GENERATION THEOREM OF SEMIGROUP FOR MULTIVALUED LINEAR OPERATORS

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1. Introduction

In the paper Favini-Yagi [8], the notion of multivalued linear operator was introduced as a tool providing a new approach toward the degenerate linear evolution equations with respect to the time derivative.

Consider, for example, an abstract degenerate equation

$$(\text{D-E.1}) \quad \begin{cases} d(Mv)/dt + Lv = f(t), & 0 \leq t \leq T, \\ Mv(0) = u_0, \end{cases}$$

in a Banach space X, where M and L are linear operators in X. If we change the unknown function $v = v(t)$ to $u = Mv(t)$, then (D-E.1) is written in a non-degenerate form

$$(\text{E}) \quad \begin{cases} du/dt + Au \ni f(t), & 0 \leq t \leq T, \\ u(0) = u_0, \end{cases}$$

using an operator $A = LM^{-1}$. Of course, A is no longer a univalent operator but conserves linearity in a multivalued sense, that is, A is a multivalued linear operator in X (cf. the Definition in Section 2). It is generally true that the degenerate linear evolution equations with respect to the time derivative are rewritten into non degenerate equations in such a way using multivalued linear operators.

This fact then leads us naturally to consider a problem of generalizing the well developed results concerning the ordinary linear evolution equations with univalent coefficient operators to those with multivalued operators and to intend handling the degenerate equations by means of analogous techniques to non degenerate ones. We have already devoted ourselves in [8] to showing that this generalization is the case for the parabolic linear equations. This paper is then devoted to establishing a multivalued version of the Hille-Yosida generation theorem of linear semigroup in a Banach space.

We shall prove, in fact, in Section 3 that a multivalued linear operator A in a Banach space X generates, under the Hille-Yosida condition on $-A$, a linear semigroup e^{-tA}, $t \geq 0$, on X. On the space $\overline{\mathcal{D}(A)}$, e^{-tA} is seen to be a C_0-

semigroup. On the space $\overline{\mathscr{D}(A)} + A0$, $e^{-tA} \in \mathscr{L}(\overline{\mathscr{D}(A)} + A0)$ defines a semigroup of bounded operators, and e^{-tA}, $t > 0$, vanishes entirely on $A0$. On the whole space X, however, we can not give any sense to e^{-tA} at each point $t > 0$, we can only see that $\int_0^t e^{-(t-\tau)A} f(\tau) d\tau$ is defined for every $f \in L^1((0,T); X)$, $0 < T < \infty$, that is, e^{-tA} defines on X a distribution semigroup the notion of which was introduced by J.L. Lions [12]. (Or we can equally consider that e^{-tA} defines an integrated semigroup on X in the sense due to Arendt [1,2].) When X is reflexive, it is verified that the Hille-Yosida condition implies $X = \overline{\mathscr{D}(A)} + A0$.

As a nonlinear version of the Hille-Yosida theory, we know the theory of nonlinear semigroup, c.f. Barbu [16] or Miyadera [22]. But our semigroups do not seem to consist in the nonlinear semigroups, although the generators of nonlinear semigroups are generally multivalued operators. Because, as was noticed above, the semigroup e^{-tA} generated by a multivalued linear operator A defines only the usual C_0-semigroup on the space $\overline{\mathscr{D}(A)}$, and therefore the generator must be a univalent operator in $\overline{\mathscr{D}(A)}$ quite differently from that of the nonlinear semigroup. In addition, the nature of multivalue of A seems to be far and away more regular than that of the generator of a nonlinear semigroup (cf. the property (ii) in Section 2). While we may utilize some techniques devised in the nonlinear semigroups for the multivalued operators.

The generation theorem of semigroup will be applied in Section 4 to abstract degenerate equations of the following three types

(D-E. 1) $\begin{cases} d(Mv)/dt + Lv = f(t), & 0 \le t \le T, \\ Mv(0) = u_0, \end{cases}$

(D-E. 2) $\begin{cases} M^* d(Mv)/dt + Lv = M^* f(t), & 0 \le t \le T, \\ Mv(0) = u_0, \end{cases}$

(D-E. 3) $\begin{cases} Mdu/dt + Lu = Mf(t), & 0 \le t \le T, \\ u(0) = u_0, \end{cases}$

in a Hilbert space X ((D-E.1) being already introduced above). Here, M and L denote linear operators in X, $f(t)$ denotes a given function, u_0 is an initial value, and u and v are unknown functions. Changing the unknown functions to $u = Mv(t)$ in (D-E.1 and 2), every equation is reduced to an equation of the form (E). In fact, multivalued linear operators in (E) are respectively LM^{-1}, $(M^*)^{-1}LM^{-1}$ and $M^{-1}L$. We shall then investigate, in each case, sufficient conditions on M and L in order that the multivalued operator satisfies the Hille-Yosida condition and that it generates a semigroup on X. As a result, we shall obtain some existence and uniqueness theorems of the strict solutions for (D-E.

1,2 and 3), i.e. solutions v and u such that Mv, $u \in C^1([0,T]; X)$, via the semi-groups. (We may note that in (D-E.2 (resp. 3)) it is not essential that the member in the right hand side of the equation is $M^*f(t)$ (resp. $Mf(t)$). It will be seen in a certain general case that $M^*f(t)$ (resp. $Mf(t)$) can be replaced by a general function $g(t)$.)

Favini studied in [7] the strict solutions of the equations (D-E.1 and 3) on the basis of a device concerning an operational equation due to Da Prato-Grisvard [3], but his assumptions are rather different from ours since his result covers mainly more general cases when the multivalued linear operators determined above do not generate semigroups in our sense. Povoas studied in [14] the strong solutions of the equations quite similar to (D-E.2), also making use of Da Prato-Grisvard [3]; her assumptions seem to be closely related to ours. Zaidman recently obtained in [15] some uniqueness result of the strict solution of the equation (D-E.3).

Some examples will be given in Section 5. Hyperbolic differential equations of degenerate type with respect to the time derivative, especially nonlinear ones, appear often in applied mathematics and have attracted interest of many mathematicians, c.f. J.L. Lions [21], Carrol-Showalter [17], Fattorini [19] etc. The author believes that our method of using the semigroup will provide a new technique in the study of these equations.

NOTATIONS. X denotes a Banach space (or a Hilbert space in Section 4) with the norm $\|\cdot\|_X$. An operator $A:X \to 2^X$ is called multivalued operator in X, the domain of A is a set $\mathcal{D}(A) = \{u \in X; Au \neq \phi\}$ and the range of A is $\mathcal{R}(A)$ $= \bigcup_{u \in \mathcal{D}(A)} Au$. $\mathcal{L}(X)$ is the space of all bounded linear operators on X equipped with the uniform operator norm denoted by $\|\cdot\|_{\mathcal{L}(X)}$. Let $0 < T < \infty$, $L^p((0,T); X)$, $1 \le p < \infty$, is the space of measurable functions f with values in X for which $\|f(\cdot)\|_X^p$ are integrable in $(0,T)$; $L^p_{\text{fin}}((0,\infty); X)$, $1 \le p < \infty$, is a space of measurable functions f defined in $(0,\infty)$ such that $f \in \bigcap_{0<T<\infty} L^p((0,T); X)$. $C([0,T]; X)$ (resp. $C^1([0,T]; X)$) is the space of continuous (resp. continuously differentiable) functions on $[0,T]$ with values in X; $C([0,\infty); X)$ (resp. $C^1([0, \infty); X))$ stands for the space $\bigcap_{0<T<\infty} C([0,T]; X)$ (resp. $\bigcap_{0<T<\infty} C^1([0,T]; X))$.

2. Multivalued linear operators

Let X be a Banach space. We define, as was done in [8,Sec.2]:

DEFINITION. A mapping A from X into 2^X is called *a multivalued linear*

operator if the domain $\mathcal{D}(A)=\{u\in X: Au\neq\phi\}$ is a linear subspace of X and if A satisfies:

$$\begin{cases} Au+Av\subset A(u+v) & \text{for} \quad u,v\in\mathcal{D}(A), \\ \lambda Au\subset A(\lambda u) & \text{for} \quad \lambda\in C, u\in\mathcal{D}(A). \end{cases}$$

Let A be a multivalued linear operator in X. The following basic properties of A are then immediate consequences of the definition (see [8, Sec. 2]):

(i) $Au+Av=A(u+v)$ for $u,v\in\mathcal{D}(A)$; and $\lambda Au=A(\lambda u)$ for $u\in\mathcal{D}(A)$ if $\lambda\neq0$.

(ii) $A0$ is a linear subspace; and, for $u\in\mathcal{D}(A)$, $Au=f+A0$ with any $f\in Au$ (in particular, A is univalent if and only if $A0=\{0\}$).

(iii) The resolvent set $\rho(A)$ of A is an open set of C and the resolvent $(\lambda-A)^{-1}$ is a holomorphic function in $\rho(A)$ with values in $\mathcal{L}(X)$. (The resolvent set is defined as a set of all numbers $\lambda\in C$ for which $\lambda-A$ has a univalent and bounded inverse on the whole space X.)

(iv) The resolvent equation

$$(\lambda-A)^{-1}-(\mu-A)^{-1} = -(\lambda-\mu)(\lambda-A)^{-1}(\mu-A)^{-1} \quad \text{for} \quad \lambda,\mu\in\rho(A)$$

holds.

In the class of multivalued linear operators we can always consider their inverses, that is, we have:

Theorem 2.1. *The inverse A^{-1} of a multivalued linear operator A is also multivalued linear. $f\in Au$ if and only if $u\in A^{-1}f$.*

For the proof, see [8, Theorem 2.3].

This result seems to be proper to the multivalued linear operators. Next one is also a proper result and plays, in fact, a very important role in the subsequence:

Theorem 2.2. *For $\lambda\in\rho(A)$,*

$$A(\lambda-A)^{-1}\supset\lambda(\lambda-A)^{-1}-1\supset(\lambda-A)^{-1}A;$$

in particular, $(\lambda-A)^{-1}A$ is univalent on $\mathcal{D}(A)$ and $(\lambda-A)^{-1}Au=(\lambda-A)^{-1}f$ with any $f\in Au$.

