\boldsymbol{R}^d\setminus B}\sum_{i=1}^d\left(\frac{\partial f}{\partial x_i}(x)\right)^2 d\lambda(x),\quad f\in C_0^\infty(\boldsymbol{R}^d).$$

Since $\mathcal{L}(u, u)=\mathcal{E}(u, u)$ for $u\in C_0^\infty(B)$, the part on $B$ of the Hunt process associated with $\mathcal{L}$ coincides with the absorbing barrier Brownian motion on $B$. Hence, we have that $\mathcal{P}f$ is $\mathcal{E}$-harmonic on $B$:

$$\int_B\sum_{i=1}^d\frac{\partial\mathcal{P}f}{\partial x_i}(x)\frac{\partial u}{\partial x_i}(x)d\lambda(x)=0\,,\qquad u\in C_0^\infty(B)\,.$$

The corresponding process $\boldsymbol{M}^{(\mu)}=(X_t^{(\mu)}, P_x^{(\mu)})$ is obtained by time changing the Brownian motion $\boldsymbol{M}=(X_t, P_x)$ on $\boldsymbol{R}^d$ by $A_t^B=\int_0^t\boldsymbol{I}_{\boldsymbol{R}^d\setminus B}(X_s)ds$ and making every points in $B$ traps. More precisely, define successively $\sigma_0=\tau_0=\tau_1=0$, $\sigma_n=\inf\{t>\tau_n\colon X_t\in B\}$, and $\tau_n=\inf\{t>\sigma_{n-1}\colon X_t\notin B\}$. Then, under $P_x$, $x\notin B$,

$$X_t^{(\mu)}=X_{t+\tau_{n+1}-\sum_{j=0}^n(\sigma_j-\tau_j)},\quad \sum_{j=0}^n(\sigma_j-\tau_j)\leq t<\sum_{j=0}^{n+1}(\sigma_j-\tau_j),\quad n=0, 1, \dots,$$

and $P_x^{(\mu)}(X_t^{(\mu)}=x)=1$, $x\in B$.

EXAMPLE 5.2. Let $D$ be a domain in $\boldsymbol{R}^d$ and consider a pre-Dirichlet form given by

$$\mathcal{E}(u, v)=\sum_{i,j=1}^d\int_D\frac{\partial u}{\partial x_i}(x)\frac{\partial v}{\partial x_j}(x)a^{ij}(x)d\lambda(x)\,,\qquad u, v\in C_0^\infty(D)\,,$$

where $(a^{ij}(x))_{1\leq i,j\leq d}$ is a nonneagtive symmetric matrix valued locally $\lambda$-integral function. Suppose that one of the following conditions is satisfied:

(a°) $\dfrac{\partial a^{ij}}{\partial x_i} \in L^2_{loc}(D: \lambda)$

(b°) $(a^{ij}) \geq \delta_K$ for some $\delta_K > 0$ on each compact $K \subset D$.

Then $(\mathcal{E}, C_0^\infty(D))$ is closable on $L^2(D; \lambda)$. Indeed, (a°) is in [2, §2.1]. To see the case (b°), let $\{u_n\}$ be an $\mathcal{E}_1$-Cauchy sequence converging to 0 in $L^2(D; \lambda)$. Applying the argument used in [2, §2.1] to $f \cdot u_n, f \in C_0^\infty(D)$, we have that $\dfrac{\partial u_n}{\partial x_i}$ converges to 0, $\lambda$-a.e. Then, repeating the argument there, we see that $(\mathcal{E}, C_0^\infty(D))$ is closable on $L^2(D; \lambda)$.

Applying Corollary 4.2, we see that $(\mathcal{E}, C_0^\infty(D))$ is closable on $L^2(D; \mu)$ if $\lambda$ is absolutely continuous with respect to $\mu$. Similar criteria can be found in [6, §4].

