Yosimura, Z.
Osaka J. Math.
27 (1990), 465–498

QUASI K-HOMOLOGY EQUIVALENCES, I

Dedicated to Professor Shôrô Araki on his sixtieth birthday

ZEN-ICHI YOSIMURA

(Received March 23, 1989)
(Revised November 7, 1989)

0. Introduction

Let KO, KU and KC be the real, complex and self-conjugate K-spectrum respectively. Following [14] we call a CW-spectrum X *a Wood spectrum* if there exists a KO-module equivalence $f: KU \to KO \wedge X$, and *an Anderson spectrum* if there exists a KO-module equivalence $g: KC \to KO \wedge X$. The elementary spectra P and Q taken to be the cofibers of the maps $\eta: \Sigma^1 \to \Sigma^0$ and $\eta^2: \Sigma^2 \to \Sigma^0$ respectively are known as typical examples of Wood and Anderson spectra [3], where $\eta: \Sigma^1 \to \Sigma^0$ is the stable Hopf map of order 2. Recently Mimura, Oka and Yasuo [14] gave some characterizations of finite CW-complexes whose suspension spectra are such spectra. The following theorem is a spectrum version of one of their results.

Theorem 0. i) *X is a Wood spectrum if and only if $KU_0X \simeq Z \oplus Z$, $KU_1X = 0$ and the conjugation t_* on KU_0X is represented by the matrix* $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
ii) *X is an Anderson spectrum if and only if $KU_0X \simeq Z$, $KU_1X \simeq Z$, $KO_2X = 0 = KO_6X$ and the conjugation t_* acts as the identity on both KU_0X and $KU_{-1}X$.*

Let E be an associative ring spectrum with unit. Given CW-spectra X, Y we say that X is *quasi E_*-equivalent to Y*, written $X \underset{E}{\simeq} Y$, if there exists a map $h: Y \to E \wedge X$ such that the composite $(\mu_\wedge 1)(1_\wedge h): E \wedge Y \to E \wedge E \wedge X \to E \wedge X$ is an equivalence. We are interested in the quasi K-homology equivalences, especially the quasi KO_*-equivalence. According to our definition, a CW-spectrum X is said to be a Wood spectrum if $X \underset{KO}{\simeq} P$ and an Anderson spectrum if $X \underset{KO}{\simeq} Q$.

Let H be a finitely generated abelian group which is 2-torsion free. If the cyclic group $Z/2$ of order 2 acts on H, then H admits a direct sum decomposition $H \simeq A \oplus B \oplus C \oplus C$ such that the action ρ behaves as $\rho = 1$ on A, $\rho = -1$ on B and $\rho = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $C \oplus C$ respectively [7]. For any abelian group G we denote by SG the Moore spectrum of type G. The Moore spectrum SZ/m is constructed by the cofiber sequence $\Sigma^0 \xrightarrow{m} \Sigma^0 \xrightarrow{i} SZ/m \xrightarrow{j} \Sigma^1$. In this note our purpose is a development of the work of Mimura-Oka-Yasuo [14]. We will first show the following results (cf. [6]) which of course contain Theorem 0.

Theorem 1. *Assume that KU_0X is finitely generated, 2-torsion free and $KU_1X=0$. Then there exist abelian groups A', A'', B', B'' and C so that $X \underset{KO}{\approx} Y \vee (P \wedge SC)$ where Y denotes the wedge sum $SA' \vee \Sigma^2 SB' \vee \Sigma^4 SA'' \vee \Sigma^6 SB''$ of the Moore spectra (Theorem 2.4).*

Theorem 2. *Assume that KU_0X and KU_1X are finitely generated, 2-torsion free. If the conjugation t_* acts as the identity on KU_0X and KU_1X, then there exist abelian groups A', A'', D', D'' and G so that $X \underset{KO}{\approx} Y \vee (\Sigma^1 Q \wedge SG)$ where Y denotes the wedge sum $SA' \vee \Sigma^1 SD' \vee \Sigma^4 SA'' \vee \Sigma^5 SD''$ of the Moore spectra (Theorem 3.4).*

As an immediate corollary of Theorem 1 we can determine the quasi KO_*-type of the complex projective n-space CP^n (Corollary 2.5), since KU_0CP^n is the free abelian group of rank n and $KU_1CP^n=0$ [1]. However we need to discuss more richly to determine the quasi KO_*-type of the real projective n-space RP^n [20, Theorem 5], since KU_1RP^n is not 2-torsion free for any $n \geqq 2$. In fact, $KU_0RP^n=0$ and $KU_1RP^n \cong Z/2^s$ or $Z \oplus Z/2^s$ according as $n=2s$ or $2s+1$ [1], and besides $KO_0RP^n=0$ if $n \equiv 1, 2, 3, 4, 5 \bmod 8$, $KO_4RP^n=0$ if $n \equiv 0, 1, 5, 6, 7 \bmod 8$ and $KO_6RP^n=0$ for all n [8].

In order to state another main result we will only need the following elementary spectra with a few cells introduced in (4.1), (4.4) and (4.16). Let M_{2m}, Q_{2m}, V_{2m} and W_{8m} $(m \geqq 1)$ denote respectively the cofibers of the maps

$$i\eta\colon \Sigma^1 \to SZ/2m\,, \qquad \tilde{\eta}\eta\colon \Sigma^3 \to SZ/2m\,,$$
$$i\bar{\eta}\colon \Sigma^1 SZ/2 \to SZ/m \quad \text{and} \quad i\bar{\eta}+\tilde{\eta}j\colon \Sigma^1 SZ/2 \to SZ/4m$$

where $\tilde{\eta}\colon \Sigma^2 \to SZ/2n$ is a coextension of η with $j\tilde{\eta}=\eta$ and $\bar{\eta}\colon \Sigma^1 SZ/2n \to \Sigma^0$ is an extension of η with $\bar{\eta}i=\eta$.

In the case when KU_0X has 2-torsion and $KU_1X=0$, we can next show a corrsponding theorem (Theorem 5.2) to Theorem 1 under certain restrictions, using these elementary spectra. This theorem implies the following result, which is useful in determining the quasi KO_*-type of such a CW-spectrum as RP^n.

Theorem 3. *Assume that $KU_1X=0$ and $KO_1X=0=KO_7X$.*
i) *If $KU_0X \cong Z/2m$ with $m=2^s$, $s \geqq 0$, then $X \underset{KO}{\approx} \Sigma^2 SZ/2m$, V_{2m}, $W_{8n}(m=4n)$ or $\Sigma^2 W_{8n}(m=4n)$.*
ii) *If $KU_0X \cong Z \oplus Z/2m$ with $m=2^s$, $s \geqq 0$, then $X \underset{KO}{\approx} \Sigma^2 \vee Y$, $\Sigma^4 \vee Y$, M_{2m}, $\Sigma^2 M_{2m}$, $\Sigma^2 Q_{2m}$ or $\Sigma^4 Q_{2m}$ where Y is one of the four elementary spectra given in* i). (Cf. [20,

Theorem 2.5].)

This paper is organized as follows. As a preliminary, in §1 we will first recall some relations among KO, KU and KC theory [3] and then give basic tools (Proposition 1.1 and Lemma 1.3) to prove our main results. After studying the KO_*-module structures of KO_*X under the situations assumed in the theorems (Propositions 2.3 and 3.2), we will prove Theorems 1 and 2 (Theorems 2.4 and 3.4) respectively in §2 and §3. In §4 we will introduce some elementary spectra with a few cells such as M_{2m}, Q_{2m}, V_{2m} and W_{8m}, and then compute their KU and KO homologies (Propositions 4.1, 4.2, 4.4 and 4.5). By making use of the results obtained in §4 we will devote ourselves to prove Theorem 5.2 in §5, and finally show Theorem 3 as a consequence of this theorem.

In this note we will work in the stable homotopy category of CW-spectra.

1. Real, complex and self-conjugate K-theory

1.1. Let KU be the BU-spectrum representing the complex K-theory and KO the BO-spectrum representing the real K-theory. Both KU and KO are associative and commutative ring spectra with unit. These spectra are related by the Bott cofiber sequence

$$\Sigma^1 KO \xrightarrow{\eta\wedge 1} KO \xrightarrow{\varepsilon_U} KU \xrightarrow{\varepsilon_O \pi_U^{-1}} \Sigma^2 KO \tag{1.1}$$

where $\eta\colon \Sigma^1 \to \Sigma^0$ is the stable Hopf map of order 2 and $\pi_U\colon \Sigma^2 KU \to KU$ denotes the Bott periodicity. The complexification $\varepsilon_U\colon KO \to KU$ and the conjugation $t\colon KU \to KU$ are both ring maps, but the realification $\varepsilon_O\colon KU \to KO$ is merely a KO-module map. As is well known, the equalities $\varepsilon_O \varepsilon_U = 2$ and $\varepsilon_U \varepsilon_O = 1+t$ hold.

Let KC be the BSC-spectrum representing the self-conjugate K-theory, which is useful in studying the relation between KO and KU theory (see [3], [6]). This spectrum KC is also an associative and commutative ring spectrum with unit, and it is obtained as the fiber of the map $1-t\colon KU \to KU$. Thus we have a cofiber sequence

$$KC \xrightarrow{\zeta} KU \xrightarrow{\pi_U^{-1}(1-t)} \Sigma^2 KU \xrightarrow{\gamma \pi_U} \Sigma^1 KC \tag{1.2}$$

(see [3, Theorem 1.2]).