For the proof, see [8, Theorem 2.7].

Let B be another multivalued linear operator. The product AB of two multivalued linear operators are defined by

$$\begin{cases} \mathcal{D}(AB) = \{v\in\mathcal{D}(B); Bv\cap\mathcal{D}(A)\neq\phi\} \\ ABv = \bigcup_{u\in Bv\cap\mathcal{D}(A)} Au. \end{cases}$$

Theorem 2.3. *The product AB is a multivalued linear operator. $f \in ABv$ if and only if there is some $u \in \mathcal{D}(A) \cap \mathcal{R}(B)$ such that $f \in Au$ and $u \in Bv$.*

Proof. The second assertion is obvious by the definition. Let $f_i \in ABv_i$ $(i=1,2)$; there are $u_i \in \mathcal{D}(A) \cap \mathcal{R}(A)$ such that $f_i \in Au_i$ and $u_i \in Bv_i$ $(i=1,2)$; since $f_1 + f_2 \in A(u_1 + u_2)$ and $u_1 + u_2 \in B(v_1 + v_2)$, $f_1 + f_2 \in AB(v_1 + v_2)$. Similarly, $\lambda f \in AB(\lambda v)$ if $f \in ABv$.

It is easy to observe from Theorems 2.1 and 2.3 that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

The rest of this section is devoted to proving some decomposition theorem of X for the multivalued linear operators A satisfying the condition:
(R) The resolvent set $\rho(A)$ contains a real half line $(-\infty, \beta)$ $(-\infty < \beta < \infty)$, and the resolvent satisfies there an estimate

$$(2.1) \qquad \|(\lambda - A)^{-1}\|_{\mathcal{L}(X)} \le M/|\lambda - \beta|, \quad \lambda < \beta,$$

with some constant M.

Theorem 2.4. *Let a multivalued linear operator A satisfy* (R) *with some β. Then the sum X_0 of the two closed subspaces $\overline{\mathcal{D}(A)}$ and $A0$ is a topologically direct sum in X, and X_0 is a closed subspace of X.*

Proof. Let us first notice that $A0$ is a closed subspace; indeed, $A0 = (\lambda_0 - A)0 = $ kernel of $(\lambda_0 - A)^{-1} \in \mathcal{L}(X)$ for $\lambda_0 < \beta$. To verify $\overline{\mathcal{D}(A)} \cap A0 = \{0\}$, we need:

Lemma 2.5. *For integers $n > -\beta$, put $J_n = n(n+A)^{-1} = (1 + n^{-1}A)^{-1}$. Then, as $n \to \infty$, J_n converges to the identity strongly on $\overline{\mathcal{D}(A)}$.*

Proof of lemma. Let $u \in \mathcal{D}(A)$; according to Theorem 2.2, $(J_n - 1)u = -(n+A)^{-1}f$ with any $f \in Au$; therefore, $J_n u \to u$ in X follows from (2.1). The assertion is then verified from the uniformly boundedness of the norms $\|J_n\|_{\mathcal{L}(X)}$ for $n > -\beta$.

Completion of Proof. Let $f \in \overline{\mathcal{D}(A)} \cap A0$; as $n \to \infty$, $J_n f \to f$; on the other hand, $f \in A0$ implies that $J_n f = 0$ for all n; hence f must be 0, i.e. $\overline{\mathcal{D}(A)} \cap A0 = \{0\}$. Consider now $g \in \overline{\mathcal{D}(A)}$ and $h \in A0$; since $J_n(g+h) = J_n g \to g$, we have: $\|g\|_X \le \lim_{n \to \infty} \|J_n(g+h)\|_X \le M\|g+h\|_X$; as a consequence, $\|h\|_X \le (M+1)\|g+h\|_X$; these shwo that the two projections from X_0 onto $\overline{\mathcal{D}(A)}$ and onto $A0$ are continuous. Let us finally verify that X_0 is closed in X. Consider sequences $g_n \in \overline{\mathcal{D}(A)}$ and $h_n \in A0$ and assume that $g_n + h_n \to f$ in X as $n \to \infty$; then, as was proved above, this implies that g_n(resp. h_n) is a Cauchy sequence in $\overline{\mathcal{D}(A)}$ (resp. in $A0$); so that, $f = g + h \in \overline{\mathcal{D}(A)} + A0 = X_0$.

When X is a reflexive Banach space, we obtain a stronger result:

Theorem 2.6. *When X is reflexive, the Condition* (R) *implies* $X = \overline{\mathscr{D}(A)} + A0 = X_0$.

Proof. Let f be an arbitrary element in X. Since $\|J_n f\|_X$, $n > -\beta$, are uniformly bounded from (2.1), it is possible to choose a subsequence $J_{n_k} f$ which is weakly convergent to an element g; g lies in $\overline{\mathscr{D}(A)}$ since $J_n f \in \mathscr{D}(A)$ (note that $\overline{\mathscr{D}(A)}$ is weakly closed). On the other hand, consider a weak limit h of the sequence $(1 - J_{n_k})f$; since Lemma 2.5 yields that $(\lambda_0 - A)^{-1}(1 - J_n)f = (1 - J_n)(\lambda_0 - A)^{-1}f \to 0$ as $n \to \infty$, we obtain that $(\lambda_0 - A)^{-1}h = 0$ with any $\lambda_0 < \beta$ (note that $(\lambda_0 - A)^{-1}$ is continuous even in the weak topology); therefore, $h \in A0$. Since $f = J_n f + (1 - J_n)f$, clearly $f = g + h \in X_0$.

3. Semigroups and Evolution Equations

Let A be a multivalued linear operator in a Banach space X. Let A satisfy the condition of Hille-Yosida type:

(H-Y) The resolvent set $\rho(A)$ contains a real half line $(-\infty, \beta)$, $-\infty < \beta < \infty$, and there the estimate

(3.1) $\|(\lambda - A)^{-n}\|_{\mathcal{L}(X)} \leq M/|\lambda - \beta|^n$, $\lambda < \beta$, $n = 1, 2, 3, \ldots$

holds with some constant $M \geq 0$.

Under (H-Y) we shall construct a semigroup e^{-tA} generated by $-A$, and shall establish the existence and uniqueness of a strict solution of an evolution equation

(E) $\begin{cases} du/dt + Au \ni f(t), & 0 \leq t \leq T, \\ u(0) = u_0 \end{cases}$

in X.

For integers $n > -\beta$, we define the Yosida approximation A_n of A by

$$A_n = n - n(1 + n^{-1}A)^{-1} = n - n^2(n + A)^{-1}.$$

By a direct calculation it is verified that $\rho(A_n) \supset (-\infty, \beta_n)$ with $\beta_n = n\beta/(n + \beta)$, and that

(3.2) $(\lambda - A_n)^{-1} = \dfrac{1}{\lambda - n} + \left(\dfrac{n}{n - \lambda}\right)^2 \left(\dfrac{n\lambda}{n - \lambda} - A\right)^{-1}$, $\lambda < \beta_n, n > -\beta$.

In addition we have:

Lemma 3.1. *As $n \to \infty$, the the resolvent $(\lambda - A_n)^{-1}$ converges to $(\lambda - A)^{-1}$ in $\mathcal{L}(X)$ for any $\lambda < \beta$. Moreover, if $u \in \mathscr{D}(A)$ and $\overline{\mathscr{D}(A)} \cap Au \neq \phi$, then*

$\overline{\mathscr{D}(A)} \cap Au$ always consists of a single point and the Yosida approximation $A_n u$ converges to the point in X.

Proof. The first assertion is immmediate from (3.2). Let $u \in \mathscr{D}(A)$ and let $g_1, g_2 \in \overline{\mathscr{D}(A)} \cap Au \neq \phi$, then it follows from the Theorem 2.4 (of course, (H-Y) implies the Condition (R)) that $g_1 - g_2 \in \overline{\mathscr{D}(A)} \cap A0 = \{0\}$, hence $g_1 = g_2$. Let then $\overline{\mathscr{D}(A)} \cap Au = \{g\}$. Theorem 2.2 jointed with Lemma 2.5 then yields that, as $n \to \infty$,

$$A_n u = n\{1 - (n+A)^{-1}\}u = n(n+A)^{-1}g \to g \text{ in } X.$$

Since A_n are bounded operators on X, the semigroups e^{-tA_n} for $n > -\beta$ are given by

$$e^{-tA_n} = e^{-nt}e^{n^2(n+A)^{-1}t} = e^{-nt}\sum_{k=0}^{\infty}\frac{\{n^2(n+A)^{-1}t\}^k}{k!}, \quad t \geq 0.$$

An immediate consequence of (3.1) is that

$$\|e^{-tA_n}\|_{\mathscr{L}(X)} \leq Me^{-nt}\sum_{k=0}^{\infty}\left(\frac{n^2t}{n+\beta}\right)^k\frac{1}{k!} \leq Me^{-\beta_n t}, \quad t \geq 0, \ n > -\beta.$$

Convergence results of these semigroups are described separately in the subspace $X_0 = \overline{\mathscr{D}(A)} + A0$ and on the whole space X. We first prove in the subspace X_0:

Theorem 3.2. *For each $t \geq 0$, e^{-tA_n} converges, as $n \to \infty$, to a bounded operator $e^{-tA} \in \mathscr{L}(X_0)$ strongly on X_0; and the convergence is, on the subsapce $\overline{\mathscr{D}(A)}$, uniform in t on any finite interval $[0, T]$. e^{-tA} defines a semigroup on X_0 with an estimate $\|e^{-tA}\|_{\mathscr{L}(X_0)} \leq Me^{-\beta t}$, and is strongly continuous for $0 < t < \infty$. Moreover, e^{-tA} satisfies: $e^{-tA} = Pe^{-tA} = e^{-tA}P$ for $t > 0$, where P denotes the projection from X_0 onto $\overline{\mathscr{D}(A)}$; therefore e^{-tA} defines a C_0 semigroup on $\overline{\mathscr{D}(A)}$ and, on the other hand, vanishes on $A0$ for every $t > 0$.*