## 6. Closable parts of symmetric forms

In this section, we will investigate the closable parts of general symmetric forms on a real Hilbert space $H$ with inner product $(\cdot, \cdot)_H$. Let $\mathcal{A}$ be a symmetric form on $H$ with domain $\mathcal{D}$ in the sense of [2]. For $\alpha > 0$, we define

$$\mathcal{A}_\alpha(u, v) = \mathcal{A}(u, v) + \alpha(u, v)_H, \qquad u, v \in \mathcal{D}. \tag{6.1}$$

Denote by $\hat{\mathcal{D}}$ the abstract completion of $\mathcal{D}$ with respect to $\sqrt{\mathcal{A}_\alpha(\cdot, \cdot)}$. The extension of $\mathcal{A}_\alpha$ is denoted by $\hat{\mathcal{A}}_\alpha$. Then, the set $\hat{\mathcal{D}}$ is independent of $\alpha > 0$ and it is a Hilbert space with inner product $\hat{\mathcal{A}}_\alpha$ for each $\alpha > 0$.

Let $\varphi: \hat{\mathcal{D}} \to H$ be the bounded linear map determined by $\varphi u = u$ for any $u \in \mathcal{D}$. We have that

$$\alpha \|\varphi u\|_H^2 \leq \hat{\mathcal{A}}_\alpha(u, u), \qquad u \in \hat{\mathcal{D}}, \tag{6.2}$$

where $\|\cdot\|_H = \sqrt{(\cdot, \cdot)_H}$. We remark that $\varphi$ is not necessarily injective and that $\mathcal{A}$ is closable if and only if $\varphi$ is injective.

It follows from (6.2) that, for any $u \in H$, there exists a unique $w \in \hat{\mathcal{D}}$ with $\hat{\mathcal{A}}_\alpha(w, v) = (u, \varphi v)_H$ for every $v \in \hat{\mathcal{D}}$. We define $\hat{G}_\alpha: H \to \hat{\mathcal{D}}$ by $w = \hat{G}_\alpha u$ and $G'_\alpha: H \to H$ by

$$G'_\alpha u = \varphi(\hat{G}_\alpha u), \qquad u \in H.$$

We will show the following.

**Theorem 6.1.** *(i) $\{G'_\alpha, \alpha > 0\}$ is a strongly continuous resolvent on $H$ in the sense of* [2].
*(ii) Let $\mathcal{A}'$ with domain $\mathcal{D}'$ be the closed symmetric form associated with $\{G'_\alpha,$*

$\alpha>0\}$. *Then, it holds that*

$$\mathcal{D}' = \varphi(\hat{\mathcal{D}}), \tag{6.3}$$

$$\mathcal{A}'(u, u) = \inf \lim_n \mathcal{A}(u_n, u_n) \qquad \textit{for} \quad u\in\mathcal{D}', \tag{6.4}$$

*where the infimum is taken over all* $\mathcal{A}_1$*-Cauchy sequences* $\{u_n\}\subset\mathcal{D}$ *with* $\|u_n-u\|_H\to 0$. *Moreover,* $(\mathcal{A}', \mathcal{D}')$ *is the closure of* $(\mathcal{A}'|_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$.

(*iii*) *Suppose that* $\mathcal{B}$ *is a closed symmetric form on* $H$ *with domain* $\mathcal{D}_{\mathcal{B}}$ *satisfying that* $\mathcal{D}_{\mathcal{B}}\supset\mathcal{D}$ *and that* $\mathcal{B}(u, u)\leq\mathcal{A}(u, u)$ *for* $u\in\mathcal{D}$. *Then* $\mathcal{D}_{\mathcal{B}}\supset\mathcal{D}'$ *and* $\mathcal{B}(u, u)\leq\mathcal{A}'(u, u)$ *for* $u\in\mathcal{D}'$.

(*iv*) *Suppose that* $H=L^2(X; \mu)$ *for some everywhere dense positive Radon measure* $\mu$ *on a locally compact separable metric space* $X$. *If* $\mathcal{A}$ *is Markovian, then so is* $\mathcal{A}'$.

We call the closable symmetric form $(\mathcal{A}'|_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$ in Theorem 6.1 the closable part of the symmetric form $\mathcal{A}$. It follows that, if $H=L^2(X; \mu)$ and $\mathcal{A}$ is a pre-Dirichlet form, then $(\mathcal{A}'|_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$ is a pre-Dirichlet form which is the closable part of the pre-Dirichlet form $\mathcal{A}$ in the sense of Section 1.