Since $\varepsilon_U \varepsilon_O \pi_U^{-1} = \pi_U^{-1}(1-t)$, we get a cofiber sequence

$$\Sigma^2 KO \xrightarrow{\eta^2\wedge 1} KO \xrightarrow{\varepsilon_C} KC \xrightarrow{\tau \pi_C^{-1}} \Sigma^3 KO \tag{1.3}$$

making the diagram below commutative

$$
\begin{tikzcd}
 & \Sigma^1 KU \arrow[r, equal] \arrow[d, "\gamma\pi_U"'] & \Sigma^1 KU \arrow[d, "\varepsilon_O \pi_U^{-1}"] & \\
KO \arrow[r, "\varepsilon_C"'] \arrow[d, equal] & KC \arrow[r, "\tau\pi_C^{-1}"'] \arrow[d, "\zeta"'] & \Sigma^3 KO \arrow[r, "\eta^2{}_\wedge 1"'] \arrow[d, "\eta_\wedge 1"] & \Sigma^1 KO \arrow[d, equal] \\
KO \arrow[r, "\varepsilon_U"] & KU \arrow[r, "\varepsilon_O \pi_U^{-1}"] \arrow[d, "\pi_U^{-1}(1-t)"'] & \Sigma^2 KO \arrow[r, "\eta_\wedge 1"] \arrow[d, "\varepsilon_U"] & \Sigma^1 KO \\
 & \Sigma^2 KU \arrow[r, equal] & \Sigma^2 KU &
\end{tikzcd}
\tag{1.4}
$$

Here π_C: $\Sigma^4 KC \rightarrow KC$ denotes the periodicity satisfying $\zeta\pi_C = \pi_U^2 \zeta$ and $\pi_C \gamma = \gamma \pi_U^2$. The maps ε_C and ζ are ring maps such that $\zeta\varepsilon_C = \varepsilon_U$, and the maps γ and τ are KO-module maps such that $\tau\gamma = \varepsilon_O$ [6].

Let P denote the suspension spectrum whose second term is the complex projective space CP^2. Thus the spectrum P is constructed by the cofiber sequence

$$
\Sigma^1 \xrightarrow{\eta} \Sigma^0 \xrightarrow{i_P} P \xrightarrow{j_P} \Sigma^2 . \tag{1.1)$'$}
$$

Take the element $u \in KU_0 P$ satisfying $(\varepsilon_{O\wedge} 1)_* u = (1_\wedge i_P)_* \iota_O$ and $(\pi_{U\wedge} j_P)_* u = \iota_U$ where $\iota_O \in KO_0 \Sigma^0$ and $\iota_U \in KU_0 \Sigma^0$ denote the units. Consider the map $W_P(u)$: $KU \rightarrow KO \wedge P$ definied to be the composite $(\varepsilon_{O\wedge} 1)(\mu_{U\wedge} 1)(1_\wedge u)$: $KU \rightarrow KU \wedge KU \wedge P \rightarrow KU \wedge P \rightarrow KO \wedge P$ where μ_U denotes the multiplication of KU. Since $W_P(u)\,\varepsilon_U = 1_\wedge i_P$ and $(1_\wedge j_P)\, W_P(u) = \varepsilon_O \pi_U^{-1}$, we can use Five lemma to show that $W_P(u)$ is an equivalence. As is well known, this result says that the Bott cofiber sequence (1.1) is produced by the cofiber sequence (1.1)$'$ smashed with KO. The map $W_P(u)$: $KU \rightarrow KO \wedge P$ is called *the Wood equivalence* [3, Theorem 2.1].

Let Q denote the suspension spectrum obtained as the cofiber of the composite square η^2. Thus

$$
\Sigma^2 \xrightarrow{\eta^2} \Sigma^0 \xrightarrow{i_Q} Q \xrightarrow{j_Q} \Sigma^3 \tag{1.3)$'$}
$$

is a cofiber sequence.

Take the element $v \in KC_{-1} Q$ satisfying $(\tau_\wedge 1)_* v = (1_\wedge i_Q)_* \iota_O$ and $(\pi_{C\wedge} j_Q)_* v = \iota_C$ where $\iota_C \in KC_0 \Sigma^0$ denotes the unit. Consider the map $W_Q(v)$: $KC \rightarrow KO \wedge Q$ defined to be the composite $(\tau_\wedge 1)(\mu_{C\wedge} 1)(1_\wedge v)$: $KC \rightarrow \Sigma^1 KC \wedge KC \wedge Q \rightarrow \Sigma^1 KC \wedge Q \rightarrow KO \wedge Q$ where μ_C denotes the multiplication of KC. The map $W_Q(v)$ is also an equivalence, since $W_Q(v)\,\varepsilon_C = 1_\wedge i_Q$ and $(1_\wedge j_Q)\, W_Q(v) = \tau\pi_C^{-1}$. Hence the cofiber sequence (1.3) is produced by the cofiber sequence (1.3)$'$ smashed with KO. The map $W_Q(v)$: $KC \rightarrow KO \wedge Q$ to be the KC-analogous of the Wood equivalence, is called *the Anderson equivalence* (see [3, Theorem 3.1]).

Combining the two cofiber sequences (1.1)′ and (1.3)′ we get the following cofiber sequence

$$(1.2)' \qquad Q \to P \xrightarrow{i_P j_P} \Sigma^2 P \to \Sigma^1 Q\,,$$

which yields the cofiber sequence (1.2) by smashing with KO.

Let R denote the suspension spectrum constructed by the cofiber sequence $\Sigma^3 \xrightarrow{\eta^3} \Sigma^0 \xrightarrow{i_R} R \xrightarrow{j_R} \Sigma^4$. Then we have two cofiber sequences

$$(1.5)' \qquad \Sigma^1 Q \to R \to P \xrightarrow{i_Q j_P} \Sigma^2 Q$$

$$(1.6)' \qquad \Sigma^2 P \to R \to Q \xrightarrow{i_P j_Q} \Sigma^3 P$$

which yield cofiber sequences

$$(1.5) \qquad \Sigma^1 KC \xrightarrow{(-\tau,\, \tau\pi_C^{-1})} KO \vee \Sigma^4 KO \xrightarrow{\varepsilon_U \vee \pi_U^2 \varepsilon_U} KU \xrightarrow{\varepsilon_C \varepsilon_O \pi_U^{-1}} \Sigma^2 KC$$

$$(1.6) \qquad \Sigma^2 KU \xrightarrow{(\varepsilon_O \pi_U,\, -\varepsilon_O \pi_U^{-1})} KO \vee \Sigma^4 KO \xrightarrow{\varepsilon_C \vee \pi_C \varepsilon_C} KC \xrightarrow{\varepsilon_U \tau \pi_C^{-1}} \Sigma^3 KU$$

(see [3, Theorems 3.2 and 3.3]).

1.2. Let E be an associative ring spectrum with unit and F any associative E-module spectrum. Given a CW-spectrum Y we denote by $[E \wedge Y, F]_E$ the subgroup of $[E \wedge Y, F]$ consisting of all the homotopy classes of E-module maps. We assign to any map $f\colon Y \to F$ the E-module map $\kappa_E(f) = \mu_F(1_\wedge f)\colon E \wedge Y \to E \wedge F \to F$ where μ_F denotes the E-module structure map of F. The assignment $\kappa_E\colon [Y, F] \to [E \wedge Y, F]_E$ is evidently an isomorphism.

A map $f\colon Y \to F$ is said to be *a quasi E_*-equivalence* if $\kappa_E(f)\colon E \wedge Y \to F$ becomes an equivalence. For any CW-spectra X, Y we say that X is *quasi E_*-equivalent to* Y if there exists a quasi E_*-equivalence $f\colon Y \to E \wedge X$. In this case we write $X \underset{E}{\simeq} Y$.

Consider the homomorphism $\tilde{\kappa}_E\colon [Y, F] \to \mathrm{Hom}_{E_*}(E_* Y, F_*)$ defined by $\tilde{\kappa}_E(f) = \kappa_E(f)_*$, where $E_* = \pi_* E$ and $F_* = \pi_* F$. Taking $E = KU$ we have a universal coefficient sequence

$$(1.7) \quad 0 \to \mathrm{Ext}_{KU_*}(KU_{*-1} Y, F_*) \to [Y, F] \xrightarrow{\tilde{\kappa}_{KU}} \mathrm{Hom}_{KU_*}(KU_* Y, F_*) \to 0$$

for any associatvie KU-module spectrum F (use [1, Theorem 13.6]). In particular, we note that

$$(1.8) \qquad \tilde{\kappa}_{KU}\colon [Y, F] \to \mathrm{Hom}_{KU_*}(KU_* Y, F_*)$$

is an isomorphism if $KU_* Y$ is free, or if $KU_1 Y = 0 = F_1$.

Taking $E = KO$ and $Y = SG$, the Moore spectrum of type G, we have a short exact sequence

$$(1.9)\quad 0 \to \mathrm{Ext}_{KO_*}(KO_{*-1}SG, F_*) \to [SG, F] \overset{\tilde{\kappa}_{KO}}{\to} \mathrm{Hom}_{KO_*}(KO_*SG, F_*) \to 0$$

for any associative KO-module spectrum F, if the abelian group G is 2-torsion free.

Given two CW-spectra X, W there exists a unique CW-spectrum $F(X, W)$, called the function spectrum, with a natural isomorphism $D_{X,W}\colon [Y, F(X, W)] \to [X \wedge Y, W]$ for any CW-spectrum Y (see [12] or [18]). Let DX denote the Spanier-Whitehead dual spectrum of X. Thus DX is just the function spectrum $F(X, S)$ where S is the sphere spectrum.