Proof. Set $Y = \{u \in \mathscr{D}(A); \overline{\mathscr{D}(A)} \cap Au \neq \phi\}$, and let us consider first the proof on Y. Since we have:

$$(e^{-tA_n} - e^{-tA_m})u = \int_0^t e^{-(t-\tau)A_m}e^{-\tau A_n}(A_m - A_n)u d\tau,$$

$$\|(e^{-tA_n} - e^{-tA_m})u\|_X \leq M^2\int_0^t e^{-(t-\tau)\beta_m}e^{-\tau\beta_n}\|(A_m - A_n)u\|_X d\tau;$$

it follows from Lemma 3.1 that, if $u \in Y$, then $e^{-tA_n}u$ is convergent in X. Obviously the convergence is uniform in t on any interval $[0, T]$. For an element g in $\overline{\mathscr{D}(A)}$, we approximate it by the element in Y; in fact, put $u_n = n^2(n+A)^{-2}g$. Then, since $-nu_n + n^2(n+A)^{-1}g \in Au_n$, $u_n \in Y$; in addition, from Lemma 2.5 it follows that $u_n \to g$ in X. This then provides the same result of convergence

of $e^{-tA_n}g$. Since $e^{-tA_n}g \in \overline{\mathscr{D}(A)}$, the limit remains in $\overline{\mathscr{D}(A)}$. Let us now take an element h in $A0$. $h \in A0$ implies that $(n+A)^{-1}h = 0$ for all $n > -\beta$; so that $e^{-tA_n}h = e^{-nt}h$; hence $e^{-tA_n}h \to 0$ in X for $t > 0$. Finally, since all the elements $f \in X_0$ can be written $f = g + h$ uniquely with $g \in \overline{\mathscr{D}(A)}$ and $h \in A0$, we conclude the strong convergence of e^{-tA_n} on the subspace X_0. As this strong limit, we define a linear operator e^{-tA} for $t \geq 0$. It is now easy to see that these $e^{-tA} \in \mathscr{L}(X_0)$ define a semigroup on X_0 and satisfy all the desired properties.

On the whole space, however, we do not know in general whether e^{-tA_n} is strongly convergent or not. It is only possible to prove a weaker result. Denote by $L^1_{\mathrm{fin}}((0, \infty); X)$ the space of functions in $(0, \infty)$ which belong to $L^1((0, T); X)$ for any $T < \infty$. For $f \in L^1_{\mathrm{fin}}((0, \infty); X)$ we consider an integral operator given by

$$(e^{-tA_n}*f)(t) = \int_0^t e^{-(t-\tau)A_n}f(\tau)d\tau, \quad 0 \leq t < \infty, n > -\beta.$$

Clearly $e^{-tA_n}*f$ is a continuous function on $[0, \infty)$ with values in X, and is estimated by

$$(3.3) \qquad \|(e^{-tA_n}*f)(t)\|_X \leq M \int_0^t e^{-(t-\tau)\beta_n}\|f(\tau)\|_X d\tau, \quad 0 \leq t < \infty.$$

We then prove:

Theorem 3.3. *For each $f \in L^1_{\mathrm{fin}}((0, \infty); X)$, $e^{-tA_n}*f$ converges, as $n \to \infty$, to a continuous function $U*f \in C([0, \infty); \overline{\mathscr{D}(A)})$ uniformly on any finite interval $[0, T]$. The mapping $U*$ is then a linear operator from $L^1_{\mathrm{fin}}((0, \infty); X)$ to $C([0, \infty); \overline{\mathscr{D}(A)})$ with an estimate*

$$(3.4) \qquad \|(U*f)(t)\|_X \leq M \int_0^t e^{-(t-\tau)\beta}\|f(\tau)\|_X d\tau, \quad 0 \leq t < \infty.$$

Proof. Let us first consider the case when $f \in C^1([0, \infty); X)$. Fix a number $\lambda_0 < \mathrm{Min}\{\beta_n; n > -\beta\}$. Since $\lambda_0 \in \bigcap_{n > -\beta} \rho(A_n)$, we can write:

$$(e^{-tA_n}*f)(t)$$
$$= \lambda_0 \int_0^t e^{-(t-\tau)A_n}(\lambda_0 - A_n)^{-1}f(\tau)d\tau - A_n \int_0^t e^{-(t-\tau)A_n}(\lambda_0 - A_n)^{-1}f(\tau)d\tau$$
$$= \lambda_0 \int_0^t e^{-(t-\tau)A_n}(\lambda_0 - A_n)^{-1}f(\tau)d\tau - \int_0^t \frac{\partial e^{-(t-\tau)A_n}}{\partial \tau}(\lambda_0 - A_n)^{-1}f(\tau)d\tau.$$

By intgtration by parts it amounts to

$$= \lambda_0 \int_0^t e^{-(t-\tau)A_n}(\lambda_0 - A_n)^{-1}f(\tau)d\tau + e^{-tA_n}(\lambda_0 - A_n)^{-1}f(0)$$
$$- (\lambda_0 - A_n)^{-1}f(t) + \int_0^t e^{-(t-\tau)A_n}(\lambda_0 - A_n)^{-1}f'(\tau)d\tau.$$

According to Lemma 3.1 and Theorem 3.2, $e^{-tA_n}(\lambda_0-A_n)^{-1} \to e^{-tA}(\lambda_0-A)^{-1}$ strongly on X and uniformly on any finite interval of t. Therefore we conclude that $e^{-tA_n}*f$ converges to a function

$$(U*f)(t) = \lambda_0 \int_0^t e^{-(t-\tau)A}(\lambda_0-A)^{-1}f(\tau)d\tau + e^{-tA}(\lambda_0-A)^{-1}f(0)$$

$$-(\lambda_0-A)^{-1}f(t) + \int_0^t e^{-(t-\tau)A}(\lambda_0-A)^{-1}f'(\tau)d\tau .$$

Obviously the convergence is uniform on any finite interval of t, and $U*f$ is in $C([0, \infty); \overline{\mathscr{D}(A)})$. The estimate (3.4) is obtained from (3.3). Let us now consider the case when f is a general function in $L^1_{fin}((0, \infty); X)$. But the proof is immediate if we notice (3.3) and a fact that $C^1([0, T]; X)$ is a dense subspace in $L^1((0, T); X)$ for any $T < \infty$.

Up to now we used $U*$ to denote a linear operator without considering what U itself means. Let us observe here that U can be interpreted as a distribution semigroup generated by $-A$. In fact, let $\varphi \in C_0^\infty(\boldsymbol{R})$. Then, for any $f \in X$,

$$\{\int_0^\infty e^{-tA_n}\varphi(t)dt\} f = \int_0^T e^{-(T-t)A_n}\varphi(T-t)fdt ,$$

where $\varphi(t) = 0$ for all $t > T$. So that there exists a limit in X

$$\langle U, \varphi \rangle f = \lim_{n \to \infty} \{\int_0^\infty e^{-tA_n}\varphi(t)dt\} f$$

with

$$\|\langle U, \varphi \rangle f\|_X \leq M \int_0^\infty e^{-\beta t} |\varphi(t)| dt \|f\|_X .$$

This U is then a distribution with values in $\mathscr{L}(X)$, and $U = 0$ for $t < 0$. Since the semigroup property of e^{-tA_n} implies that

$$\int_0^\infty e^{-tA_n}(\varphi*\psi)(t)dt = \int_0^\infty e^{-tA_n}\varphi(t)dt \int_0^\infty e^{-tA_n}\psi(t)dt$$

for any $\varphi, \psi \in C_0^\infty(\boldsymbol{R})$ such that $\varphi(t) = \psi(t) = 0$ for $t \leq 0$, we have: $\langle U, \varphi*\psi \rangle = \langle U, \varphi \rangle \langle U, \psi \rangle$. According to J.L.Lions [12], an $\mathscr{L}(X)$ valued distribution with this property is called the distribution semigroup. It is also immediate to see that

$$(U*\varphi)(t)f = \lim_{n \to \infty} \int_0^t e^{-(t-\tau)A_n}\varphi(\tau)fd\tau , \quad 0 \leq t < \infty ,$$

for all $\varphi \in C_0^\infty(\boldsymbol{R})$, $\varphi(t) = 0$ for $t \leq 0$, and for all $f \in X$; and this justifies the definition of $U*$ in Theorem 3.3.

In view of the above remark we may also denote the distribution semi-

group U by e^{-tA}, i.e.

$$(e^{-tA} * f)(t) = \lim_{n \to \infty} \int_0^t e^{-(t-\tau)A_n} f(\tau) d\tau , \quad 0 \leq t < \infty ,$$

for $f \in L^1_{\text{fin}}((0, \infty); X)$.

REMARK. For $n > -\beta$, put

$$S_n(t) = \int_0^t e^{-\tau A_n} d\tau , \quad 0 \leq t < \infty .$$

It is immediate from Theorem 3.3 to verify that, for each $t, S_n(t)$ converges strongly, as $n \to \infty$, to a bounded operator $S(t) \in \mathcal{L}(X)$. The semigroup property of e^{-tA_n} now implies that

$$S(t)S(s) = \int_0^t \{S(s+\tau) - S(\tau)\} d\tau , \quad 0 \leq t, s < \infty .$$

Such a family $S(t)$ of operator is called by Arendt [1, 2] integrated semigroup. For the univalent linear operator, it is known that an operator A is the generator of a locally Lipschitz continuous, integrated semigroup if and only if A satisfies the Hille-Yosida Condition, cf. Kellerman and Hieber [10].

We now proceed to the study of the evolution equation

(E) $\begin{cases} du/dt + Au \ni f(t) , & 0 \leq t \leq T , \\ u(0) = u_0 , \end{cases}$

with a multivalued linear operator A satisfying the Condition (H-Y). Here, $f: [0, T] \to X$ is a given continuous function, $u_0 \in \mathcal{D}(A)$ is an initial value, and $u: [0, T] \to X$ is an unknown function.