Combining Theorem 6.1 with Lemma 4.3, we see that, for a pre-Dirichlet form $(\mathcal{E}, \mathcal{C})$ on $L^2(X; \mu)$, the closable part $(\mathcal{E}_0^{(\mu)}, \mathcal{C})$ is realized also in the manner as stated in Theorem 6.1. It is also possible to see directly that $\mathcal{E}^{(\mu)}$ coincides with $\mathcal{E}'$ constructed in Theorem 6.1. See Proposition 6.1 at the end of this section.

We now proceed to the proof of Theorem 6.1.

Proof of (i). The symmetry of $G'_\alpha$ is an immediate consequence of the definition. The definition of $G'_\alpha$ and (6.2) imply the contraction property of $G'_\alpha$:

$$\alpha\|G'_\alpha u\|_H\leq\|u\|_H, \qquad \text{for every} \quad u\in H. \tag{6.5}$$

By the definition of $\hat{G}_\alpha$, we have that

$$\hat{\mathcal{A}}_\alpha(\hat{G}_\alpha u-\hat{G}_\beta u, v) = -(\alpha-\beta)\hat{\mathcal{A}}_\alpha(\hat{G}_\alpha(G'_\beta u), v), \qquad \text{for} \quad v\in\mathcal{D}.$$

Thus it holds that $\hat{G}_\alpha u-\hat{G}_\beta u+(\alpha-\beta)\hat{G}_\alpha(G'_\beta u)=0$, which implies the resolvent equation for $\{G'_\alpha, \alpha>0\}$.

To see the strong continuity, notice that (6.5) implies that $\alpha(u, G'_\alpha u)_H\leq \|u\|_2^H$. Hence it follows from (6.2) that, for $u\in\mathcal{D}$,

$$\begin{aligned}\alpha\|\alpha G'_\alpha u-u\|_H^2 &\leq \hat{\mathcal{A}}_\alpha(\alpha\hat{G}_\alpha u-u, \alpha\hat{G}_\alpha u-u)\\ &= \alpha^2(u, G'_\alpha u)_H-\alpha(u, u)_H+\mathcal{A}(u, u)\\ &\leq \mathcal{A}(u, u).\end{aligned} \tag{6.6}$$

Thus

(6.7) $$\lim_{\alpha\to\infty} \|\alpha G'_\alpha u - u\|_H = 0 .$$

for every $u\in\mathcal{D}$. Since $\mathcal{D}$ is dense in $H$, (6.5) yields (6.7) for all $u\in H$.

Proof of (ii). We first show that (6.3) holds. Let

$$\mathcal{A}'_\alpha(u, u) = \mathcal{A}'(u, u)+\alpha(u, u)_H , \qquad u\in\mathcal{D}' ,$$

for $\alpha>0$. Notice that

(6.8) $$\hat{\mathcal{A}}_\alpha(\hat{G}_\alpha u, \hat{G}_\alpha u) = \mathcal{A}'_\alpha(G'_\alpha u, G'_\alpha u) , \qquad u\in H .$$

If $u\in\mathcal{D}$, then (6.6) implies that

$$\alpha(u-\alpha G'_\alpha u, u)_H\le\mathcal{A}(u, u) .$$

Letting $\alpha\uparrow\infty$, we see that $\mathcal{D}\subset\mathcal{D}'$ and that $\mathcal{A}'_\alpha(u, u)\le\mathcal{A}_\alpha(u, u)$ for $u\in\mathcal{D}$.

Let $u\in\hat{\mathcal{D}}$. Take $u_n\in\mathcal{D}$ such that $\hat{\mathcal{A}}_\alpha(u_n-u, u_n-u)\to 0$. Then, $u_n\to\varphi u$ in $H$. By the above observation, $\{u_n\}$ is an $\mathcal{A}'_\alpha$-Cauchy sequence in $\mathcal{D}'$ and satisfies that

(6.9) $$\mathcal{A}'_\alpha(u_n, u_n)\le\hat{\mathcal{A}}_\alpha(u_n, u_n) .$$

The closedness of $\mathcal{A}'$ then implies that $\varphi u\in\mathcal{D}'$ and hence

