Maehara, K.
Osaka J. Math.
28 (1991), 255-261

ON THE HIGHER DIMENSIONAL MORDELL CONJECTURE OVER FUNCTION FIELDS

KAZUHISA MAEHARA

(Received November 2, 1989)

Introduction

The purpose of this note is to give a partial answer to the following conjecture which is a function theoretic analogue of Mordell conjecture and was formulated by S. Lang, E. Bombieri and J.Noguchi ([6], [10], [11]):

Let K be a function field over the complex number field $\boldsymbol{C}$. Let V be a projective variety defined over K, $\Omega_{V/K}$ the sheaf of regular differential 1-forms ω_V the canonical invertible sheaf. Recall that V is called a variety of general type if the rational mapping associated with the l-th pluri-canonical system $|\omega_V^l|$ for an integer $l>0$ is birational. We say that V is isotrivial if there exist a projective variety V_0 defined over $\boldsymbol{C}$ and a finite extention K' of K such that $V\otimes_K K'$ is birationally equivalent to $V_0\otimes_{\boldsymbol{C}} K'$.

Conjecture M. *Let V be a projective variety of general type defined over K. Suppose that V is not isotrivial. Then the set of K-rational points of V cannot be Zariski dense in V.*

(i) Mordell conjectured that any curve of genus ≥ 2 defined over a number field $\mathfrak{K}$ does not admit an infinite number of $\mathfrak{K}$-rational points, which is proved by G. Faltings. An analogue of Modell conjecture over function fields was proved by Y. Manin and H. Grauert ([2], [3], [6]).

In this case a curve is assumed to be not isotrivial over the definition function field.

(ii) J. Noguchi ([11]) and M. Deschamps ([1]) proved Conjecture M under the assumption that $\Omega_{V/K}$ is ample, in other words the fundamental sheaf $\mathcal{O}_{\boldsymbol{P}(\Omega_{V/K})}(1)$ of the projective bundle $\boldsymbol{P}(\Omega_{V/K})$ is ample. Note that if $\mathcal{O}_{\boldsymbol{P}(\Omega_{V/K})}(1)$ is ample then $\mathcal{O}_{\boldsymbol{P}(\Omega_{V/K})}(\alpha)\otimes\omega_{\boldsymbol{P}(\Omega_{V/K})}^{-1}$ for some $\alpha>0$ is ample, which turns out to be nef and big (for the definition see §1).

(iii) A compact analytic space X is said to be hyperbolic if any holomorphic map from $\boldsymbol{C}$ into X is constant, i.e., X does not contain any singular elliptic curve as well as any rational curve. It is conjectured that a hyperbolic variety is a variety of general type.

(iv) D. Riebesehl ([12]) proved Conjecture M under the hypothesis of negative curvature and the assumption that all the fibres have negative curvature. Further J. Noguchi ([10]) proved it under the hypothesis that V is hyperbolic with the Chern class $c(\omega_V)$ represented by a semipositive (1, 1)-form.

(v) Conjecture M follows from the boundendness hypothesis to the effect that the intersection number (Γ, ω_X) is bounded above for any non-singular curve Γ with fixed genus contained in a given variety ([9]).

The main result of this paper is the following:

Let K be a function field over $\boldsymbol{C}$ and let V be a projective non-singular variety over K.

Theorem. *Assume that V is of general type and that the fundamental sheaf $\mathcal{O}_{P(\Omega_{V/K})}(1)$ of the projective bundle $\boldsymbol{P}(\Omega_{V/K})$ is K-nef and K-big and that there exists $\alpha>0$ such that $\mathcal{O}_{P(\Omega_{V/K})}(\alpha)\otimes\omega_{P(\Omega_{V/K})}^{-1}$ is K-nef. The set of K-rational points $\{s_\lambda(K)\}$ is not dense in V provided that V is not isotrivial over K.*

REMARKS. (a) Under the same assumption as above, V does not contain any rational curve but may contain a singular elliptic curve.

(b) In the previous paper ([9]), the same result was proved under the assumption that $\mathcal{O}(\alpha)\otimes p^*\omega_X^{-1}$ is $f\circ p$-nef over the whole X, not only over the generic fibre.