We shall prove the existence and uniqueness of strict solution of (E) by using the semigroup e^{-tA}. By the strict solution we mean:

DEFINITION. A function u is called *strict solution of* (E) if $u \in C^1([0, T]; X)$ with $u(0) = u_0$ and if u satisfies the equation of (E) at every point $0 \leq t \leq T$; in particular, $u(t) \in \mathcal{D}(A)$ for every $0 \leq t \leq T$.

If a strict solution u exists, then $u'(0) \in \{f(0) - Au_0\} \cap \overline{\mathcal{D}(A)}$; this shows that the condition

(C) $\overline{\mathcal{D}(A)} \cap \{f(0) - Au_0\} \neq \phi$

is a compatibility condition to be always satisfied in seeking the strict solution of (E). Furthermore, if (C) takes place, then the set $\overline{\mathcal{D}(A)} \cap \{f(0) - Au_0\}$ always consists of a single point; the proof is the same as in Lemma 3.1.

Theorem 3.4. *Let A satisfy the Condition* (H-Y). *If $f \in C^1([0, T]; X)$ and if $u_0 \in \mathcal{D}(A)$ with the compatibility condition* (C), *then the function given by*

$$(3.5) \qquad\qquad u(t) = e^{-tA}u_0 + (e^{-tA}*f)(t), \quad 0 \leq t \leq T,$$

is a strict solution of (E). *Conversely, any strict solution of* (E) *with* $f \in C([0, T];$
$X)$ *and with* $u_0 \in \mathscr{D}(A)$ *is necessarily of the form* (3.5), *and hence is unique.*

Proof. Let us first prove the existence. Let $\{g_0\} = \overline{\mathscr{D}(A)} \cap \{f(0) - Au_0\}$,
and consider a sequence of function defined by

$$u_n(t) = e^{-tA_n}u_{0,n} + \int_0^t e^{-(t-\tau)A_n}f(\tau)d\tau, \quad 0 \leq t \leq T, n > -\beta,$$

where $u_{0,n} = u_0 + n^{-1}\{f(0) - g_0\}, n > -\beta$. By Theorems 3.2 and 3.3, u_n converges
to the function u of (3.5) pointwisely. In addition, operating A_n, we have:

$$A_n u_n(t) = e^{-tA_n}A_n u_{0,n} + \int_0^t \frac{\partial e^{-(t-\tau)A_n}}{\partial \tau}f(\tau)d\tau$$

$$= -e^{-tA_n}g_0 + f(t) - \int_0^t e^{-(t-\tau)A_n}f'(\tau)d\tau.$$

Here we used Theorem 2.2 to obtain that

$$A_n u_{0,n} = -n\{n(n+A)^{-1} - 1\}\{u_0 + n^{-1}(f(0) - g_0)\} = f(0) - g_0$$

(note that $f(0) - g_0 \in Au_0$). As a consequence, we conclude that $A_n u_n(t)$ also con-
verges to a function $f(t) - g(t)$ pointwisely, where

$$g(t) = e^{-tA}g_0 + (e^{-tA}*f')(t), \quad 0 \leq t \leq T,$$

is a continuous function. Then the proof is immediate. Indeed, from $u_n(t) = (\lambda_0 - A_n)^{-1}(\lambda_0 - A_n)u_n(t)$ for some $\lambda_0 \leq \text{Min}\{\beta_n; n > -\beta\}$, it is verified that $u(t) = (\lambda_0 - A)^{-1}\{\lambda_0 u(t) - f(t) + g(t)\}$, i.e. $u(t) \in \mathscr{D}(A)$ and $g(t) \in f(t) - Au(t)$ for every $0 \leq t \leq T$. On the other hand, letting $n \to \infty$ in

$$u_n(t) - u_{0,n} = \int_0^t \{f(\tau) - A_n u_n(\tau)\}d\tau,$$

we see that $u(t) - u_0 = \int_0^t g(\tau)d\tau$, i.e. $u \in C^1([0, T]; X)$ and $u' = g$.

Now, let us prove the uniqueness of strict solution. Let u be a strict solu-
tion of (E). Then we have:

$$\frac{\partial e^{-(t-\tau)A_n}u(\tau)}{\partial \tau} = e^{-(t-\tau)A_n}A_n u(\tau) + e^{-(t-\tau)A_n}u'(\tau), \quad 0 \leq \tau \leq t \leq T.$$

Note here that, since $f(\tau) - u'(\tau) \in Au(\tau)$, Theorem 2.2 yields that

$$A_n u(\tau) = -n\{n(n+A)^{-1} - 1\}u(\tau) = n(n+A)^{-1}\{f(\tau) - u'(\tau)\}, 0 \leq \tau \leq T.$$

Therefore by integration it follows that

$$u(t) - e^{-tA_n} u_0 = n(n+A)^{-1} \int_0^t e^{-(t-\tau)A_n} f(\tau) \, d\tau$$

$$- \{n(n+A)^{-1} - 1\} \int_0^t e^{-(t-\tau)A_n} u'(\tau) \, d\tau \, .$$

Let $n \to \infty$, then the formula (3.5) is verified from the Theorems 3.2 and 3.3 and from the Lemma 2.5. Indeed, note that

$$\int_0^t e^{-(t-\tau)A_n} f(\tau) \, d\tau \left(\text{resp.} \int_0^t e^{-(t-\tau)A_n} u'(\tau) \, d\tau \right)$$

$$\to (e^{-tA} * f)(t) \, (\text{resp.} \, (e^{-tA} * u')(t)) \qquad \text{for} \quad 0 \le t \le T \, ,$$

and that $e^{-tA} * f$(resp. $e^{-tA} * u'$) takes values in $\overline{\mathcal{D}(A)}$.

REMARK. In the case when A is univalent linear operator, the existence and uniqueness of strict solution of (E) under (H-Y) had been first proved by Da Prato and Sinestrari [4] but without using any semigroup generated by A. Afterward, Kellerman and Hieber [10] published a simplified proof using the integrated semigroup.

4. Abstract Degenerate Evolution Equations

This section is devoted to studying abstract degenerate linear evolution equations in a Hilbert space. Throughout this section X denotes a Hilbert space with the scalar product $(\cdot, \cdot)_X$.

We begin with generalizing a known fact that the maximal accretive linear operators in X generate contraction semigroups on X to the multivalued linear operators. The definition of "maximal accretive" for the multivalued linear operators is quite analogous to that for the univalent linear operators. Indeed,

DEFINITION. A multivalued linear operator A in X is called *accretive operator* if

$$\text{Re} \, (f, u)_X \ge 0 \qquad \text{for all} \quad u \in \mathcal{D}(A) \quad \text{and all} \quad f \in Au \, .$$

An accretive multivalued linear operator which has no strict extension of accretive operator is called *maximal accretive operator*.

It is proved by the similar argument as for the univalent operator (therefore we may omit the proof) that a multivalued linear operator A is maximal accretive if and only if A is accretive with a range condition $\mathcal{R}(\lambda_0 - A) = X$ for some $\lambda_0 < 0$.

We then have:

Theorem 4.1. *Let A be a multivalued linear operator in X such that $A - \beta$ is maximal accretive with some real number β; more precisely, let*

(4.1) $\text{Re} \, (f, u)_X \ge \beta \|u\|^2$ *for all* $u \in \mathcal{D}(A)$ *and all* $f \in Au$

with

(4.2) $\mathscr{R}(\lambda_0 - A) = X$ *for some* $\lambda_0 < \beta$.

Then, $\rho(A) \supset (-\infty, \beta)$ and an estimate

$$\|(\lambda - A)^{-1}\|_{\mathscr{L}(X)} \leq 1/|\lambda - \beta| \quad for   \lambda < \beta$$

holds; that is, A satisfies the Hille-Yosida Condition (H-Y) with $M = 1$.

Proof.   Let $\lambda < \beta$.   For $u \in \mathscr{D}(A)$ and $f \in (\lambda - A) u$, it follows from (4.1) that

$$\text{Re } (\lambda u - f, u)_X \geq \beta \|u\|_X^2   \quad \text{or}   \quad \text{Re } (f, u)_X \leq (\lambda - \beta) \|u\|_X^2.$$

Therefore, $\|u\|_X \leq \|f\|_X / |\lambda - \beta|$.   This shows that $(\lambda - A)^{-1}$ is a univalent operator satisfying

$$\|(\lambda - A)^{-1} f\|_X \leq |\lambda - \beta|^{-1} \|f\|_X   \quad \text{for}   f \in \mathscr{R}(\lambda - A).$$

To complete the proof it therefore suffices to verify that $\mathscr{R}(\lambda - A) = X$ for all $\lambda < \beta$.   By assumption this is the case when $\lambda = \lambda_0$.   Consider then λ such that $|\lambda - \lambda_0| < |\lambda_0 - \beta|$.   For any $f \in X$, put $f_1 = \{1 + (\lambda - \lambda_0)(\lambda_0 - A)^{-1}\}^{-1} f$ (note that $\|(\lambda_0 - A)^{-1}\|_{\mathscr{L}(X)} \leq |\lambda_0 - \beta|^{-1}$) and put $u = (\lambda_0 - A)^{-1} f_1$; then, $f = f_1 + (\lambda - \lambda_0) u$ and $f \in (\lambda - A) u$; i.e. $\mathscr{R}(\lambda - A) = X$.   We shall then repeat the same kind of argument for λ_1 such that $\mathscr{R}(\lambda_1 - A) = X$ has been established and shall complete the proof.

By virtue of Theorem 3.2, a multivalued linear operator A such that $A - \beta$ is maximal accretive generates a semigroup e^{-tA} on the whole space X with an estimate $\|e^{-tA}\|_{\mathscr{L}(X)} \leq e^{-\beta t}$, $0 \leq t < \infty$ (remember that $X = \overline{\mathscr{D}(A)} + A0$ from Theorem 2.6).