The elementary spectra P and Q are both self-dual in the sense that $DP = \Sigma^{-2}P$ and $DQ = \Sigma^{-3}Q$. So there exist duality isomorphisms $D_P\colon [\Sigma^2 Y, P \wedge X] \to [P \wedge Y, X]$ and $D_Q\colon [\Sigma^3 Y, Q \wedge X] \to [Q \wedge Y, X]$ for any CW-spectra X, Y. Let $\tilde{u} \in KU^0P$ be the dual element of $(\pi_{U\wedge}1)_* u \in KU_2P$ and $\tilde{v} \in KC^0Q$ the dual element of $(\pi_{C\wedge}1)_* v \in KC_3Q$. Then the element $\tilde{u}$ satisfies $i_P^* \tilde{u} = \iota_U$ and $(\varepsilon_O \pi_U^{-1})_* \tilde{u} = j_P^* \iota_O$, and similarly the element $\tilde{v}$ satisfies $i_Q^* \tilde{v} = \iota_C$ and $(\tau\pi_C^{-1})_* \tilde{v} = j_Q^* \iota_O$. Making use of these equalities and Five lemma we can show that $\kappa_{KO}(\tilde{u})\colon KO \wedge P \to KU$ and $\kappa_{KO}(\tilde{v})\colon KO \wedge Q \to KC$ are both equivalences, which give the inverses of $W_P(u)$ and $W_Q(v)$ respectively. Thus

(1.10) *$\tilde{u}\colon P \to KU$ and $\tilde{v}\colon Q \to KC$ are both quasi KO_*-equivalences.*

Moreover we note that the following diagram is commutative

$$(1.11)\qquad \begin{array}{ccccccc} \Sigma^1 P & \to & Q & \to & P & \to & \Sigma^2 P \\ \tilde{u}\downarrow & & \tilde{v}\downarrow & & \downarrow\tilde{u} & & \downarrow\tilde{u} \\ \Sigma^1 KU & \to & KC & \to & KU & \to & \Sigma^2 KU \end{array}$$

in which the cofiber sequences (1.2), (1.2)′ are involved (cf. [3, Lemma 3.2]).

For any maps $f\colon Y \to KU \wedge X$ and $g\colon Y \to KC \wedge X$ we define a map $e_P(f)\colon P \wedge Y \to KU \wedge X$ to be the composite $(\mu_{U\wedge}1)(1_\wedge f)(\tilde{u}_\wedge 1)\colon P \wedge Y \to KU \wedge Y \to KU \wedge KU \wedge X \to KU \wedge X$, and similarly a map $e_Q(g)\colon Q \wedge Y \to KC \wedge X$ to be the composite $(\mu_{C\wedge}1)(1_\wedge g)(\tilde{v}_\wedge 1)\colon Q \wedge Y \to KC \wedge Y \to KC \wedge KC \wedge X \to KC \wedge X$. Obviously $\kappa_{KO}(e_P(f)) = \kappa_{KU}(f)(\kappa_{KO}(\tilde{u})_\wedge 1)$ and $\kappa_{KC}(e_Q(g)) = \kappa_{KC}(g)(\kappa_{KO}(\tilde{v})_\wedge 1)$. Therefore it follows immediately from (1.10) that

(1.12) i) *$f\colon Y \to KU \wedge X$ is a quasi KU_*-equivalence if and only if $e_P(f)\colon P \wedge Y \to KU \wedge X$ is a quasi KO_*-equivalence.*
ii) *$g\colon Y \to KC \wedge X$ is a quasi KC_*-equivalence if and only if $e_Q(g)\colon Q \wedge Y \to KC \wedge X$ is a quasi KO_*-equivalence.*

The following result, which states a relation between quasi KU_*- and KO_*-equivalences, is very useful in proving our main theorems.

Proposition 1.1. *A map h: $Y \to KO \wedge X$ is a quasi KO_*-equivalence if and only if the composite $(\varepsilon_U \wedge 1)\, h$: $Y \to KO \wedge X \to KU \wedge X$ is a quasi KU_*-equivalence.* (Cf. [15, Theorem 8.14] or [13].)

Proof. Given a quasi KO_*-equivalence h: $Y \to KO \wedge X$ we consider the commutative diagram

$$\begin{array}{ccccccc} \Sigma^1 Y & \to & Y & \to & P \wedge Y & \to & \Sigma^2 Y \\ h\downarrow & & h\downarrow & & \downarrow h_1 & & \downarrow h \\ \Sigma^1 KO \wedge X & \to & KO \wedge X & \to & KU \wedge X & \to & \Sigma^2 KO \wedge X \end{array}$$

involving the cofiber sequences (1.1), (1.1)′, where $h_1 = e_P((\varepsilon_U \wedge 1)\, h)$. Applying Five lemma we see that h_1 is a quasi KO_*-equivalence. Thus (1.12) i) shows that $(\varepsilon_U \wedge 1)\, h$ is a quasi KU_*-equivalence.

Conversely we assume that $(\varepsilon_U \wedge 1)\, h$: $Y \to KU \wedge X$ is a quasi KU_*-equivalence. Use the two commutative diagrams

$$\begin{array}{ccccccc} \Sigma^1 P \wedge Y & \to & Q \wedge Y & \to & P \wedge Y & \to & \Sigma^2 P \wedge Y \\ h_1\downarrow & & h_2\downarrow & & \downarrow h_1 & & \downarrow h_1 \\ \Sigma^1 KU \wedge X & \to & KC \wedge X & \to & KU \wedge X & \to & \Sigma^2 KU \wedge X \end{array}$$

$$\begin{array}{ccccccc} \Sigma^2 P \wedge Y & \to & R \wedge Y & \to & Q \wedge Y & \to & \Sigma^3 P \wedge Y \\ h_1\downarrow & & h_3\downarrow & & \downarrow h_2 & & \downarrow h_1 \\ \Sigma^2 KU \wedge X & \to & KO \wedge R \wedge X & \to & KC \wedge X & \to & \Sigma^3 KU \wedge X \end{array}$$

involving the cofiber sequences (1.2), (1.2)′, (1.6) and (1.6)′, where $h_1 = e_P((\varepsilon_U \wedge 1) h)$, $h_2 = e_Q((\varepsilon_C \wedge 1)\, h)$ and $h_3 = (T \wedge 1)\,(1 \wedge h)$ for the switching map T: $R \wedge KO \to KO \wedge R$. Then Five lemma shows that h_2 and hence h_3 is a quasi KO_*-equivalence as h_1 is. This implies that h_*: $KO_* Y \to KO_* X$ is an epimorphism as well as a monomorphism, because $KO \wedge R = KO \vee \Sigma^4 KO$. Thus h: $Y \to KO \wedge X$ is a quasi KO_*-equivalence.

1.3. Let f: $Y \to KU \wedge X$ be a map satisfying $(t \wedge 1) f = f$. Then there exists a map g: $Y \to KC \wedge X$ such that $(\zeta \wedge 1)\, g = f$. Given such maps f, g we have a commutative diagram

$$\begin{array}{ccccccc} \Sigma^1 Y & \to & Y & \to & P \wedge Y & \to & \Sigma^2 Y \\ f\downarrow & & g\downarrow & & \downarrow e_P(f) & & \downarrow f \\ \Sigma^1 KU \wedge X & \to & KC \wedge X & \to & KU \wedge X & \to & \Sigma^2 KU \wedge X \end{array} \tag{1.13}$$

involving the cofiber sequences (1.1), (1.1)′, because $\gamma \pi_U \zeta = \eta \wedge 1$: $\Sigma^1 KC \to KC$.

In other words, there exists a commutative diagram

$$\begin{array}{ccccccc} \Sigma^1 P \wedge Y & \to & Q \wedge Y & \to & P \wedge Y & \to & \Sigma^2 P \wedge Y \\ e_P(f)\downarrow & & e_Q(g)\downarrow & & \downarrow e_P(f) & & \downarrow e_P(f) \\ \Sigma^1 KU \wedge X & \to & KC \wedge X & \to & KU \wedge X & \to & \Sigma^2 KU \wedge X \end{array} \tag{1.14}$$

involving (1.2), (1.2)′, since [4, Theorem 1.3] says that $\gamma\mu_U(1_\wedge\zeta)=\mu_C(\gamma_\wedge 1)$: $KU\wedge KC\to\Sigma^1 KC$. Applying Five lemma and (1.12) we see that

(1.15) *g: $Y\to KC\wedge X$ is a quasi KC_*-equivalence if f: $Y\to KU\wedge X$ is a quasi KU_*-equivalence.*

Lemma 1.2. *Assume that $[Y,\Sigma^1 KU\wedge X]=0$ and the map η_*^2: $[Y,\Sigma^4 KO\wedge X]\to[Y,\Sigma^2 KO\wedge X]$ is trivial. If a map f: $Y\to KU\wedge X$ satisfies $(t_\wedge 1)f=f$, then there exists a map h: $Y\to KO\wedge X$ such that $(\varepsilon_{U\wedge}1)h=f$.*