(6.10) $$\varphi(\hat{\mathcal{D}})\subset\mathcal{D}'.$$

Moreover, letting $n\uparrow\infty$ in (6.9), we see that

(6.11) $$\mathcal{A}'_\alpha(\varphi u, \varphi u)\le\hat{\mathcal{A}}_\alpha(u, u) \qquad \text{for any} \quad u\in\hat{\mathcal{D}} .$$

To see the converse inclusion, let $u\in\mathcal{D}'$. Since $G'_\alpha(H)$ is $\mathcal{A}'_\alpha$-dense in $\mathcal{D}'$, there is a sequence $\{u_n\}\subset H$ such that

$$\mathcal{A}'_\alpha(G'_\alpha u_n-u, G'_\alpha u_n-u)\to 0 .$$

It then follows from (6.8) that $\{\hat{G}_\alpha u_n\}$ is an $\hat{\mathcal{A}}_\alpha$-Cauchy sequence. Let $v\in\hat{\mathcal{D}}$ be its limit. By virtue of (6.10) and (6.11), $G'_\alpha u_n-\varphi v=\varphi(\hat{G}_\alpha u_n-v)$ is in $\mathcal{D}'$ and converges to 0 with respect to $\mathcal{A}'_\alpha$. Hence $u=\varphi v$. Thus $\mathcal{D}'\subset\varphi(\hat{\mathcal{D}})$ and (6.3) is established.

We now show that (6.4) holds. Let

$$v\in(\mathrm{Ker}\,\varphi)^\perp\equiv\{v\in\hat{\mathcal{D}}: \hat{\mathcal{A}}_\alpha(v, w)=0 \quad \text{for every } w\in\hat{\mathcal{D}} \text{ with } \varphi w=0\} .$$

By (6.3), there is a sequence $\{f_n\}\subset H$ such that

$$\mathcal{A}'_\alpha(G'_\alpha f_n-\varphi v, G'_\alpha f_n-\varphi v)\to 0 .$$

Hence, by (6.8), $\{\hat{G}_\alpha f_n\}$ is an $\hat{\mathcal{A}}_\alpha$-Cauchy sequence. Let $v'\in\hat{\mathcal{D}}$ be its limit.

Then, we have that

$$\varphi v' = \varphi v . \tag{6.12}$$

Note that $\hat{G}_\alpha f_n \in (\mathrm{Ker}\,\varphi)^\perp$ by the definition of $\hat{G}_\alpha$. Hence $v' \in (\mathrm{Ker}\,\varphi)^\perp$. Combining this with (6.12), we see that $v'=v$ since $\varphi\colon (\mathrm{Ker}\,\varphi)^\perp \to H$ is injective. Thus, applying (6.8) we have that

$$\begin{aligned} \hat{\mathcal{A}}_\alpha(v, v) &= \lim_n \mathcal{A}_\alpha(\hat{G}_\alpha f_n, \hat{G}_\alpha f_n) \\ &= \lim_n \mathcal{A}'_\alpha(G'_\alpha f_n, G'_\alpha f_n) \\ &= \mathcal{A}'_\alpha(\varphi v, \varphi v) . \end{aligned} \tag{6.13}$$

Let $u \in \mathcal{D}'$. Choose $v \in \hat{\mathcal{D}}$ such that $u = \varphi v$ and decompose it as $v = v_1 + v_2$, where $v_1 \in (\mathrm{Ker}\,\varphi)^\perp$ and $\varphi v_2 = 0$. It then follows from (6.13) that

$$\mathcal{A}'_\alpha(u, u) = \mathcal{A}'_\alpha(\varphi v_1, \varphi v_1) = \hat{\mathcal{A}}_\alpha(v_1, v_1) .$$

On the other hand, we have that

$$\begin{aligned} \hat{\mathcal{A}}_\alpha(v_1, v_1) &= \inf \{\hat{\mathcal{A}}_\alpha(v-w, v-w)\colon w \in \hat{\mathcal{D}} \text{ with } \varphi w = 0\} \\ &= \inf \{\hat{\mathcal{A}}_\alpha(w, w)\colon w \in \hat{\mathcal{D}} \text{ with } \varphi w = u\} \\ &= \inf \{\lim_n \mathcal{A}_\alpha(w_n, w_n)\colon \{w_n\} \subset \mathcal{D} \text{ is } \mathcal{A}_\alpha\text{-Cauchy and } \|w_n - u\|_H \to 0\} . \end{aligned}$$

This implies the equality (6.4).