In addition, let

(E) $\begin{cases} du/dt + Au \ni f(t), \ 0 \leq t \leq T, \\ u(0) = u_0 \end{cases}$

be an evolution equation in X with A such that $A - \beta$ is maximal accretive.   In the present case the compatibility condition (C) is only that $u_0 \in \mathscr{D}(A)$.   Indeed, since $Au_0 = f_0 + A0$ with an arbitrary $f_0 \in Au_0$ and since $A0$ is a linear subspace (from (ii) in Sec. 2), we observe from Theorem 2.6 that

$$f(0) - Au_0 = f(0) - f_0 - A0 = g_0 + h_0 - A0 = g_0 - A0 \ni g_0,$$

where $f(0) - f_0 = g_0 + h_0$ with $g_0 \in \overline{\mathscr{D}(A)}$ and $h_0 \in A0$; hence the condition (C).   We then obtain by virtue of Theorme 3.4 that, if $f \in C^1([0, T]; X)$ and if $u_0 \in \mathscr{D}(A)$, then the function

$$u(t) = e^{-tA} u_0 + \int_0^t e^{-(t-\tau)A} f(\tau) \, d\tau, \ 0 \leq t \leq T,$$

is a unique strict solution of (E).

Applying these results, let us now study abstract evolution equations in X which are degenerate with respect to the time derivative.

Consider first:

(D-E.1) $\begin{cases} d(Mv)/dt + Lv = f(t),\ 0 \leq t \leq T\,, \\ Mv(0) = u_0\,, \end{cases}$

where M and L are univalent linear operators in X with $\mathcal{D}(L) \subset \mathcal{D}(M)$. $f\colon$ $[0, T] \to X$ is a given continuous function, u_0 is an initial value, and $v\colon [0, T] \to \mathcal{D}(L)$ is an unknown function. We change the unknown function to $u(t) = Mv(t)$; then, (D-E.1) is rewritten in the non degenerate form (E) with $A = LM^{-1}$. Assume here that

(4.3) $\mathrm{Re}\,(Lv, Mv)_X \geq \beta\,\|Mv\|_X^2$ for all $v \in \mathcal{D}(L)\,;$ and

(4.4) $\mathcal{R}(\lambda_0 M - L) = X$ for some $\lambda_0 < \beta\,.$

Then $A - \beta$ is shown to be maximal accretive in X. Indeed, let $f \in Au$; then, $(f, u)_X = (Lv, Mv)_X$ with some $v \in \mathcal{D}(L)$ such that $f = Lv$ and $Mv = u$; so that, (4.1) follows immediately from (4.3). On the other hand, let $f \in X$; from (4.4), $f = (\lambda_0 M - L)v$ with some $v \in \mathcal{D}(L)$; put here $u = Mv$, then $u \in M(\mathcal{D}(L)) = \mathcal{D}(A)$ and $f \in (\lambda_0 - A)u$, i.e. (4.2). Therefore we have:

Theorem 4.2. *Let* (4.3) *and* (4.4) *be satisfied. For any* $f \in C^1([0, T]; X)$ *and any* $u_0 \in M(\mathcal{D}(L))$, *there exists a unique strict solution v of* (D-E.1) *such that*

(4.5) $Mv \in C^1([0, T]; X)$ *and* $Lv \in C([0, T]; X)\,.$

Proof. Rewrite (D-E.1) into (E) in the way described above. It is then easy to see that $u = Mv$ is a strict solution of (E) if and only if v is a strict solution of (D-E.1) in the sense of (4.5). Similarly, $u_0 \in \mathcal{D}(A)$ if and only if $u_0 \in M(\mathcal{D}(L))$.

Consider next:

(D-E.2) $\begin{cases} M^* d(Mv)/dt + Lv = M^* f(t)\,,\quad 0 \leq t \leq T\,, \\ Mv(0) = u_0\,, \end{cases}$

where M is a bounded linear operator in X the adjoint of which is denoted by M^* and where L is a univalent linear operator in X. $f\colon [0, T] \to X$ is a given function, u_0 is an initial value, and $v\colon [0, T] \to \mathcal{D}(L)$ is an unknown function.

Changing the unknown function to $u(t) = Mv(t)$, (D-E.2) is written in the form

$$(4.6) \qquad \begin{cases} M^* du/dt + LM^{-1} u \ni M^* f(t), & 0 \leq t \leq T, \\ u(0) = u_0. \end{cases}$$

Moreover, introducing a multivalued linear operator

$$(4.7) \qquad A = (M^*)^{-1} LM^{-1},$$

(4.6) is finally written in the non degenerate form (E).

Assume here that there exists a real number β with the following conditions:

$$(4.8) \qquad \mathrm{Re}\,(Lv, v)_X \geq \beta \,||Mv||_X^2 \qquad \text{for all} \quad v \in \mathscr{D}(L); \quad \text{and}$$

$$(4.9) \qquad \mathscr{R}(\lambda_0 M^* M - L) \supset \mathscr{R}(M^*) \qquad \text{for some} \quad \lambda_0 < \beta.$$

Then $A - \beta$ is maximal accretive in X. Indeed, if $f \in Au$, there exists by (4.7) some $v \in \mathscr{D}(L)$ such that $M^* f = Lv$ and $Mv = u$; so that $(f, u)_X = (f, Mv)_X = (Lv, v)_X$; hence (4.1) is verified from (4.8). In addition, for any $f \in X$, there exists from (4.9) $v \in \mathscr{D}(L)$ such that $M^* f = (\lambda_0 M^* M - L) v$; put $u = Mv$, then $v \in M^{-1} u$ and $\lambda_0 u - f \in (M^*)^{-1} Lv$; therefore, in view of Theorem 2.3, $u \in \mathscr{D}(A)$ and $f \in (\lambda_0 - A) u$, i.e. (4.2).

Thus we obtain:

Theorem 4.3. *Let* (4.8) *and* (4.9) *be satisfied. For any* $f \in C^1([0, T]; X)$ *and any* $u_0 \in X$ *such that*

$$(4.10) \qquad u_0 = Mv_0 \quad \text{and} \quad Lv_0 \in \mathscr{R}(M^*) \quad \text{with some} \quad v_0 \in \mathscr{D}(L),$$

there exists a unique strict solution v of (D-E.2) *in a sense that*

$$(4.11) \qquad Mv \in C^1([0, T]; X) \quad \text{and} \quad Lv \in C([0, T]; X).$$

Proof. It is immediate to verify that a function $u = Mv$ is a strict solution of (E), where A is defined by (4.7), if and only if v is a strict solution of (D-E.2) satisfying (4.11). On the other hand, $u_0 \in \mathscr{D}(A)$ is equivalent to (4.10).

Under a stronger condition than (4.9) it is possible to handle an equation

$$(\text{D-E.2})' \qquad \begin{cases} M^* d(Mw)/dt + Lw = g(t), & 0 \leq t \leq T, \\ Mw(0) = v_0. \end{cases}$$

In fact, assume in addition to (4.8) that

$(4.12) \quad \lambda_0 M^* M - L$ has a univalent and bounded inverse on X with some $\lambda_0 < \beta$.

Then, changing the unknown function $w(t)$ to $v(t) = w(t) + (\lambda_0 M^* M - L)^{-1} g(t)$,

we can rewite (D-E.2)' into (D-E.2) with

$$f(t) = M(\lambda_0 M^* M - L)^{-1} \{g'(t) + \lambda_0 g(t)\}$$

and with

(4.13) $u_0 = v_0 + M(\lambda_0 M^* M - L)^{-1} g(0)$.

Therefore, the Theorem 4.3 provides:

Theorem 4.4. *Let (4.8) and (4.12) be satisfied. For any $g \in C^2([0, T]; X)$ and any $v_0 \in X$ such that*

(4.14) $v_0 = M w_0$ *and* $L w_0 - g(0) \in \mathcal{R}(M^*)$ *with some* $w_0 \in \mathcal{D}(L)$,

there exists a unique strict solution w of (D-E.2)'.

Proof. The only thing to be verified is that, if v_0 satisfies (4.14), then u_0 determined by (4.13) satisfies (4.10). But this is an easy calculation.

REMARK. Povoas studied in [14] the strong solution of the equation quite similar to (D-E.2)'; she proved, under analogous conditions to ours, existence and uniqueness of the strong solution in the Hilbert space $L^2((0, T); X)$ for $g \in L^2((0, T); X)$ and (essentially) for $v_0 \in \overline{M(\mathcal{D}(L))}$.

Let us finally consider

(D-E.3) $\begin{cases} M du/dt + Lu = M f(t) , & 0 \le t \le T , \\ u(0) = u_0 , \end{cases}$

where M and L are densely defined univalent linear operators in X with $\mathcal{D}(L^*) \subset \mathcal{D}(M^*)$, M^* and L^* being the adjoint operators of M and L respectively. (D-E.3) is the adjoint problem of (D-E.1). Using a multivalued linear operator $A = M^{-1} L$, (D-E.3) is also written in the form (E).

We assume, with some real number β, the conditions:

(4.15) $\mathrm{Re}\,(L^*\xi, M^*\xi)_X \ge \beta \,\|M^*\xi\|_X^2$ for $\xi \in \mathcal{D}(L^*)$,   and

(4.16) $\mathcal{R}(\lambda_0 M^* - L^*) = X$ for some $\lambda_0 < \beta$.

Then, by the same argument as for (D-E.1), we conclude that $L^*(M^*)^{-1} - \beta$ is a maximal accretive operator in X. As a consequence, its adjoint $\{L^*(M^*)^{-1}\}^* - \beta$ is also maximal accretive (for the definition of the adjoint of multivalued linear operators, see the Remark below). Denote $\mathcal{A} = \{L^*(M^*)^{-1}\}^*$ and consider

(E) $\begin{cases} d\mathcal{U}/dt + \mathcal{A}\mathcal{U} \ni f(t) , & 0 \le t \le T , \\ \mathcal{U}(0) = \mathcal{U}_0 . \end{cases}$

From the Theorem 4.1, there exists a unique strict solution of (E) for any $f \in C^1$

cable to various hyperbolic differential equations. The main purpose of this paper is, however, only to check the Assumptions we made in Section 4 in each example. It remains, as a subsequent work, to analyze more detailed properties of the strict solutions and to apply them to other problems.