Proof. Under the assumption that $[Y,\Sigma^1 KU\wedge X]=0$, $(\zeta_\wedge 1)_*$: $[P\wedge Y,\Sigma^2 KC\wedge X]\to[P\wedge Y,\Sigma^2 KU\wedge X]$ is a monomorphism. Then (1.14) implies that $(\varepsilon_C\varepsilon_O\pi\bar{v}^{-1}{}_\wedge 1)\,e_P(f)=e_Q(g)\,(i_Q j_{P\wedge}1)$. Hence there exists a map h_R: $R\wedge Y\to KO\wedge R\wedge X$ making the diagram below commutative

$$\begin{array}{ccccccc}
\Sigma^1 Q\wedge Y & \to & R\wedge Y & \to & P\wedge Y & \xrightarrow{i_Q j_{P\wedge}1} & \Sigma^2 Q\wedge Y\\
e_Q(g)\downarrow & & h_R\downarrow & & \downarrow e_P(f) & & \downarrow e_Q(g)\\
\Sigma^1 KC\wedge X & \to & KO\wedge R\wedge X & \to & KU\wedge X & \xrightarrow[\varepsilon_C\varepsilon_O\pi\bar{v}^{-1}{}_\wedge 1]{} & \Sigma^2 KC\wedge X
\end{array}$$

where the rows are induced by the cofiber sequences (1.5), (1.5)′. We here consider the commutative diagram

$$\begin{array}{ccccccccc}
Y & \xrightarrow{i_{R\wedge}1} & R\wedge Y & \to & P\wedge Y & \xrightarrow{j_{P\wedge}1} & \Sigma^2 Y & \xrightarrow{i_{Q\wedge}1} & \Sigma^2 Q\wedge Y\\
 & & h_R\downarrow & & \downarrow e_P(f) & & & & \downarrow e_Q(g)\\
KO\wedge X & \xrightarrow[1_\wedge i_{R\wedge}1]{} & KO\wedge R\wedge X & \to & KU\wedge X & \xrightarrow[\varepsilon_O\pi\bar{v}^{-1}{}_\wedge 1]{} & \Sigma^2 KO\wedge X & \xrightarrow[\varepsilon_{C\wedge}1]{} & \Sigma^2 KC\wedge X
\end{array}$$

Since ε_{C*}: $[Y,\Sigma^2 KO\wedge X]\to[Y,\Sigma^2 KC\wedge X]$ is a monomorphism by our second as sumption, the composite $(\varepsilon_O\pi\bar{v}^{-1}{}_\wedge 1)\,e_P(f)\,(i_{P\wedge}1)$: $Y\to\Sigma^2 KO\wedge X$ is trivial. So we can find a map h: $Y\to KO\wedge X$ such that $(\varepsilon_{U\wedge}1)h=f$.

In proving our main theorems we shall often use the following result, whose proof is given in [20, Lemma 1.1 and (1.7)].

Lemma 1.3. *Let f: $Y\to KU\wedge X$ be a map satisfying $(t_\wedge 1)f=f$ and k: $W\to Y$ be a map inducing an epimorphism k^*: $[Y,\Sigma^1 KU\wedge X]\to[W,\Sigma^1 KU\wedge X]$. Then there exist maps h_0: $W\to KO\wedge X$ and g: $Y\to KC\wedge X$ making the diagram below commutative*

$$\begin{array}{ccccc}
W & \xrightarrow{k} & Y & & \\
h_0\downarrow & & g\downarrow & \searrow^{f} & \\
KO\wedge X & \xrightarrow[\varepsilon_C\wedge 1]{} & KC\wedge X & \xrightarrow[\zeta_\wedge 1]{} & KU\wedge X
\end{array}$$

if the composite $(\varepsilon_O\pi\bar{v}^{-1}{}_\wedge 1)\,fk$: $W\to\Sigma^2 KO\wedge X$ is trivial, in particular if $(\eta_\wedge 1)_*$: $[W,\Sigma^3 KO\wedge X]\to[W,\Sigma^2 KO\wedge X]$ is trivial.

1.4. Let ∇E denote the Anderson dual spectrum of E (see [4], [5], [9] or [19, *I* and *II*]). The CW-spectra E and ∇E are related by the following universal coefficient sequence

$$0 \to \mathrm{Ext}(E_{*-1}X, Z) \to \nabla E^*X \to \mathrm{Hom}(E_*X, Z) \to 0\,.$$

The Anderson dual spectrum ∇E is just the function spectrum $F(E, \nabla S)$ where ∇S is the Anderson dual of the sphere spectrum S.

We now assume that E is an associative ring spectrum with unit. Note that the Anderson dual ∇E is an associative E-module spectrum [19, *II*]. To any map $f\colon Y \to E \wedge X$ we may assign the E-module map $\kappa_E(f)^*\colon F(X, \nabla E) \to F(Y, \nabla E)$ where $F(W, \nabla E) = F(W, F(E, \nabla S)) = F(E \wedge W, \nabla S)$. Evidently it follows that

(1.16) *the E-module map* $\kappa_E(f)^*$ *is an equivalence whenever* $f\colon Y \to E \wedge X$ *is a quasi* E_**-equivalence.*

For any CW-spectra X, Y we say that X is *quasi* E^**-equivalent to* Y if there exists an E-module map $g\colon F(X, E) \to F(Y, E)$ which is an equivalence. Recall that $\nabla KU = KU$ as KU-module spectra, $\nabla KO = \Sigma^4 KO$ as KO-module spectra and also $\nabla KC = \Sigma^1 KC$ as KC-module spectra (see [4] or [19, *I*]). Then we obtain

Proposition 1.4. *Let E denote the K-spectrum KU, KO or KC. If X is quasi* E_**-equivalent to Y, then X is quasi* E^**-equivalent to Y.*

Proof. If a map $f\colon Y \to E \wedge X$ is a quasi E_*-equivalence, then the E-module map $f^*\colon F(X, E) \to F(Y, E)$ induced by f is an equivalence because we may replace E with ∇E in this case.

A CW-spectrum W is said to be of finite type if $\pi_i W$ is finitely generated for each i. Notice that $E \wedge W = \nabla\nabla(E \wedge W) = F(F(W, \nabla E), \nabla S)$ if $E \wedge W$ is of finite type (see [19, *I*] or [5]). Then we obtain

Proposition 1.5. *Let E denote the K-spectrum KU, KO or KC. Assume that both* $E \wedge X$ *and* $E \wedge Y$ *are of finite type. Then X is quasi* E_**-equivalent to Y if and only if X is quasi* E^**-equivalent to Y.*

Proof. We have only to prove the "if" part. Let $g\colon F(X, E) \to F(Y, E)$ be an E-module equivalence. Under the finiteness assumption on $E \wedge X$ and $E \wedge Y$ we get an E-module map $g^*\colon E \wedge Y \to E \wedge X$ which is also an equivalence, by replacing E with ∇E.

For the Spanier-Whitehead dual spectrum $DW = F(W, S)$ there exists an equivalence $\delta\colon DW \wedge E \to F(W, E)$ if W is finite. Note that the equivalence δ is an E-module map when E is an associative ring spectrum. As is easily seen, we have

Corollary 1.6. *Let E denote the K-spectrum KU, KO or KC. Assume that X and Y are finite CW-spectra. Then X is quasi E_*-equivalent to Y if and only if DY is quasi E_*-equivalent to DX.*

2. Wood spectra

2.1. Let H be a finitely generated abelian group which is 2-torsion free. Assume that the cyclic group $Z/2$ of order 2 acts on H. Thus the abelian group H possesses an automorphism $\rho: H\rightarrow H$ with $\rho^2=1$. By applying the integral representation theory of the cyclic group $Z/2$ [7] we observe that H has a direct sum decomposition $H\cong A\oplus B\oplus C\oplus C$ with C free, on which the $Z/2$-action ρ behaves as follows:

$$(2.1)\qquad \rho=1 \text{ on } A\,,\quad \rho=-1 \text{ on } B \quad\text{and}\quad \rho=\begin{pmatrix}0&1\\1&0\end{pmatrix} \text{ on } C\oplus C\,.$$

The conjugation $t: KU\rightarrow KU$ gives rise to a $Z/2$-action t_* on KU_*X for any CW-spectrum X. We first deal with a CW-spectrum X such that KU_0X and KU_1X are decomposed into the forms $KU_0X\cong A\oplus B\oplus C\oplus C$ and $KU_1X\cong D\oplus E\oplus F\oplus F$ respectively, on which the conjuagtion t_* behaves as follows:

$$(2.2)\qquad \begin{aligned} &t_*=1 \text{ on } A \text{ or } D\,,\quad t_*=-1 \text{ on } B \text{ or } E\,,\quad\text{and}\\ &t_*=\begin{pmatrix}0&1\\1&0\end{pmatrix} \text{ on } C\oplus C \text{ or } F\oplus F\,.\end{aligned}$$

For such a CW-spectrum X we will study K-homologies KC_*X and KO_*X.