1. Notation

We recall the following

DEFINITION ([4]). Let $f: X\rightarrow S$ be a proper morphism onto a variety S and L an invertible sheaf on X. Let η be the generic point of S and L_η denote the restriction of L to the generic fibre X_η. An invertible sheaf L is f-ample if for any coherent sheaf $\mathcal{F}$, the natural homomorphisms $f^*f_*(\mathcal{F}\otimes L^m)\rightarrow\mathcal{F}\otimes L^m$ for some m_0 and any $m\geq m_0$ are surjective. An invertible sheaf L is said to be *f-big*, if for any invertible sheaf M on X, the natural homomorphism $f^*f_*(M\otimes L^m)\rightarrow M\otimes L^m$ for some $m>0$ is not zero, in other words $f_*(M\otimes L^m)\neq 0$. And an invertible sheaf L is said to be *f-nef* if $\deg_D L_D\geq 0$ for every curve D which is mapped to a point on S by f. When $S=\operatorname{Spec} K$, *f-big* and *f-nef* are said to be *K-big* and *K-nef*, respectively.

Let $f: X\rightarrow S$ be a proper surjective morphism of projective complex manifolds. Let K be the function field of S and V the generic fibre of f. We let $\Omega_{V/K}$ denote the sheaf of the Kähler differential on V, let $\boldsymbol{P}(\Omega_{V/K})$ denote the projective bundle associated to $\Omega_{V/K}$ over V and let $\mathcal{O}_{P(\Omega_{V/K})}(1)$ denote the fundamental sheaf over $\boldsymbol{P}(\Omega_{V/K})$. We denote by ω_V the canonical invertible sheaf, i.e., $\det \Omega_{V/K}$. We have the exact sequence

$$0 \to f^*\Omega_K \to \Omega_V \to \Omega_{V/K} \to 0\,.$$

Then $\boldsymbol{P}(\Omega_{V/K}) \subset \boldsymbol{P}(\Omega_V)$. We have $\Omega_V = \mathcal{O}_V \otimes_{\mathcal{O}} (\Omega_X|_V)$ and $\Omega_K = K \otimes_{\mathcal{O}} \Omega_S$;

$$\begin{array}{ccc} \boldsymbol{P}(\Omega_{X/S}) & \supset & \boldsymbol{P}(\Omega_{V/K}) \\ \downarrow & \square & \downarrow \\ X & \supset & V \\ F\downarrow & \square & \downarrow f \\ S & \supset & \operatorname{Spec} K \end{array} \qquad \begin{array}{ccc} \boldsymbol{P}(\Omega_X) & \supset & \boldsymbol{P}(\Omega_V) \\ \downarrow & \square & \downarrow \\ X & \supset & V \end{array}$$

Here $\square$ means that the diagram is cartesian.

2. Proof of the main theorem

In order to prove the theorem, we first consider the case in which $\operatorname{tr.deg} K/\boldsymbol{C}=1$. In this case, we denote S by C.

Lemma 1. *Some power $\mathcal{O}(\beta)$ of the fundamental sheaf $\mathcal{O}(1)$ on $\boldsymbol{P}(\Omega_V)$ is generated by its global sections for any $\beta \gg 0$.*

Proof. We will use the following Kawamata-Shoklov's base point free theorem (see [4], Base Point Free Theorem):

Let X be a compact manifold and $f: X \to S$ a proper surjective morphism onto a variety. Assume that $L^\alpha \otimes \omega_X^{-1}$ is f-nef and f-big for some $\alpha > 0$ and that L is f-nef. Then there exists a positive integer m_0 such that $f^*f_*L^m \to L^m$ is surjective for any $m \geq m_0$.

We return to the proof.

Observing the exact sequence $0 \to \mathcal{O}_V \to \Omega_V \to \Omega_{V/K} \to 0$, one sees that $\boldsymbol{P}(\Omega_{V/K})$ is identified with a member D of the complete linear system $|\mathcal{O}(1)|$ on $\boldsymbol{P}(\Omega_V)$. One has the following exact sequence:

$$0 \to \mathcal{O}(\beta-1) \to \mathcal{O}(\beta) \to \mathcal{O}_D(\beta) \to 0\,.$$

By the assumption of the theorem, one has $H^1(\mathcal{O}_D(\beta))=0$ for $\beta > \alpha$ using Kawamata-Viehweg vanishing theorem ([4]). Hence $\dim H^1(\mathcal{O}(\beta))$ is a monotonous decreasing function in β if $\beta \gg 0$. Thus $H^0(\mathcal{O}(\beta)) \to H^0(\mathcal{O}_D(\beta))$ is surjective for sufficiently large number β. On the other hand, applying Kawamata-Shoklov's base point free theorem [4] to $\mathcal{O}_D(1)$, one sees that $\mathcal{O}_D(\beta)$ is base point free for $\beta > \beta_0 \gg 0$. Combining these observations, one proves the lemma. $\square$