FUNCTION SPACES. Let Ω be a region in $\boldsymbol{R}^n$. $C^k(\Omega)$, $k=0, 1, 2, \cdots$, (resp. $C^\infty(\Omega)$) denotes the function space of k-times continuously (resp, infinitely) differentiable functions in Ω. $\mathscr{B}^k(\Omega)$, $k=0, 1, 2, \cdots$, is a subsapce of $C^k(\Omega)$ consisting of functions with bounded derivatives up to order k. $C_0^\infty(\Omega)$ is the space of functions in $C^\infty(\Omega)$ with compact supports in Ω, $C_0^\infty(\Omega)'$ is the space of distributions in Ω. $H^k(\Omega)$, $k=0, 1, 2, \cdots$, denotes the Sobolev space with the norm $\|\cdot\|_k$ and the scalar product $(\cdot, \cdot)_k$; $H^0(\Omega)=L^2(\Omega)$ and the norm $\|\cdot\|_0$ (resp. the product $(\cdot, \cdot)_0$) is abbreviated as $\|\cdot\|$ (resp. by $(\cdot, \cdot)$) if there is no fear of confusion. The closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$ is denoted by $H_0^k(\Omega)$; $H^{-k}(\Omega)=H_0^k(\Omega)'$ and the scalar product between $H_0^k(\Omega)$ and $H^{-k}(\Omega)$ is denoted by $\langle\cdot, \cdot\rangle_{H_0^k(\Omega)\times H^{-k}(\Omega)}$.

EXAMPLE 5.1.

$$(5.1) \quad \begin{cases} \partial(m(x)\,w)/\partial t+\partial w/\partial x = g(t, x), & 0\leq t\leq T, \; -\infty<x<\infty, \\ m(x)\,w(0, x) = v_0(x), & -\infty<x<\infty. \end{cases}$$

Here, $m(x)$ is a characteristic function of some measurable set $A\subset\boldsymbol{R}$ (i.e. $m(x)=1$ for $x\in A$, $m(x)=0$ for $x\notin A$), $g(t, x)$ is a given function, $v_0(x)$ is an initial function, and $w=w(t, x)$ is an unknown function.

Let us consider the problem in $X=L^2(\boldsymbol{R})$. Let M be the multiplication operator of the function $m(x)$, M is a bounded operator on X, and clearly $M^*=M$ and $M^2=M$. Therefore (5.1) can be formulated in the form

$$(5.2) \quad \begin{cases} M^*d(Mw)/dt+Lw = g(t), & 0\leq t\leq T, \\ Mw(0) = v_0, \end{cases}$$

in X, where $L=d/dx$ with $\mathscr{D}(L)=H^1(\boldsymbol{R})$, L being a closed linear operator in X. Obviously, (5.2) is of form (D-E.2)' in Section 4 or of form (D-E.2) if $g(t, x)=m(x)f(t, x)$.

Since $\mathrm{Re}\,(Lv, v)=0$ for every $v\in\mathscr{D}(L)$, these M and L satisfy (4.8) with $\beta=0$. So the question is whether (4.9) or (4.12) is satisfied. We shall investigate this problem for some specific characteristic functions $m(x)$.

1) $A=(-\infty, a)\cup(b, \infty)$ $(a<b)$. In this case, (4.12) is valid. Indeed, consider a problem

$$(5.3) \quad\quad\quad m(x)\,v+dv/dx = f(x), \quad -\infty<x<\infty,$$

and seek for any $f\in L^2(\boldsymbol{R})$ a solution v in $H^1(\boldsymbol{R})$. Obviously, (5.3) consists of three problems: $v_1+dv_1/dx=f(x)$ in $x<a$, under $v_1(-\infty)=0$; $dv_2/dx=f(x)$ in $a<x<b$, under $v_2(a)=v_1(a)$; and $v_3+dv_3/dx=f(x)$ in $b<x$, under $v_3(b)=v_2(b)$.

$([0, T]; X)$ and any $\mathcal{U}_0 \in \mathcal{D}(\mathcal{A})$.

On the other hand, it is verified that

$$\mathcal{A} = \{L^*(M^*)^{-1}\}^* \supset (M^{**})^{-1} L^{**} \supset M^{-1} L = A .$$

This then shows that the equation (E) with $A = M^{-1} L$ which was induced from the original equation (D-E.3) can be extended to a well-posed equation $(\mathcal{E})$. Thus we have proved:

Theorem 4.5. *Let* (4.15) *and* (4.16) *be satisfied. For any* $f \in C^1([0, T]; X)$ *and any* $u_0 \in \mathcal{D}(L)$ *such that* $L u_0 \in \mathcal{R}(M)$, *there exists a unique strict solution of the extended equation* $(\mathcal{E})$ *induced from* (D-E.3).

REMARK. Let A be a multivalued linear operator in X. The adjoint operator A^* of A is defined by

$$\left\{ \begin{array}{l} \xi \in \mathcal{D}(A^*) \text{ if and only if } \xi \text{ is orthogonal to } A0 \text{ (hence, from (ii)} \\ \text{in Sec. 2 the product } (Au, \xi)_X \text{ is defined as a single value for} \\ \text{every } u \in \mathcal{D}(A)) \text{ and there exists } \phi \in X \text{ such that} \\ \qquad (Au, \xi)_X = (u, \phi)_X \qquad \text{for all } u \in \mathcal{D}(A) ; \\ A^* \xi = \{\phi \in X; (Au, \xi)_X = (u, \phi)_X \qquad \text{for all } u \in \mathcal{D}(A)\} . \end{array} \right.$$

It is not difficult to verfy that A^* is also a multivalued linear operator. Moreover, analogous results to the univalent operator are verified: $A \subset A^{**}$, $(A^*)^{-1} = (A^{-1})^*$, $(AB)^* \subset B^* A^*$ for two multivalued linear operators, etc. If A is maximal accretive, so is the adjoint A^*.

It is also possible to handle an equation

$$(\text{D-E.3})' \quad \left\{ \begin{array}{l} M dw/dt + Lw = g(t) , \quad 0 \le t \le T , \\ w(0) = w_0 , \end{array} \right.$$

in X, if we assume in addition a condition:

(4.17) $\quad \lambda_0 M - L$ has a univalent and bounded inverse on X with some number $\lambda_0 < \beta$.

Indeed, change the unknown function $w(t)$ to $u(t) = w(t) + (\lambda_0 M - L)^{-1} g(t)$, then (D-E.3)$'$ is reduced to (D-E.3) with

$$f(t) = (\lambda_0 M - L)^{-1} \{g'(t) + \lambda_0 g(t)\}$$

and with $u_0 = w_0 + (\lambda_0 M - L)^{-1} g(0)$.

5. Examples

We should like to show that the results in the preceding section are appli-

And by an elementary calculation, a function v given by $v=v_1$ for $x<a$, $v=v_2$ for $a<x<b$ and $v=v_3$ for $b<x$, is seen to be a solution of (5.3) and to belong to $H^1(\mathbf{R})$ with $\|v\|\leq$ Const. $\|f\|$. This shows that, for any $f\in X$, $(M+L)v=f$ has a unique solution $v\in\mathscr{D}(L)$ with $\|v\|\leq$ Const. $\|f\|$; hence (4.12) with $\lambda_0=-1$.

2) $A=(a,\infty)$. Consider similarly the problem (5.3). We observe in this case that the first problem of (5.3): $dv_1/dt=f(x)$ in $x<a$, does not admit any solution in $H^1((-\infty,a))$ in general for $f\in L^2((-\infty,a))$; this means that (4.12) is no longer valid. To the contrary, (4.9) is valid. In fact, it suffices to show that an equation

$$(5.4) \qquad m(x)\,v+dv/dx = m(x)\,f(x), \quad -\infty<x<\infty\,,$$

admits a solution $v\in H^1(\mathbf{R})$ for any $f\in L^2(\mathbf{R})$. And this is true since the equation is essentially: $v+dv/dx=f(x)$ in $a<x$, under an initial condition $v(a)=0$.

3) $A=(-\infty,a)$. Neither of (4.9) and (4.12) is valid. In fact, consider the equation (5.4); then, since $dv/dx=0$ for $a<x$, any solution v in $H^1(\mathbf{R})$ must be: $v(x)=v(a)$ for all $a<x$; so that $v(a)$ must be 0; but this is impossible in general, since $v+dv/dx=f(x)$ in $x<a$, under the conditions $v(-\infty)=v(a)=0$.

EXAMPLE 5.2.   Consider Maxwell's equation

$$\text{rot } E = -\frac{\partial B}{\partial t}\,, \quad \text{rot } H = \frac{\partial D}{\partial t}+J$$

in $\mathbf{R}^3$, where E (resp. H) denotes the electric (resp. magnetic) field intensity, B (resp. D) denotes the electric (resp. magnetic) flux density, and where J is the current density.

We assume that the medium which fills the space $\mathbf{R}^3$ is linear but may be anisotropic and nonhomogeneous, that is,

$$D = \mathcal{E}E\,, \quad B = \mu H\,, \quad J = \sigma E+J'$$

with some 3×3 matrices $\mathcal{E}(x)$, $\mu(x)$ and $\sigma(x)$, $x\in\mathbf{R}^3$, where J' is a (given) forced current density. Then the equation is:

$$\frac{\partial}{\partial t}\begin{pmatrix}\mathcal{E}(x) & 0 \\ 0 & \mu(x)\end{pmatrix}\begin{pmatrix}E \\ H\end{pmatrix}+\begin{pmatrix}0 & -\text{rot} \\ \text{rot} & 0\end{pmatrix}\begin{pmatrix}E \\ H\end{pmatrix}+\begin{pmatrix}\sigma(x) & 0 \\ 0 & 0\end{pmatrix}\begin{pmatrix}E \\ H\end{pmatrix} = -\begin{pmatrix}J'(t,x) \\ 0\end{pmatrix},$$

or

$$(5.5) \qquad \frac{\partial c(x)\,w}{\partial t}+\sum_{i=1}^{3} a_i\,\frac{\partial w}{\partial x_i}+b(x)\,w = g(t,x) \quad \text{in } [0,T]\times\mathbf{R}^3$$

with

$$w = \begin{pmatrix}E \\ H\end{pmatrix}, c(x) = \begin{pmatrix}\mathcal{E}(x) & 0 \\ 0 & \mu(x)\end{pmatrix}, b(x) = \begin{pmatrix}\sigma(x) & 0 \\ 0 & 0\end{pmatrix}, g(t,x) = -\begin{pmatrix}J'(t,x) \\ 0\end{pmatrix}$$

and with some 6×6 symmetric matrices a_i, $i=1, 2, 3$.