To prove that $(\mathcal{A}', \mathcal{D}')$ is the closure of $(\mathcal{A}' |_{\mathcal{D}\times\mathcal{D}}, \mathcal{D})$, let $u \in \mathcal{D}'$. Choose $v \in \hat{\mathcal{D}}$ and $\{v_n\} \subset \mathcal{D}$ such that $\varphi v = u$ and $\hat{\mathcal{A}}_\alpha(v_n - v, v_n - v) \to 0$. Then, $\|v_n - u\|_H \to 0$ by (6.2) and $\{v_n\}$ is $\mathcal{A}'_\alpha$-Cauchy by (6.11). Hence the closedness of $\mathcal{A}'$ implies that $\mathcal{A}'_\alpha(v_n - u, v_n - u) \to 0$.

Proof of (iii). Define $\mathcal{B}_1(u, v) = \mathcal{B}(u, v) + (u, v)_H$. Let $u \in \mathcal{D}'$ and $\{u_n\} \subset \mathcal{D}$ be an $\mathcal{A}_1$-Cauchy sequence with $u_n \to u$ in $H$. Then, $\{u_n\}$ is a $\mathcal{B}_1$-Cauchy sequence. The closedness of $\mathcal{B}$ implies that $\mathcal{B}_1(u_n - w, u_n - w) \to 0$ for some $w \in \mathcal{D}_{\mathcal{B}}$. It follows that $u = w$ and $\mathcal{D}' \subset \mathcal{D}_{\mathcal{B}}$. Since $\mathcal{B}(u_n, u_n) \le \mathcal{A}(u_n, u_n)$, we have

$$\mathcal{B}(u, u) \le \lim_n \mathcal{A}(u_n, u_n) .$$

Combining this with (6.4), we see that

$$\mathcal{B}(u, u) \le \mathcal{A}'(u, u) \qquad \text{for every} \quad u \in \mathcal{D}' .$$

Proof of (iv). Let $f \in H$ and $0 \le f \le 1$ $\mu$-a.e. Define a quadratic form $\psi$ on $\hat{\mathcal{D}}$ by

$$\psi(u) = \hat{\mathcal{A}}(u, u) + \alpha(\varphi u - f, \varphi u - f)_H , \qquad u \in \hat{\mathcal{D}} ,$$

where $\hat{\mathcal{A}}$ is the extension of $\mathcal{A}$ to $\hat{\mathcal{D}}$. Then, it holds that

$$\psi(u)-\psi(\alpha\hat{G}_\alpha f) = \hat{\mathcal{A}}_\alpha(u-\alpha\hat{G}_\alpha f, u-\alpha\hat{G}_\alpha f), \qquad u\in\hat{\mathcal{D}}.$$

See [2, p. 23].

Choose $u_n\in\mathcal{D}$ such that

$$\hat{\mathcal{A}}_\alpha(u_n-\alpha\hat{G}_\alpha f, u_n-\alpha\hat{G}_\alpha f)\to 0.$$

For $\varepsilon>0$, we define

$$u_n^\varepsilon = \beta_\varepsilon(u_n).$$

It then holds that $u_n^\varepsilon\in\mathcal{D}$, $\mathcal{A}(u_n^\varepsilon, u_n^\varepsilon)\leq\mathcal{A}(u_n, u_n)$ and $|u_n^\varepsilon-f|\leq|u_n-f|$. Hence we have that

$$\psi(\alpha\hat{G}_\alpha f)\leq\psi(u_n^\varepsilon)\leq\psi(u_n)\to\psi(\alpha\hat{G}_\alpha f),$$

which implies that

$$\hat{\mathcal{A}}_\alpha(\alpha\hat{G}_\alpha f-u_n^\varepsilon, \alpha\hat{G}_\alpha f-u_n^\varepsilon) = \psi(\alpha\hat{G}_\alpha f)-\psi(u_n^\varepsilon)\to 0.$$

Thus, a subsequence of $\{u_n^\varepsilon\}$ converges to $\alpha G'_\alpha f$ $\mu$-a.e. and it holds that

$$-\varepsilon\leq\alpha G'_\alpha f\leq 1+\varepsilon, \qquad \mu\text{-a.e.}$$

Letting $\varepsilon\downarrow 0$, we see that $0\leq\alpha G'_\alpha f\leq 1$ $\mu$-a.e. This shows that $\mathcal{A}'$ is Markovian (cf. [2, Theorem 1.4.1]).