Lemma 2.1. i) *There are short exact sequences*

$$\begin{aligned}
&0\rightarrow D\oplus(E\otimes Z/2)\oplus F\rightarrow KC_0X\rightarrow A\oplus(B*Z/2)\oplus C\rightarrow 0\\
&0\rightarrow (A\otimes Z/2)\oplus B\oplus C\rightarrow KC_1X\rightarrow D\oplus(E*Z/2)\oplus F\rightarrow 0\\
&0\rightarrow (D\otimes Z/2)\oplus E\oplus F\rightarrow KC_2X\rightarrow (A*Z/2)\oplus B\oplus C\rightarrow 0\\
&0\rightarrow A\oplus(B\otimes Z/2)\oplus C\rightarrow KC_3X\rightarrow (D*Z/2)\oplus E\oplus F\rightarrow 0\,.
\end{aligned}$$

ii) *$KO_iX\otimes Z[1/2]\cong(A\oplus C)\otimes Z[1/2]$, $(D\oplus F)\otimes Z[1/2]$, $(B\oplus C)\otimes Z[1/2]$ or $(E\oplus F)\otimes Z[1/2]$ corresponding to $i\equiv 0, 1, 2$ or 3 mod 4.*
iii) *If KU_iX is 2-torsion free, then the 2-torsion subgroup $KO_iX*Z/2^\infty$ of KO_iX is a $Z/2$-module.*

Proof. i) Use the long exact sequence induced by the cofiber sequence (1.2).

ii) Use the exact sequence $0\rightarrow KO_iX\otimes Z[1/2]\rightarrow KU_iX\otimes Z[1/2]\rightarrow KU_{i-2}X\otimes Z[1/2]\rightarrow KO_{i-4}X\otimes Z[1/2]\rightarrow 0$ induced by the cofiber sequence (1.1).

iii) Under the 2-torsion freeness assumption on KU_iX, the complexification $\varepsilon_{U*}: KO_iX\rightarrow KU_iX$ restricted to the 2-torsion subgroup $KO_iX*Z/2^\infty$ is trivial. Then it follows that $2(KO_iX*Z/2^\infty)=0$ because $\varepsilon_O\varepsilon_U=2$.

Lemma 2.2. *Assume that* $KU_1X=0$. *Then*
i) $KO_1X\oplus KO_5X\cong(A\otimes Z/2)\oplus(B*Z/2)$ *and*
$KO_3X\oplus KO_7X\cong(A*Z/2)\oplus(B\otimes Z/2)$.
ii) $0\to A\oplus(B\otimes Z/2)\oplus C\to KO_0X\oplus KO_4X\to A\oplus(B*Z/2)\oplus C\to 0$
$0\to(A\otimes Z/2)\oplus B\oplus C\to KO_2X\oplus KO_6X\to(A*Z/2)\oplus B\oplus C\to 0$
are short exact sequences.

Proof. Consider the exact sequences

$$0\to KC_3X\to KO_4X\oplus KO_0X\to KU_4X\xrightarrow{\varphi_2}KC_2X\to KO_3X\oplus KO_7X\to 0$$

$$0\to KC_1X\to KO_2X\oplus KO_6X\to KU_2X\xrightarrow{\varphi_0}KC_0X\to KO_1X\oplus KO_5X\to 0$$

induced by the cofiber sequence (1.5). Here the homomorphisms φ_2: $A\oplus B\oplus C\oplus C\to(A*Z/2)\oplus B\oplus C$ and φ_0: $A\oplus B\oplus C\oplus C\to A\oplus(B*Z/2)\oplus C$ induced by the map $\varepsilon_C\varepsilon_O\pi_U^{-1}$: $KU\to\Sigma^2KC$, are respectively expressed as $\varphi_2(a,b,c_1,c_2)=(0,2b,c_1-c_2)$ and $\varphi_0(a,b,c_1,c_2)=(2a,0,c_1+c_2)$ because $\zeta\varepsilon_C\varepsilon_O\pi_U^{-1}=\pi_U^{-1}(1-t)$. The result is now immediate.

2.2. We here deal with a CW-spectrum X such that KU_0X is finitely generated, 2-torsion free and $KU_1X=0$. In this case KU_0X has a direct sum decomposition $KU_0X\cong A\oplus B\oplus C\oplus C$ with C free, on which the conjugation t_* behaves as (2.2).

Proposition 2.3. *There are direct sum decompositions* $A\cong A'\oplus A''$ *and* $B\cong B'\oplus B''$ *with* A'', B'' *free, so that* $KO_*X\cong(KO_*\otimes A')\oplus(KO_{*-2}\otimes B')\oplus(KO_{*-4}\otimes A'')\oplus(KO_{*-6}B'')\oplus(KU_*\otimes C)$ *as* KO_**-modules.*

Proof. Consider the exact sequences $KU_{2i+2}X\to KC_{2i}X\xrightarrow{\psi_{2i}}KO_{2i+1}X\oplus KO_{2i+5}X\to 0$ induced by the cofiber sequence (1.5). Set $KO_1X=A_1$, $KO_5X=A_5$, $KO_3X=B_3$ and $KO_7X=B_7$, all of which are $Z/2$-modules by Lemma 2.2 i). Since A and B are both 2-torsion free, we can choose direct sum decompositions $KC_0X\cong A'\oplus A''\oplus C$ and $KC_2X\cong B'\oplus B''\oplus C$ so that $A'\otimes Z/2\cong A_1$, $A''\otimes Z/2\cong A_5$, $B'\otimes Z/2\cong B_3$ and $B''\otimes Z/2\cong B_7$, and moreover ψ_0, ψ_2 are both the canonical epimorphisms (use [11, §20]). Here A'', B'' may be taken to be free.

The commutative diagram (1.4) gives rise to the following diagram

$$\begin{array}{ccccccccccc}
 & & KO_{2i-2}X & \to & KO_{2i}X & \to & KC_{2i}X & \to & KO_{2i-3}X & \to & 0\\
 & & \downarrow & & \| & & \downarrow & & \downarrow & & \\
0 & \to & KO_{2i-1}X & \to & KO_{2i}X & \to & KU_{2i}X & \to & KO_{2i-2}X & \to & KO_{2i-1}X \to 0
\end{array}$$

with exact rows. Denote by L_{2i} the cokernel of η_*: $KO_{2i-1}X\to KO_{2i}X$. It is just the kernel of $(\tau\pi_{C}^{-1})_*\colon KC_{2i}X\to KO_{2i-3}X$. Since the homomorphism ψ_{2i} is induced by the pair $(-\tau, \tau\pi_{C}^{-1})\colon \Sigma^1KC\to KO\vee\Sigma^4KO$, we observe that $L_{2i}\cong KC_{2i}X$, and the inclusions $l_{2i}\colon L_{2i}\to KC_{2i}X$ are expressed as $l_0(a_1, a_2, c)=(a_1, 2a_2, c)$, $l_4(a_1, a_2, c)=(2a_1, a_2, c)$ for any $(a_1, a_2, c)\in A'\oplus A''\oplus C$, and so on.

In order to determine the KO_*-module structure of KO_*X we will describe explicitly the complexification $\varepsilon_{U*}=\varepsilon_{2i}\colon KO_{2i}X\to KU_{2i}X$, admitting a factorization $KO_{2i}X\to L_{2i}\to KC_{2i}X\to KU_{2i}X$. Note that $KO_{2i}X\cong L_{2i}\oplus KO_{2i-1}X$. As is easily computed, $\varepsilon_{2i}\colon KO_{2i}X\to KU_{2i}X$ are given by the following homomorphisms:

$$\begin{aligned}
&\varepsilon_0\colon A'\oplus A''\oplus(B''\otimes Z/2)\oplus C \to A'\oplus A''\oplus B\oplus C\oplus C\\
&\varepsilon_2\colon (A'\otimes Z/2)\oplus B'\oplus B''\oplus C \to A\oplus B'\oplus B''\oplus C\oplus C\\
&\varepsilon_4\colon A'\oplus A''\oplus(B'\otimes Z/2)\oplus C \to A'\oplus A''\oplus B\oplus C\oplus C\\
&\varepsilon_6\colon (A''\otimes Z/2)\oplus B'\oplus B''\oplus C \to A\oplus B'\oplus B''\oplus C\oplus C
\end{aligned}$$

defined by $\varepsilon_0(a_1, a_2, b, c)=(a_1, 2a_2, 0, c, c)$, $\varepsilon_2(a, b_1, b_2, c)=(0, b_1, 2b_2, c, -c)$, $\varepsilon_4(a_1, a_2, b, c)=(2a_1, a_2, 0, c, c)$ and $\varepsilon_6(a, b_1, b_2, c)=(0, 2b_1, b_2, c, -c)$.

We moreover investigate the induced homomorphism $\eta_*=\eta_j\colon KO_jX\to KO_{j+1}X$. Obviously η_{2i-1} is the canonical monomorphism. On the other hand, η_{2i} is obtained as the composite $KO_{2i}X\to L_{2i}\to KC_{2i}X\xrightarrow{\cong} KC_{2i+4}X\to KO_{2i+1}X$ because $\eta_\wedge 1=\tau\varepsilon_C\colon \Sigma^1KO\to KO$. Therefore η_{2i} is the canonical epimorphism.

The above investigations about ε_{U*} and η_* show that $KO_*X\cong(KO_*\otimes A')\oplus(KO_{*-2}\otimes B')\oplus(KO_{*-4}\otimes A'')\oplus(KO_{*-6}\otimes B'')\oplus(KU_*\otimes C)$ as KO_*-modules.