$\mathcal{E}(x)$, $\mu(x)$ and $\sigma(x)$ are assumed to be real matrices the components of which are bounded measurable functions in $\boldsymbol{R}^3$. In addition, it is assumed that

(5.6) $\mathcal{E}(x)$ is symmetric and ≥ 0 for every $x \in \boldsymbol{R}^3$;

(5.7)   there exist some $\delta > 0$ and $\gamma \geq 0$ such that

$(\{\gamma \mathcal{E}(x) + \sigma(x)\} \, \xi, \xi) \geq \delta \, \|\xi\|^2$ for $\xi \in \boldsymbol{R}^3$

uniformly in $x \in \boldsymbol{R}^3$; and

(5.8) $\mu(x)$ is symmetric and $\geq \delta > 0$ uniformly in $x \in \boldsymbol{R}^3$.

Then we can formulate the problem (5.5) in an abstract form

(5.9)
$$\begin{cases} M d(Mw)/dt + Lw = g(t) \\ Mw(0) = v_0 \end{cases}$$

in the space $X = \{L^2(\boldsymbol{R}^3)\}^6$, using a bounded multiplication operator $M = \sqrt{c(x)}$ on X and a closed linear operator L

(5.10)
$$\begin{cases} \mathcal{D}(L) = \{v \in X; \ \sum_{i=1}^{3} a_i \dfrac{\partial v}{\partial x_i} \in X\} \\[2mm] Lv = \sum_{i=1}^{3} a_i \dfrac{\partial v}{\partial x_i} + b(x) \, v . \end{cases}$$

Since $M = M^*$, the equation (5.9) is of the form (D-E.2)′. Let us verify so that the Conditions (4.8) and (4.12).

Let first $v \in Y = \{H^1(\boldsymbol{R}^3)\}^6 \subset \mathcal{D}(L)$. Then it is observed that

$$(Lv, v) = \left(\sum_{i=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x) \, v, \, v \right)$$
$$= -\left(v, \sum_{j=1}^{3} a_i \frac{\partial v}{\partial x_i} + b(x) \, v \right) + (b(x) \, v, v) + (v, b(x) \, v) ,$$

so that

$$\mathrm{Re} \, (Lv, v) = \mathrm{Re} \, (b(x) \, v, v) = \mathrm{Re} \, (\sigma(x) \, E, E) ,$$

where $v = (E, H)$. The conditions (5.7) and (5.8) then yield that, for any $\lambda \leq -\mathrm{Max} \, \{\gamma, 1\}$,

(5.11) $\mathrm{Re} \, (Lv, v) \geq \delta (\|E\|^2 + \|H\|^2) - \gamma (\mathcal{E}(x) \, E, E) - (\mu(x) \, H, H)$
$$\geq \delta (\|E\|^2 + \|H\|^2) + \lambda \{(\mathcal{E}(x) \, E, E) + (\mu(x) \, H, H)\}$$

(5.12) $= \delta \, \|v\|^2 + \lambda \, \|Mv\|^2 .$

This estimate in fact holds for all the functions $v \in \mathscr{D}(L)$, for, if $v \in \mathscr{D}(L)$, there exists a sequence $v_n \in Y$ such that $v_n \to v$ and $Lv_n \to Lv$ in X (consider for example the mollifier of v). Thus it has been in particular verified that (4.8) is valid with $\beta = -\mathrm{Max}\ \{\gamma, 1\}$.

Let us next verify (4.12), i.e. $\lambda M^2 - L$ has a bounded inverse on X for $\lambda < -\mathrm{Max}\ \{\gamma, 1\}$. Since (5.12) yields that

$$\|(\lambda M^2 - L)\, v\| \geq \delta\, \|v\| \qquad \text{for all}\quad v \in \mathscr{D}(L)\,,$$

it is observed that $\lambda M^2 - L$ is one to one and has a closed range. Therefore it surfices to verify that $\mathscr{R}(\lambda M^2 - L)^{\perp} = \{0\}$. Let $w \in \mathscr{R}(\lambda M^2 - L)^{\perp}$; then, clearly $w \in \mathscr{D}(L^*)$ and $(\lambda M^2 - L^*)\, w = 0$. On the other hand, since

$$\left(\sum_{i=1}^{3} a_i\, \frac{\partial \varphi}{\partial x_i},\, w \right) = (\varphi,\, \lambda M^2 w - b(x)^* w) \qquad \text{for all}\quad \varphi \in C_0^\infty(\mathbf{R}^3)\,.$$

we observe that $w \in \mathscr{D}(L)$. Hence, from (5.12)

$$\delta\, \|w\|^2 + \lambda\, \|Mw\|^2 \leq \mathrm{Re}\ (Lw,\, w) = \mathrm{Re}\ (w,\, L^*w) = \lambda\, \|Mw\|^2\,,$$

i.e. $w = 0$.

REMARK. Equations of form (5.5), even with some boundary condition, have been already studied by Duvaut-Lions [18], Lax-Phillips [11], Povoas [13], etc. Our assumptions are analogous to Povoas [13] in which she studied the strong solution of (5.5) in $\Omega \subset \mathbf{R}^3$ with some boundary condition.

EXAMPLE 5.3. Consider an equation of the form

$$(5.13) \qquad \begin{cases} \partial u/\partial t + (a(x) \cdot \nabla)\, u + \nabla p = f(t, x) & \text{in}\quad [0, T] \times \mathbf{R}^3 \\ \mathrm{div}\ u = 0 & \text{in}\quad [0, T] \times \mathbf{R}^3 \\ u(0, x) = u_0(x) & \text{in}\quad \mathbf{R}^3\,. \end{cases}$$

Here, $a(x)$ (resp. $f(t, x)$) is a given function of $x \in \mathbf{R}^3$ (resp. of $(t, x) \in [0, T] \times \mathbf{R}^3$) with values in $\mathbf{R}^3$, $u = (u_1(t, x), u_2(t, x), u_3(t, x))$ and $p = p(t, x)$ are unknown functions, and $u_0(x)$ is an initial function.

Under the assumption that

$$(5.14) \qquad \begin{cases} a(x) \in \{\mathscr{B}^1(\mathbf{R}^3)\}^3 \quad \text{and is a real valued function with} \\ \mathrm{div}\ a = 0 \quad \text{in}\quad \mathbf{R}^3\,; \end{cases}$$

let us verify that (5.13) can be formulated as an abstract evolution equation of the form

$$(5.15) \qquad \begin{cases} du/dt + Au \ni f(t)\,, \quad 0 \leq t \leq T\,, \\ u(0) = u_0\,, \end{cases}$$

in the space $X=\{L^2(\mathbf{R}^3)\}^3$, using a suitable maximal accretive, multivalued linear operator A.

Let $X=X_\sigma+X_\mathscr{G}$ be the orthogonal decomposition of X with

$$X_\sigma = \text{clousure of } \{u\in(C_0^\infty(\mathbf{R}^3))^3; \text{ div } u = 0 \text{ in } \mathbf{R}^3\} \text{ in } X,$$

$$X_\mathscr{G} = X_\sigma^\perp = \{\nabla p\in X; p\in L^2_{loc}(\mathbf{R}^3)\}.$$

Let $P: X\to X_\sigma$ denote the projection. Since $(1-P)u=(\mathscr{F}^{-1}[\sum_{i=1}^{3}\frac{\xi_i\,\xi_j}{|\xi|^2}\mathscr{F}$ $[u_i]])_{j=1,2,3}$, it is observed that $1-P$ is continuous in the H^1-norm also; this means that P is bounded on $\{H^1(\mathbf{R}^3)\}^3$ and defines a decomposition of $X^1=$ $\{H^1(\mathbf{R}^3)\}^3$ with $X^1=X_\sigma^1+X_\mathscr{G}^1$, where $X_\sigma^1=PX^1$ and $X_\mathscr{G}^1=(1-P)X^1$. By the similar reason, $1-P$ and P are continuous in the H^{-1}-norm and can be extended on $\{H^{-1}(\mathbf{R}^3)\}^3$; this then defines a decomposition of $X^{-1}=\{H^{-1}(\mathbf{R}^3)\}^3$ with $X^{-1}=X_\sigma^{-1}+X_\mathscr{G}^{-1}$, where $X_\sigma^{-1}=PX^{-1}$ and $X_\mathscr{G}^{-1}=(1-P)X^{-1}$. Moreover,

$$X_\sigma^{-1} = \text{closure of } \{u\in(C_0^\infty(\mathbf{R}^3))^3; \text{ div } u = 0 \text{ in } \mathbf{R}^3\} \text{ in } X^{-1},$$

$$X_\mathscr{G}^{-1} = \{\nabla p\in X^{-1}; p\in L^2_{loc}(\mathbf{R}^3)\}.$$

Now let us define a multivalued linear operator A in X by

$$(5.16) \quad \begin{cases} \mathscr{D}(A) = \{u\in X_\sigma; (a(x)\cdot\nabla)u+g\in X \text{ with some } g\in X_\mathscr{G}^{-1}\} \\ Au = \{(a(x)\cdot\nabla)u+g; g\in X_\mathscr{G}^{-1} \text{ for which } (a(x)\cdot\nabla)u+g\in X\}. \end{cases}$$

Then the equation (5.15) with this operator A can be considered as an abstract formulation of (5.13) in X.