Set $g = f \circ p$. Then the surjection $g^*g_*\mathcal{O}(l) \to \mathcal{O}(l)$ for $l \gg 0$ gives a g-birational morphism $\varphi: \boldsymbol{P}(\Omega_V) \to \boldsymbol{P}(g_*\mathcal{O}(l))$. Thus one obtains the following diagram:

$$\begin{array}{ccc} \boldsymbol{P}(\Omega_V) & & \\ p\downarrow & \searrow^{\varphi} & \\ & & \boldsymbol{P}(g_*\mathcal{O}(l)) \\ V & \swarrow & \\ f\downarrow & \swarrow & \\ K & & \end{array}$$

Let $\mathcal{F}$ be a coherent sheaf over V and let $T\to V$ be a map such that there exists a surjection $\mathcal{F}_T\to\mathcal{L}$, where $\mathcal{L}$ is an invertible sheaf over T. Then there exists a unique map $T\to\boldsymbol{P}(\mathcal{F})$ over V such that $\mathcal{F}_T\to\mathcal{L}$ is the pull-back to T of the fundamental surjection $\mathcal{F}_{\boldsymbol{P}(\mathcal{F})}\to\mathcal{O}_{\boldsymbol{P}(\mathcal{F})}(1)$. Applying this to the natural surjections $\Omega_V|_{s_\lambda(K)}\to\Omega_{s_\lambda(K)}$, we have the Gauss maps $\sigma_\lambda\colon s_\lambda(K)\to\boldsymbol{P}(\Omega_V)$. Let Z be a component of the Zariski closure of the set of K-rational points $\{\sigma_\lambda(s_\lambda(K))\}$ defined by Gauss map such that $p(Z)=V$. For each l, the l multiple of the divisor $D=\boldsymbol{P}(\Omega_{V/K})$ is the pull-back of a hyperplane Σ of $\boldsymbol{P}(g_*(l))$. We denote $\varphi(Z)$ by W. We divide into two cases:

(i) $\dim W=0$,

(ii) $\dim W>0$.

We prove some preliminary lemmas.

Lemma 2. *Let U denote $\boldsymbol{P}(\Omega_V)-\boldsymbol{P}(\Omega_{V/K})$. Put $\sigma_\lambda(s_\lambda(K))=$ the rational point defined by the natural surjection $\Omega_V|_{s_\lambda(K)}\to\Omega_{s_\lambda(K)}$ defining $\sigma_\lambda\colon s_\lambda(K)\to\boldsymbol{P}(\Omega_V)$. Then σ_λ factors through U. Let T be any scheme over V such that there exist an invertible sheaf L and a surjection $\Omega_V|_T\to L$. Then we have a V-morphism ϕ: $T\to\boldsymbol{P}(\Omega_V)$. We have the following diagram:*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{O}_V|_T & \longrightarrow & \Omega_V|_T & \longrightarrow & \Omega_{V/K}|_T & \longrightarrow & 0 \\ & & a(T) & \searrow & \downarrow & & & & \\ & & & & \mathcal{O}(1)|_T & & & & \end{array}$$

Let t be a point of T. If $\phi(t)\in D$, we have $a(t)=0$ and if $\phi(t)\in U$, $a(t)$ is bijective. Hence if $T\subset U$, $a(T)$ is bijective and the exact sequence above splits over T.

Proof. Since $f^*\Omega_K=\sigma_\lambda^*\mathcal{O}(1)$, the result follows. (cf. [1])

Lemma 3. *Let $u\colon M\to N$ be a proper surjective morphism between varieties. Suppose that N is a normal variety. Then the exact sequence $0\to F\to E\to Q\to 0$ of locally free sheaves of finite rank on N splits if and if the pull back of this sequence splits on M.*

Proof. It follows from the injectivity of the natural map $H^1(L)\to H^1(u^*L)$ for any locally free coherent sheaf L. □

Case (i).

Note that $\varphi(Z)$ consists of a single point. From Lemmas 2 and 3, one has the splitting of the exact sequence $0 \to f^*\Omega_K \to \Omega_V \to \Omega_{V/K} \to 0$. We take a projective non-singular model of $f: V \to \mathrm{Spec}\, K$, denoted by $f: X \to C$. Thus $f: X \to C$ is locally trivial in the sense of etale topology.