2.3. Using the cofiber sequences (1.1), (1.1)$'$ we consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dl] \\
& KU_0\Sigma^0 \arrow[dr, "i_{P*}"] \arrow[dd] & & KO_2P \arrow[dl, "(\pi_U^{-1}\varepsilon_U)_*"'] \arrow[dd] & \\
& & KU_0P \arrow[dl, "\varepsilon_{O*}"] \arrow[dr, "j_{P*}"'] & & \\
& KO_0P \arrow[dl] & & KU_0\Sigma^2 \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

Here both of the two vertical arrows are identified with multiplication by 2 on Z. Evidently $KU_0P\cong KU_0\Sigma^2\oplus KU_0\Sigma^0\cong Z\oplus Z$. Set $(\pi_U^{-1}\varepsilon_U)_*(1)=(2, -n)$ for some integer n. Then $\varepsilon_{O*}(0, 1)=2$ and $\varepsilon_{O*}(1, 0)=n$. Note that n is odd because ε_{O*} is an epimorphism. We may take n to be 1 by replacing suitably the splitting of j_{P*}. Since $\varepsilon_O t=\varepsilon_O$, the conjugation t_* on KU_0P is represented by the matrix $\begin{pmatrix}-1 & 0\\ 1 & 1\end{pmatrix}$ where the matrix behaves as left action on $Z\oplus Z$. Thus

(2.3) $KU_0P \cong KU_0\Sigma^2 \oplus KU_0\Sigma^0 \cong Z \oplus Z$ *on which* $t_* = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$, *and* $KU_1P=0$.

After changing the isomorphism $KU_0P \cong Z \oplus Z$ suitably we obtain

(2.3)′ $KU_0P \cong Z \oplus Z$ *on which* $t_* = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *and* $KU_1P = 0$

because the matrix $\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ is congruent to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We can now prove one of our main results concerning Wood spectra (cf. [20, Theorem 1.6] or [16]).

Theorem 2.4. *Let X be a CW-spectrum such that KU_0X is finitely generated, 2-torsion free and $KU_1X=0$. Then there exist abelian groups A', A'', B', B'' and C so that X is quasi KO_*-equivalent to the wedge sum $SA' \vee \Sigma^2 SB' \vee \Sigma^4 SA'' \vee \Sigma^6 SB'' \vee (P \wedge SC)$.*

Proof. We may write $KU_0X \cong A \oplus B \oplus C \oplus C$ with C free, on which t_* acts as (2.2). By Proposition 2.3 we admit direct sum decompositions $A \cong A' \oplus A''$ and $B \cong B' \oplus B''$ so that $KO_*X \cong (KO_* \otimes A') \oplus (KO_{*-2} \otimes B') \oplus (KO_{*-4} \otimes A'') \oplus (KO_{*-6} \otimes B'') \oplus (KU_* \otimes C)$ as KO_*-modules.

Set $Y = SA' \vee \Sigma^2 SB' \vee \Sigma^4 SA'' \vee \Sigma^6 SB''$, the wedge sum of the Moore spectra. Then we can choose a map $h_Y\colon Y \to KO \wedge X$ whose induced homomorphism $\kappa_{KO}(h_Y)_*\colon KO_*Y \to KO_*X$ is the canonical inclusion, by means of (1.9). Putting $f_Y = (\varepsilon_U \wedge 1)\, h_Y$, its induced homomorphism $\kappa_{KU}(f_Y)_*\colon KU_*Y \to KU_*X$ is of course the canonical inclusion.

We next choose a map $f_P\colon P \wedge SC \to KU \wedge X$ whose induced homomorphism $\kappa_{KU}(f_P)_*\colon KU_*(P \wedge SC) \to KU_*X$ is the canonical inclusion. Because of (1.8) such a map f_P is uniquely chosen, and hence $(t_\wedge 1) f_P = f_P$. Note that $\eta_*\colon [P, \Sigma^{i+1} KO \wedge X] \to [P, \Sigma^i KO \wedge X]$ is always trivial as $\eta_\wedge 1 = 3 i_P \nu j_P\colon \Sigma^1 P \to P$ where $\nu\colon \Sigma^3 \to \Sigma^0$ is the stable Hopf map. We may here apply Lemma 1.2 to obtain a map $h_P\colon P \wedge SC \to KO \wedge X$ satisfying $(\varepsilon_U \wedge 1)\, h_P = f_P$.

Set $h = h_Y \vee h_P\colon Y \vee (P \wedge SC) \to KO \wedge X$. Obviously $(\varepsilon_U \wedge 1)\, h\colon Y \vee (P \wedge SC) \to KU \wedge X$ is a quasi KU_*-equivalence. By making use of Proposition 1.1 we can show that the map h is a quasi KO_*-equivalence as desired.

Let CP^n be the complex projective n-space. As is well known, KU_0CP^n is the free abelian group of rank n and $KU_1CP^n = 0$ [1]. So we can apply Theorem 2.4 to show

Corollary 2.5. $CP^n \underset{KO}{\approx} \bigvee_t P$ *or* $\bigvee_t P \vee \Sigma^{2n}$ *according as* $n = 2t$ *or* $2t+1$. (Cf. [10].)

Proof. KO^*CP^n has been computed by Fujii [8, Theorem 2]. So we can determine the additive structure of KO_*CP^n, by applying the universal coefficient sequence $0 \to \mathrm{Ext}(KO^{*+5}X, Z) \to KO_*X \to \mathrm{Hom}(KO^{*+4}X, Z) \to 0$ for any finite CW-spectrum X. Then the result follows immediately from Theorem 2.4.

3. Anderson spectra

3.1. We here deal with a CW-spectrum X such that $KU_0X \cong A$ and $KU_1X \cong D$ are finitely generated, 2-torsion free and $t_*=1$ on both KU_0X and KU_1X. Then it follows from [20, Lemma 1.9] that

(3.1) i) *KO_iX is 2-torsion free for each $i \equiv 0$ mod 4, and*
ii) *KO_jX is a $Z/2$-module for each $j \equiv 2, 3$ mod 4.*

We will first calculate K-homologies KC_*X and KO_*X by means of Lemma 2.1 and (3.1).

Lemma 3.1. i) *$KC_iX \cong A \oplus D$, $(A \otimes Z/2) \oplus D$, $D \otimes Z/2$ or A corresponding to $i \equiv 0, 1, 2$ or 3 mod 4.*
ii) *$KO_iX \cong A$, $A_i \oplus D$, $A_{i-1} \oplus D_{i+1} \oplus G_0$ or D_i for some $Z/2$-modules A_1, A_5, D_3, D_7 and G_0, corresponding to $i \equiv 0, 1, 2$ or 3 mod 4. Here these $Z/2$-modules hold the relations $A_1 \oplus A_5 \oplus G_0 \cong A \otimes Z/2$ and $D_3 \oplus D_7 \oplus G_0 \cong D \otimes Z/2$.*

Proof. i) Consider the short exact sequence $0 \to KU_{-1}X \to KC_0X \to KU_0X \to 0$ induced by the cofiber sequence (1.2). This sequence splits if tensored with $Z[1/2]$, since $\varepsilon_U = \zeta\varepsilon_C$ and $\varepsilon_{U*}: KO_0X \otimes Z[1/2] \to KU_0X \otimes Z[1/2]$ becomes an isomorphism by (3.1) ii). So we observe that this sequence remains split even if not tensored with $Z[1/2]$, because it is a pure exact sequence. Thus $KC_0X \cong A \oplus D$. The other cases when $i \not\equiv 0$ mod 4 are immediate from Lemma 2.1 i).

ii) The $i \not\equiv 2$ mod 4 cases follow immediately from Lemma 2.1 ii), iii) and (3.1).

To show the remainders we first consider the two exact sequences

$$KC_4X \xrightarrow{\varphi_1} KU_1X \xrightarrow{\psi_1} KO_3X \oplus KO_7X \to 0$$

$$0 \to KC_3X \xrightarrow{\varphi_0} KU_0X \xrightarrow{\psi_0} KO_2X \oplus KO_6X \to KC_2X \to 0$$

induced by the cofiber sequence (1.6). The former gives rise to an epimorphism $D \otimes Z/2 \to KO_3X \oplus KO_7X$, and the latter a short exact sequence $0 \to A \otimes Z/2 \to KO_2X \oplus KO_6X \to D \otimes Z/2 \to 0$ since $\varphi_0: A \to A$ is just multiplication by 2. Thus $KO_3X \oplus KO_7X \oplus G_0 \cong D \otimes Z/2$ for some $Z/2$-module G_0, and $KO_2X \oplus KO_6X \cong (A \oplus D) \otimes Z/2$.

Let j be a fixed integer with $j \equiv 1$ mod 4. Combine the two exact sequences $0 \to KU_jX \to KO_jX \to KO_{j+1}X \to 0$ and $KO_jX \to KU_jX \to KO_{j-2}X \to 0$ induced by the cofiber sequence (1.1). Then we get a short exact sequence $0 \to KO_{j-2}X$

$\to KO_jX\otimes Z/2\to KO_{j+1}X\to 0$ because $\varepsilon_0\varepsilon_U=2$. Thus $KO_{j-2}X\oplus KO_{j+1}X\cong A_j\oplus(D\otimes Z/2)$ with $A_j=KO_jX*Z/2^\infty$ the 2-torsion subgroup of KO_jX. On the other hand, the cofiber sequence (1.3) gives an exact sequence $KO_{j+1}X\to KC_{j+1}X\to KO_{j-2}X\to 0$. Therefore we get immediately that $KO_{j+1}X\cong A_j\oplus D_{j+2}\oplus G_0$, since $KC_{j+1}X\cong D\otimes Z/2\cong D_3\oplus D_7\oplus G_0$ where $D_3=KO_3X$ and $D_7=KO_7X$. Then it is easily verified that $A_1\oplus A_5\oplus G_0\cong A\otimes Z/2$ because $KO_2X\oplus KO_6X\cong(A\oplus D)\otimes Z/2$.