The operator A is in fact shown to be maximal accretive in X. Let $u\in\mathscr{D}(A)$ and $f\in Au$; then, $f=(a(x)\cdot\nabla)u+g$ with $g\in X_\mathscr{G}^{-1}$. Let $\rho_\delta*$ be the mollifier, and denote $\varphi_\delta=\rho_\delta*\varphi$ for $\varphi\in C_0^\infty(\mathbf{R}^3)'$. Then if follows from (5.14) that

$$\begin{aligned}(f_\delta, u_\delta) &= ((a\cdot\nabla)u_\delta, u_\delta)+([\rho_\delta*, a\cdot\nabla]u, u_\delta)+(g_\delta, u_\delta) \\ &= -(u_\delta, (a\cdot\nabla)u_\delta)+([\rho_\delta*, a\cdot\nabla]u, u_\delta)+(g_\delta, u_\delta) \\ &= -(u_\delta, f_\delta)+2\text{Re}([\rho_\delta*, a\cdot\nabla]u, u_\delta)+2\text{Re}(g_\delta, u_\delta).\end{aligned}$$

$[\rho_\delta*, a\cdot\nabla]$ is the commutator of $\rho_\delta*$ and $a\cdot\nabla$, and it is known that $[\rho_\delta*, a\cdot\nabla]\to$ 0 strongly on $L^2(\mathbf{R}^3)$ as $\delta\to 0$ from $a\in\{\mathscr{B}^1(\mathbf{R}^3)\}^3$. Since $g_\delta=\nabla p_\delta$ with some $p_\delta\in C(\mathbf{R}^3)$, and since $u_n\to u_\delta$ in X with some $u_n\in\{C_0^\infty(\mathbf{R}^3)\}^3$, div $u_n=0$ ($n=$ 1, 2, 3, $\cdots$); we see that $(g_\delta, u_\delta)=0$. Therefore, Re $(f, u)=0$ i.e. A is accretive.

In order to verify that $\mathscr{R}(\lambda-A)=X$ for $\lambda<0$, we first notice that $\mathscr{R}(\lambda-A)$ is a closed subspace of X. Indeed, Re $(f, u)=0$ for $f\in Au$ implies that $\|u\|\leq\|f\|/|\lambda|$ for $f\in(\lambda-A)u$; therefore, if $f_n\in(\lambda-A)u_n$ and $f_n\to f$ in X, then u_n converges to some u in X; while, A is a closed operator by the definition (5.16); hence, $u\in\mathscr{D}(A)$ and $f\in(\lambda-A)u$. Let us next consider an element $v\in X$ which is orthogonal to $\mathscr{R}(\lambda-A)$. Since v is orthogonal in particular to

$A0=X_{\varphi}$, v must be in X_{σ}. In addition, since $X_{\sigma}^1 \subset \mathcal{D}(A)$ and $(a \cdot \nabla) u \in Au$ for $u \in X_{\sigma}^1$, we have:

$$0 = (\lambda u - (a \cdot \nabla) u, v) = \langle u, \lambda v - (a \cdot \nabla) v \rangle_{X^1 \times X^{-1}} ;$$

therefore, $0 = \langle Pw, \lambda v - (a \cdot \nabla) v \rangle_{X^1 \times X^{-1}} = \langle w, \lambda v - P(a \cdot \nabla) v \rangle_{X^1 \times X^{-1}}$ for any $w \in X^1$, i.e. $\lambda v - P(a \cdot \nabla) v = 0$ in X^{-1}. This then shows that $v \in \mathcal{D}(A)$ and $0 \in (\lambda - A) v$; so that $v = 0$, i.e. $\mathcal{R}(\lambda - A) = X$ for $\lambda < 0$.

EXAMPLE 5.4. Let

$$(5.17) \quad \begin{cases} \left(m(x) \dfrac{\partial}{\partial t} \right)^2 v - \Delta v = g(t, x) & \text{in} \quad [0, T] \times \Omega \\ v = 0 & \text{on} \quad [0, T] \times \partial \Omega \\ v(0, x) = v_0(x), \ m(x) \dfrac{\partial v}{\partial t}(0, x) = v_1(x) & \text{in} \quad \Omega \end{cases}$$

be a Poisson-wave equation in a (bounded or unbounded) region $\Omega \subset \mathbf{R}^n$ with a smooth boundary $\partial \Omega$. $m(x)$ and $g(t, x)$ are given functions; in particular, $m(x) \geq 0$ is allowed to vanish in a bounded subset of Ω. $v = v(t, x)$ is an unkown function. And $v_0(x)$ and $v_1(x)$ are initial functions.

Setting $v_0 = v$ and $v_1 = \partial v / \partial t$, we rewrite the equation in a system

$$\begin{pmatrix} 1 & 0 \\ 0 & m(x) \end{pmatrix} \frac{\partial}{\partial t} \begin{pmatrix} 1 & 0 \\ 0 & m(x) \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix} \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} = \begin{pmatrix} 0 \\ g(t, x) \end{pmatrix}.$$

Then an abstract formulation of (5.17) in the space $L^2(\Omega)$ is:

$$(5.18) \quad \begin{cases} M d(MW)/dt + LW = G(t), & 0 \leq t \leq T, \\ MW(0) = V_0, \end{cases}$$

in the product space $X = \begin{matrix} H_0^1(\Omega) \\ \times \\ L^2(\Omega) \end{matrix}$, where

$$W = \begin{pmatrix} v_0(t, x) \\ v_1(t, x) \end{pmatrix}, \ G(t) = \begin{pmatrix} 0 \\ g(t, x) \end{pmatrix} \quad \text{and} \quad V_0 = \begin{pmatrix} v_0(x) \\ v_1(x) \end{pmatrix}.$$

$M = \begin{pmatrix} 1 & 0 \\ 0 & m(x) \end{pmatrix}$ is a multiplication operator in X. $L = -\begin{pmatrix} 0 & 1 \\ \Delta & 0 \end{pmatrix}$ is a closed linear operator in X with $\mathcal{D}(L) = \begin{matrix} H^2(\Omega) \cap H_0^1(\Omega) \\ \times \\ H_0^1(\Omega) \end{matrix}$. Thus (5.18) is an equation of form (D-E.2)' in Section 4.

Assume here that

$$(5.19) \qquad m \in L^\infty(\Omega); \ m(x) \geq 0, \ x \in \Omega; \quad \text{and for sufficiently large } R,$$
$$m(x) \geq \delta > 0, \ x \in \Omega - \Omega_R, \text{ where } \Omega_R = \{x \in \Omega; \ |x| \leq R\} .$$

Then M and L are shwon to satisfy the conditions (4.8) and (4.12).

To see this let us first verify:

Lemma 5.1. $\|u\|_{\tilde{1}} = \sqrt{\|\nabla u\|_0^2 + \|mu\|_0^2}$ *defines an equivalent norm in the space* $H_0^1(\Omega)$.

Proof. Clearly, $\|u\|_{\tilde{1}} \leq \mathrm{Max}\ \{1, \|m\|_{L^\infty(\Omega)}\} \|u\|_1$, $u \in H_0^1(\Omega)$. On the other hand, let $\psi \in C_0^\infty(\mathbf{R}^n)$, $0 \leq \psi \leq 1$, such that supp $\psi \subset \{|x| \leq R+1\}$ and $\psi \equiv 1$ on $\{|x| \leq R\}$. Then, from (5.19),

$$\|u\|_0 = \|\psi u\|_0 + \|(1-\psi) u\|_0 \leq \|\psi u\|_{L^2(\Omega_{R+1})} + \|u\|_{L^2(\Omega - \Omega_R)}$$
$$\leq \|\psi u\|_{L^2(\Omega_{R+1})} + \delta^{-1} \|mu\|_0 .$$

In addition, according to the Poincaré inequality, we have:

$$\|\psi u\|_{L^2(\Omega_{R+1})} \leq C \|\nabla(\psi u)\|_{L^2(\Omega_{R+1})}$$
$$\leq C \ \{\|(\nabla \psi) u\|_0 + \|\psi(\nabla u)\|_0\} \leq C \ \{\delta^{-1} \|mu\|_0 + \|\nabla u\|_0\}$$

(note that $\psi u \in H_0^1(\Omega_{R+1})$). Hence, $\|u\|_1 \leq C \|u\|_{\tilde{1}}$, $u \in H_0^1(\Omega)$, with some constant C.

Hereafter we shall equip the space $H_0^1(\Omega)$ with the norm $\|\cdot\|_{\tilde{1}}$ in stead of $\|\cdot\|_1$, therefore for $F = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in X$, $\|F\|_X = \sqrt{\|f_0\|_{\tilde{1}}^2 + \|f_1\|_0^2}$. Let $V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{D}(L)$. Then,

$$(LV, V)_X = -(\nabla v_1, \nabla v_0)_0 - (mv_1, mv_0)_0 - (\Delta v_0, v_1)_0$$
$$= 2i \ \mathrm{Im}\,(\nabla v_0, \nabla v_1)_0 - (mv_1, mv_0)_0.$$

Hence, $\mathrm{Re}\,(LV, V)_X \geq -\{\|mv_0\|_0^2 + \|mv_1\|_0^2\} \geq -\|MV\|_X^2$, i.e. (4.8) is valid with $\beta = -1$.

Let us next verify (4.12) for $\lambda < 0$. Consider a sesquilinear form

$$a_\lambda(u, v) = (\nabla u, \nabla v)_0 + \lambda^2(mu, mv)_0 , \quad u, v \in H_0^1(\Omega)$$

on $H_0^1(\Omega)$. Since $a_\lambda(u, u) \geq \mathrm{Min}\ \{1, \lambda^2\} \|u\|_{\tilde{1}}^2$, Lax-Milgram's theorem yields that for any $F = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix} \in X$, there exists a function $v_0 \in H_0^1(\Omega)$ such that

$$a_\lambda(v_0, v) = (\lambda m^2 f_0 - f_1, v)_0 \qquad \text{for all} \quad v \in H_0^1(\Omega) .$$

Thanks to the a priori estimate of elliptic operator, this implies that $v_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $-\Delta v_0 + \lambda^2 m^2 v_0 = \lambda m^2 f_0 - f_1$. Set $v_1 = f_0 - \lambda v_0 \in H_0^1(\Omega)$, then it is immediate to observe that $V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{D}(L)$ and $(\lambda M^2 - L) V = F$. Thus (4.12) is valid for any $\lambda < 0$.

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