Case (ii).

Note that $Z \cap D \neq \phi$.

Lemma 4. *The K-rational points* $\{\sigma_\lambda \circ s_\lambda(K)\}$ *on* $\boldsymbol{P}(\Omega_V)$ *are not contained in* $\mathrm{Bs}\,|\mathcal{O}(\beta)\otimes\omega_P^{-k}|$ *for general* λ *nad some* β *and* $k>0$.

Proof. Observing the exact sequence $0 \to \mathcal{O}_V \to \Omega_V \to \Omega_{V/K} \to 0$, one sees that $\boldsymbol{P}(\Omega_{V/K})$ on $\boldsymbol{P}(\Omega_V)$ is a divisor of the complete linear system $|\mathcal{O}(1)|$. One has the following exact sequence:

$$0 \to \mathcal{O}(\beta-1)\otimes\omega_P^{-k} \to \mathcal{O}(\beta)\otimes\omega_P^{-k} \to \mathcal{O}_D(\beta+k)\otimes\omega_D^{-k} \to 0 .$$

By the assumption of the theorem we can apply Kawamata-Viehweg's vanishing theorem to obtain $H^1(\mathcal{O}_D(\beta+k)\otimes\omega_D^{-k})=0$, if $\beta>\alpha(k+1)-k$. Hence $\dim H^1(\mathcal{O}(\beta)\otimes\omega_P^{-k})$ is a monotonous decreasing function in β if $\beta\gg 0$. Thus $H^0(\mathcal{O}(\beta)\otimes\omega_P^{-1}) \to H^0(\mathcal{O}_D(\beta+k)\otimes\omega_D^{-k})$ is surjective for sufficiently large number β. By the hypothesis of the theorem, applying Kawamata's base point free theorem [4] to $\mathcal{O}_D(\alpha')\otimes\omega_D^{-1}$ for $\alpha'>2\alpha$, one concludes that $\mathcal{O}_D(k\alpha')\otimes\omega_C^{-k}$ is base point free for sufficiently large $k\gg 0$. On the other hand some power of $\mathcal{O}_D(1)$ is generated by its global sections by Kawamata's theorem. Thus $\mathcal{O}_D(\beta+k)\otimes\omega_D^{-k}$ is generated by its global sections for sufficiently large β and $k\gg 0$. Hence $\mathrm{Bs}\,|\mathcal{O}(\beta)\otimes\omega_P^{-k}|\cap D=\phi$. Since $Z\cap D\neq\phi$, we 'conclude that $\mathrm{Bs}\,|\mathcal{O}(\beta)\otimes\omega_P^{-k}|$ does not include Z.

Considering $f: X \to C$, we have some ample invertible sheaf L on C such that the natural map

$$\mathcal{O}_{\sigma_\lambda\circ s_\lambda(C)}\otimes H^0(\sigma_\lambda\circ s_\lambda(C), \mathcal{O}(\beta)\otimes\mathcal{O}(\omega_P^{-k})\otimes p^*f^*L) \to \mathcal{O}(\beta)\otimes\mathcal{O}(\omega_P^{-k})\otimes p^*f^*L\,|_{\sigma_\lambda\circ s_\lambda(C)}$$

is generically surjective for suitable β, $k>0$. Hence we have a dense set of curves $\{\sigma_\lambda(s_\lambda(C))\}$ in Z such that the intersection $(\mathcal{O}(\beta)\otimes\omega_P^{-k}\otimes p^*f^*L,\ \sigma_\lambda\circ s_\lambda(C))\geq 0$. recalling that

$$(\mathcal{O}(1),\ \sigma_\lambda\circ s_\lambda(C)) = 2g-2\,, \quad \omega_{P/X} = \mathcal{O}(-n-1)\otimes p^*\det\Omega_X\,,$$

one has

$$\deg_{\sigma_\lambda(s_\lambda(C))} p^*\omega_X^k = (\sigma_\lambda(s_\lambda(C)),\ p^*\omega_X^k) \leq (g(C)-1)(\beta+n-1)+\frac{1}{2}\deg_C L\,.$$

By the projection formula, one obtains

$$(s_\lambda(C),\, \omega_X) \leq \frac{\beta+n-1}{k}(g(C)-1)+\frac{1}{2k}\deg_C L.$$