We again consider the exact sequences

$$KC_4X\xrightarrow{\varphi_1}KU_1X\xrightarrow{\psi_1}KO_3X\oplus KO_7X\to 0$$

$$0\to KC_3X\xrightarrow{\varphi_0}KU_0X\xrightarrow{\psi_0}KO_2X\oplus KO_6X\to KC_2X\to 0\,.$$

As is easily seen, KU_0X and KU_1X admit direct sum decompositions such that ψ_0 and ψ_1 are given as the canonical morphisms (use [11]). Thus they are written into the forms $KU_0X\cong A'\oplus A''\oplus G$ and $KU_1X\cong D'\oplus D''\oplus G$ so that $A'\otimes Z/2\cong A_1$, $A''\otimes Z/2\cong A_5$, $D'\otimes Z/2\cong D_3$, $D''\otimes Z/2\cong D_7$ and $G\otimes Z/2\cong G_0$ where A'', D'' and G are taken to be free. Besides

$\psi_0\colon A'\oplus A''\oplus G\to A_1\oplus D_3\oplus G_0\oplus A_5\oplus D_7\oplus G_0$ and $\psi_1\colon D'\oplus D''\oplus G\to D_3\oplus D_7$

are expressed as

(3.2) $\psi_0(a_1,a_2,g)=([a_1],0,[g],[a_2],0,[g])$ and $\psi_1(d_1,d_2,g)=([d_1],[d_2])$

where [ ] stands for the mod 2 reduction.

Hence Lemma 3.1 says that

(3.3) *KO_*X is decomposed as an abelian group into the direct sum $(KO_*\otimes A')\oplus(KO_{*-1}\otimes D')\oplus(KO_{*-4}\otimes A'')\oplus(KO_{*-5}\otimes D'')\oplus(KC_{*-1}\otimes G)$ for some abelian groups A', A'', D', D'' and G.*

3.2. Let X be a CW-spectrum such that KU_0X and KU_1X are finitely generated, 2-torsion free. Assume that $t_*=1$ on both KU_0X and KU_1X. By studying the KO_*-module structure of KO_*X as in Proposition 2.3 we will show

Proposition 3.2. *There are direct sum decompositions $KU_0X\cong A'\oplus A''\oplus G$ and $KU_1X\cong D'\oplus D''\oplus G$ with A'', D'' and G free, so that $KO_*X\cong(KO_*\otimes A')\oplus(KO_{*-1}\otimes D')\oplus(KO_{*-4}\otimes A'')\oplus(KO_{*-5}\otimes D'')\oplus(KC_{*-1}\otimes G)$ as KO_*-modules.*

Proof. In order to determine the KO_*-module structure of KO_*X, we will describe explicitly the complexification $\varepsilon_{U*}=\varepsilon_i\colon KO_iX\to KU_iX$ and the induced homomorphism $\eta_*=\eta_i\colon KO_iX\to KO_{i+1}X$. It is sufficient to show that

$$\varepsilon_0\colon A'\oplus A''\oplus G \to A'\oplus A''\oplus G \qquad \varepsilon_4\colon A'\oplus A''\oplus G \to A'\oplus A''\oplus G$$
$$\varepsilon_1\colon A_1\oplus D'\oplus D''\oplus G \to D'\oplus D''\oplus G \quad \varepsilon_5\colon A_5\oplus D'\oplus D''\oplus G \to D'\oplus D''\oplus G$$

are given by $\varepsilon_0(a_1, a_2, g)=(a_1, 2a_2, 2g)$, $\varepsilon_4(a_1, a_2, g)=(2a_1, a_2, 2g)$, $\varepsilon_1([a_1], d_1, d_2, g)=(d_1, 2d_2, g)$ and $\varepsilon_5([a_2], d_1, d_2, g)=(2d_1, d_2, g)$, and moreover

$$\eta_0\colon A'\oplus A''\oplus G \to A_1\oplus D \qquad \eta_4\colon A'\oplus A''\oplus G \to A_5\oplus D$$

are given by $\eta_0(a_1, a_2, g)=([a_1], 0)$, $\eta_4(a_1, a_2, g)=([a_2], 0)$ and also η_i the canonical epimorphisms when $i\equiv 1, 2 \bmod 4$.

Let j be a fixed integer with $j\equiv 1 \bmod 4$ as in the proof of Lemma 3.1. Recall (3.2) that $\psi_1\colon KU_1X\to KO_3X\oplus KO_7X$ is given as the canonical epimorphism $D'\oplus D''\oplus G\to D_3\oplus D_7$. Then $\varepsilon_j\colon KO_jX\to KU_jX$ is immediately determined since ψ_1 is induced by $(\varepsilon_O\pi_U, -\varepsilon_O\pi_{\bar{U}}^{1})$. Note that $\varepsilon_{C*}\colon KO_{j+1}X\to KC_{j+1}X$ is given as the canonical morphism $A_j\oplus D_{j+2}\oplus G_0\to D_3\oplus D_7\oplus G_0$, and $\tau_*\colon KC_{j+1}X\to KO_{j+2}X$ as the canonical epimorphism $D_3\oplus D_7\oplus G_0\to D_{j+2}$. Thus $\eta_{j+1}\colon KO_{j+1}X\to KO_jX$ is just the canonical epimorphism because $\eta_\wedge 1=\tau\varepsilon_C$.

We next use the exact sequences $0\to KO_{j+3}X\xrightarrow{\varepsilon_{j+3}}KU_{j+3}X\to KO_{j+1}X\xrightarrow{\eta_{j+1}}KO_{j+2}X\to 0$, $0\to KU_jX\xrightarrow{e_j}KO_jX\xrightarrow{\eta_j}KO_{j+1}X\to 0$ and $0\to KU_{j+1}X\to KO_{j-1}X\xrightarrow{\eta_{j-1}}KO_jX\xrightarrow{\varepsilon_j}KU_jX\to KO_{j-2}X\to 0$. Then ε_{j+3} and η_{j-1} are easily determined by means of η_{j+1} and ε_j respectively. Moreover it follows that η_j is the canonical epimorphism since $e_j\varepsilon_j$ is multiplication by 2 on KO_jX.

These investigations imply that $KO_*X\cong(KO_*\otimes A')\oplus(KO_{*-1}\otimes D')\oplus(KO_{*-4}\otimes A'')\oplus(KO_{*-5}\otimes D'')\oplus(KC_{*-1}\otimes G)$ as KO_*-modules.

3.3. Making use of the cofiber sequence (1.3)′ we see immediately

(3.4) *$KU_0\Sigma^1Q\cong Z$ and $KU_1\Sigma^1Q\cong Z$, on both of which $t_*=1$.*

Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dl] \\
& KO_3Q \arrow[dr, "\varepsilon_{C*}"] \arrow[dd] & & KU_2Q \arrow[dl] \arrow[dd] & \\
& & KC_3Q \arrow[dl, "\zeta_*"] \arrow[dr] & & \\
& KU_3Q \arrow[dl] & & KO_0Q \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

induced by the cofiber sequences (1.2) and (1.3). Here both of the vertical arrows are identified with multiplication by 2 on Z. Evidently $KC_3Q\cong KU_3Q\oplus KU_2Q\cong Z\oplus Z$, and then $\varepsilon_{C*}(1)=(2, 2m+1)$ for some integer m. We may take m to be 0 by replacing suitably the splitting of ζ_*. Thus

(3.5) ε_{C*}: $KO_3Q \to KC_3Q$ *is represented by the row* $(2\ 1)$: $Z \to Z \oplus Z$.

Let X be a CW-spectrum as in Proposition 3.2. Choose a map f: $\Sigma^1 Q \wedge SG \to KU \wedge X$ whose induced homomorphism $\kappa_{KU}(f)_*$: $KU_{*-1}(Q \wedge SG) \to KU_* X$ is the canonical inclusion. By means of (1.8) we note that such a map f is uniquely chosen, and hence $(t_\wedge 1) f = f$. Then there exists a map g: $\Sigma^1 Q \wedge SG \to KC \wedge X$ satisfying $(\zeta_\wedge 1) g = f$. The diagram (1.14) gives a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & KU_2(Q \wedge SG) & \to & KC_3(Q \wedge SG) & \to & KU_3(Q \wedge SG) & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & KU_3 X & \to & KC_4 X & \to & KU_4 X & \to & 0\,. \end{array}$$

The two rows are split exact sequences by Lemma 3.1 i), so $KC_3(Q \wedge SG) \cong KU_3(Q_\wedge SG) \oplus KU_2(Q \wedge SG)$ and $KC_4 X \cong KU_4 X \oplus KU_3 X$. The central arrow $\kappa_{KC}(g)_*$: $KC_3(Q \wedge SG) \to KC_4 X$ is represented by the matrix $\begin{pmatrix} 0 & 0 & 1 & u & v & w \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$: $G \oplus G \to A' \oplus A'' \oplus G \oplus D' \oplus D'' \oplus G$ for some homomorphisms u, v and w. Combine this expression with (3.5) to obtain

(3.6) $\kappa_{KC}(g)_* \varepsilon_{C*}$: $KO_3(Q \wedge SG) \to KC_4 X$ *is represented by the row* $(0\ 0\ 2\ 2u\ 2v\ 2w+1)$: $G \to A' \oplus A'' \oplus G \oplus D' \oplus D'' \oplus G$.

Lemma 3.3. $(\tau\pi\bar{c}^1)_* \kappa_{KC}(g)_* \varepsilon_{C*}$: $KO_3(Q \wedge SG) \to KO_1 X$ *is represented by the row* $(0\ 4x\ 2y\ 4z)$: $G \to (A' \otimes Z/2) \oplus D' \oplus D'' \oplus G$ *for some homomorphisms* x, y *and* z.