**Proposition 6.1.** *Let $\mu\in\mathcal{M}'$, $\mathcal{D}=\mathcal{C}$, $\mathcal{A}=\mathcal{E}$ and $H=L^2(X;\mu)$. Then the resolvents $\{G_\alpha^{(\mu)}, \alpha>0\}$ and $\{G'_\alpha, \alpha>0\}$ coincide.*

Proof. This is a consequence of Theorems 4.1 and 6.1. We, however, will give a direct proof below.

It suffices to show that

$$G_\alpha^{(\mu)}u = G'_\alpha u \qquad \text{for any bounded} \quad u\in L^2(X;\mu). \tag{6.14}$$

For the $\mathcal{E}_1$-polar set $N$ with $\mu_0=\boldsymbol{I}_{X\setminus N}\cdot\mu$, by Lemma 4.1 and (4.12), it holds that

$$G_\alpha^{(\mu)}(\boldsymbol{I}_N u) = G'_\alpha(\boldsymbol{I}_N u) = \frac{1}{\alpha}\boldsymbol{I}_N u \qquad \mu\text{-a.e.,} \quad u\in L^2(X;\mu). \tag{6.15}$$

Thus, according to Lemma 4.5, it suffices to show that, for every bounded $u\in L^2(X;\mu)$,

$$G_\alpha^0 u = G'_\alpha u \qquad \mu_0\text{-a.e.} \tag{6.16}$$

By the definition of $\mathcal{L}$, there is a bounded linear map $\varphi'\colon\hat{\mathcal{C}}\to\mathcal{K}_e$ such that $\varphi' v=v$ for $v\in\mathcal{C}$. Since $L^2(X;\mu)\subset L^2(S;\mu_0)$, we then obtain that

$$\varphi' u = \varphi u \qquad \mu_0\text{-a.e. for every} \quad u\in\hat{\mathcal{C}}. \tag{6.17}$$

Moreover, it f holds that

$$(6.18)\qquad \hat{\mathcal{E}}(v, w) = \mathcal{L}(\varphi' v, \varphi' w) - (\varphi' v, \varphi' w)_{\mu_0}, \qquad v, w \in \hat{\mathcal{C}}.$$

Combining these with (6.15) and the definition of $\hat{G}_\alpha$, we have that

$$(6.19)\qquad \mathcal{L}(\varphi' \hat{G}_\alpha u, v) = (u - \alpha G'_\alpha u, v)_{\mu_0} + (\varphi' \hat{G}_\alpha u, v)_{\mu_0}, \qquad v \in \mathcal{C}.$$

Recall that a set is $\mathcal{E}_1$-polar if and only if $\mathcal{L}$-polar. See [4]. Moreover, note that $\mathcal{C}$ is dense in $\mathcal{K}_e$ and that if $f_n \to f$ in $\mathcal{K}_e$ then a subsequence converges to $f$ in $L^2(S; \mu_0)$. Thus, it follows from (3.3) and (6.19) that

$$(6.20)\qquad \mathcal{L}(\varphi' \hat{G}_\alpha u, v) = 0 \qquad \text{for any } v \in \mathcal{K}_e \text{ with } v = 0 \ \mathcal{L}\text{-q.e. on } \tilde{S}.$$

Hence it hold that

$$(6.21)\qquad \mathcal{P}(\varphi' \hat{G}_\alpha u) = \varphi' \hat{G}_\alpha u.$$

By virtue of (6.17), (6.19) and (6.21), we can conclude that $G'_\alpha u \in \mathcal{F}^0$ and, for every $v \in \mathcal{C}$,

$$\begin{aligned} \mathcal{E}^0(G'_\alpha u, v) &= \mathcal{L}(\varphi' \hat{G}_\alpha u, v) - (\varphi' \hat{G}_\alpha u, v)_{\mu_0} \\ &= (u - \alpha G'_\alpha u, v). \end{aligned}$$

This implies that (6.16) holds.

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