By the Viehweg formula ([14]), one has $\kappa(\omega_X \otimes f^*L)=\kappa(\omega_V)+1$. Hence for any ample invertible sheaf H over X there exist a positive integer ν and an effective divisor F such that $(\omega_X \otimes f^*L)^\nu = H+F$. Thus we can bound the degree of sections $C_\lambda = \sigma_\lambda(C)$ which are not contained in F of X and we have at most a finite number of Hilbert polynomials of the graphs Γ_λ of sections C_λ in $C\times X$. Thus we let H be a Hilbert scheme parametrizing proper subschemes in $C\times X$ with the Hilbert polynomials mentioned above. Thus we have a subvariety T^0 which parametrizes the graphs Γ_λ of sections C_λ, whose set is dense in X. Let T be a compactification of T^0. Hence we have the following commutative diagram:

$$\begin{array}{ccc} & \{\Gamma_\lambda\} \subset X\times T & \\ \swarrow & & \downarrow \\ C\times T & & X \\ & \searrow \quad \swarrow & \\ & C & \end{array}$$

Thus $f: X \to C$ is birationally trivial over C from the lemma ([7], section 5(*p.* 115), Appendix (*p.* 119)):

Let T be a complete variety and $\phi: T\times S \to X$ be a dominant S-rational map. Then X is birationally trivial over S.

We can easily reduce the general case to the case when tr. deg $K/\boldsymbol{C}=1$. Considering the pluri-S-canonical mapping $X/S \to \boldsymbol{P}_S(f_*\omega_{X/S}^{\otimes k})$ for $k \gg 0$ and noting that varieties of general type have no infinitesimal automorphisms except for finite automorphisms, we have a dense open S^0 in S such that every fibre of X/S is birational, since we can join any two points in S^0 by a non singular curve in S^0. Hence one can find etale covering S' over S such that the pull-back of the pluri-S-canonical mapping $X/S \to \boldsymbol{P}_S(f_*\omega_{X/S}^{\otimes k})$ is trivial. Q.E.D.

References

[1] M. Deschamps: *Propriétés de descente des variétés à fibre cotangent ample,* Ann. Inst. Fourier, Grenoble, **33** (1984), 39–64.

[2] M. Deschamps: *La construction de Kodaira-Parshin,* Seminaire sur les pinceaux arithmetiques: la conjecture de Mordell, Soc. Math. France, Asterisque **127** (1985), 261–271.

[3] H. Grauert: *Mordells Vermtung über rationale Punkte auf Algebraischen Kurven und Functionenkörper,* Publ. Math. IHES **25** (1965), 1–95.

[4] Y. Kawamata, K. Matsuda, K. Matsuki: *Introduction to the minimal model problem,* Advanced Studies in Pure Mathematics 10, Algebraic Geometry, 283–360,

Sendai (1985), 1987.

[5] K. Kodaira and D.C. Spencer: *On deformation of complex analytic structures I*, Ann. of Math. **68** (1958), 328–401.

[6] S. Lang: *Hyperbolic and Diophantine analysis*, Bull. Amer. Math. Soc. **14** (1986), 159–205.

[7] K. Maehara: *Finiteness property of varieties of general type*, Math., Ann., **265** (1983), 101–123.

[8] K. Maehara: *The weak* 1*-positivity of direct image sheaves*, J. Reine und Angewante Math. **364** (1986), 112–129.

[9] K. Maehara: *The Mordell-Bombieri-Noguchi conjecture over function fields*, Kodai Math. J. **11** (1988), 1–4.

[10] J. Noguchi: *Hyperbolic fibre spaces and Mordell's conjecture over function fields*, Publ. R.I.M.S. Kyoto Univ., **21** (1985), 27–46.

[11] J. Noguchi: *A higher dimensional analogue of Mordell's conjecture over function fields*, Math. Ann. **258** (1981), 207–212.

[12] D. Riebesehl: *Hyperbolische komplexe Raüme und die Vermutung von Mordell*, Math. Ann **257** (1981), 99–110.

[13] P. Samuel: *Lectures on old and new results on algebraic curves*, Tata Inst. F.R., Bombay, 1966.

[14] E. Viehweg: *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces*, in Algebraic Varieties and Analytic varieties, Advanced Studies in Pure Math. 1, 329–353: Tokyo and Amsterdam, 1983.

Tokyo Institute of Polytechnics
1583 Iiyama,
Atsugi city,
Kanagawa, 243–02
Japan