Proof. Let i_U: $G \to KU_4 X \cong A' \oplus A'' \oplus G$ be the canonical inclusion and i_C: $G \to KC_4 X \cong A' \oplus A'' \oplus G \oplus D' \oplus D'' \oplus G$ the injection into the former G. First we will show that $(\tau\pi\bar{c}^1)_* i_C$: $G \to KO_1 X \cong (A' \otimes Z/2) \oplus D' \oplus D'' \oplus G$ is represented by the row $(0\ 2p\ q\ 2r+1)$ for some homomorphisms p, q and r. Express $(\tau\pi\bar{c}^1)_* i_C$: $G \to (A' \otimes Z/2) \oplus D' \oplus D'' \oplus G$ into a form $([s]\ p'\ q'\ r')$, and then note that $(\eta_\wedge 1) \tau\pi\bar{c}^1 = \varepsilon_O \pi \bar{U}^1 \zeta$ and $\zeta_* i_C = i_U$. Proposition 3.2 asserts that η_*: $KO_1 X \to KO_2 X$ and $(\varepsilon_O \pi \bar{U}^1)_*$: $KU_4 X \to KO_2 X$ are respectively the canonical morphisms η_1: $(A' \otimes Z/2) \oplus D' \oplus D'' \oplus G \to (A' \oplus D' \oplus G) \otimes Z/2$ and e_2: $A' \oplus A'' \oplus G \to (A' \oplus D' \oplus G) \otimes Z/2$ (or see the proof of Proposition 3.2). Since $\eta_1(\tau\pi\bar{c}^1)_* i_C = e_2 i_U$, we then see that $([s]\ [p']\ [r']) = (0\ 0\ [1])$: $G \to (A' \oplus D' \oplus G) \otimes Z/2$ where $[\ \ ]$ denotes the mod 2 reduction. Thus $[s]=0$, $p'=2p$, $q'=q$ and $r'=2r+1$ for some homomorphisms p, q and r.

On the other hand, $\tau\pi\bar{c}^1 \gamma \pi_U = \varepsilon_O \pi \bar{U}^1$ and $(\varepsilon_O \pi \bar{U}^1)_*$: $KU_3 X \to KO_1 X$ is identified with the homomorphism e_1: $D' \oplus D'' \oplus G \to (A' \otimes Z/2) \oplus D' \oplus D'' \oplus G$ defined by $e_1(d_1, d_2, g) = (0, 2d_1, d_2, 2g)$. Combining the above observations with (3.6), we can easily show that $(\tau\pi\bar{c}^1)_* \kappa_{KC}(g)_* \varepsilon_{C*}$: $KO_3(Q \wedge SG) \to KO_4 X$ is expressed as the sum $(0\ 4p\ 2q\ 4r+2) + (0\ 4u\ 2v\ 4w+2)$: $G \to (A' \otimes Z/2) \oplus D' \oplus D'' \oplus G$.

We can now prove another main result concerning Anderson spectra (cf. [20, Theorem 1.7]).

Theorem 3.4. *Let X be a CW-spectrum such that KU_0X and KU_1X are finitely generated, 2-torsion free. Assume that $t_*=1$ on both KU_0X and KU_1X. Then there exist abelian groups A', A'', D', D'' and G so that X is quasi KO_*-equivalent to the wedge sum $SA'\vee\Sigma^1SD'\vee\Sigma^4SA''\vee\Sigma^5SD''\vee(\Sigma^1Q_\wedge SG)$.*

Proof. By Proposition 3.2 we have direct sum decompositions $KU_0X\cong A'\oplus A''\oplus G$ and $KU_1X\cong D'\oplus D''\oplus G$ so that $KO_*X\cong(KO_*\otimes A')\oplus(KO_{*-1}\otimes D')\oplus(KO_{*-4}\otimes A'')\oplus(KO_{*-5}\otimes D'')\oplus(KC_{*-1}\otimes G)$ as KO_*-modules. Here A'', D'' and G may be taken to be free. Set $Y=SA'\vee\Sigma^1SD'\vee\Sigma^4SA''\vee\Sigma^5SD''$, the wedge sum of the Moore spectra, and choose a map $h_Y\colon Y\to KO\wedge X$ whose induced homomorphism $\kappa_{KO}(h_Y)_*\colon KO_*Y\to KO_*X$ is the canonical inclusion. Then the homomorphism $\kappa_{KU}(f_Y)_*\colon KU_*Y\to KU_*X$ induced by the composite $f_Y=(\varepsilon_U{}_\wedge 1)\,h_Y$ is the canonical inclusion, too.

We next choose a map $f_Q\colon \Sigma^1Q\wedge SG\to KU\wedge X$ whose induced homomorphism $\kappa_{KU}(f_Q)_*\colon KU_{*-1}(Q\wedge SG)\to KU_*X$ is the canonical inclusion. Because of (1.8) it is obvious that $(t_\wedge 1)f_Q=f_Q$. First we will find vertical arrows g, h_0 and h_1 making the diagram below commutative

$$\begin{array}{ccccc}
\Sigma^1SG & \xrightarrow{i_Q{}_\wedge 1} & \Sigma^1Q\wedge SG & \xrightarrow{j_Q{}_\wedge 1} & \Sigma^4SG\\
h_0\downarrow & & \downarrow g & & \downarrow h_1\\
KO\wedge X & \xrightarrow{\varepsilon_C{}_\wedge 1} & KC\wedge X & \xrightarrow{\tau\pi\bar{c}^1{}_\wedge 1} & \Sigma^3KO\wedge X\\
\| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1\\
KO\wedge X & \xrightarrow[\varepsilon_U{}_\wedge 1]{} & KU\wedge X & \xrightarrow[\varepsilon_O\pi\bar{U}^1{}_\wedge 1]{} & \Sigma^2KO\wedge X
\end{array}$$

with $(\zeta_\wedge 1)\,g=f_Q$, where the cofiber sequence (1.3)$'$ and a part of the commutative diagram (1.4) are involved. Consider the composite $f'_Q=(\varepsilon_O\pi\bar{U}^1{}_\wedge 1)f_Q(i_Q{}_\wedge 1)\colon \Sigma^1SG\to\Sigma^2KO\wedge X$. The composite homomorphism $(\varepsilon_O\pi\bar{U}^1)_*\,\kappa_{KU}(f_Q)_*\colon KU_0(Q_\wedge SG)\to KU_1X\to KO_7X$ becomes trivial, since $(\varepsilon_O\pi\bar{U}^1)_*\colon KU_1X\to KO_7X$ is given by the canonical epimorphism $e_7\colon D'\oplus D''\oplus G\to D''\otimes Z/2$. Hence $\kappa_{KO}(f'_Q)_*\colon KO_0SG\to KO_7X$ is trivial. This triviality means that the composite map f'_Q is in fact trivial. So we can apply Lemma 1.3 to obtain the required maps $g\colon \Sigma^1Q\wedge SG\to KC\wedge X$ and $h_0, h_1\colon \Sigma^1SG\to KO\wedge X$.

In order to show that the composite $(\eta_\wedge 1)\,h_1(j_Q{}_\wedge 1)\colon Q\wedge SG\to\Sigma^1KO\wedge X$ becomes trivial, we will find a map $k\colon SG\to KO\wedge X$ satisfying $(\eta^2{}_\wedge 1)\,k=(\eta_\wedge 1)\,h_1$. Consider the commutative square

$$\begin{array}{ccc}
[SG,\Sigma^{-1}KO\wedge X] & \xrightarrow{\tilde{\kappa}} & \mathrm{Hom}(KO_0(SG), KO_1X)\\
(j_Q{}_\wedge 1)^*\downarrow & & \downarrow (j_{Q*})^*\\
[\Sigma^{-3}Q\wedge SG,\Sigma^{-1}KO\wedge X] & \xrightarrow{\tilde{\kappa}} & \mathrm{Hom}(KO_3(Q\wedge SG), KO_1X)
\end{array}$$

in which the arrows $\tilde{\kappa}$ assign to any map f the induced homomorphism $\kappa_{KO}(f)_*$

in dimension 0. Obviously $\tilde{\kappa}(h_1(j_{Q\wedge}1))$ coincides with the composite $(\tau\pi\bar{c}^{1})_*$ $\kappa_{KC}(g)_*\,\varepsilon_{C*}$. Since the right vertical arrow $(j_{Q*})^*$ is just multiplication by 2 on $\mathrm{Hom}(G, KO_1X)$, Lemma 3.2 asserts that $\tilde{\kappa}(h_1)$ is written into the form $([s]\ 2x\ y\ 2z)$: $G\to(A'\otimes Z/2)\oplus D'\oplus D''\oplus G$. Recall that η_*: $KO_1X\to KO_2X$ is the canonical epimorphism η_1: $(A'\otimes Z/2)\oplus D'\oplus D''\oplus G\to(A'\oplus D'\oplus G)\otimes Z/2$. So $\eta_*\kappa(h_1)$: $KO_0(SG)\to KO_2X$ is represented by the row $([s]\ 0\ 0)$: $G\to(A'\oplus D'\oplus G)\otimes Z/2$. On the other hand, η_*^2: $KO_0X\to KO_2X$ is identified with the composite homomorphism $\eta_1\eta_0$: $A'\oplus A''\oplus G\to(A'\oplus D'\oplus G)\otimes Z/2$ defined by $\eta_1\eta_0(a_1, a_2, g)=([a_1], 0, 0)$. Therefore the homomorphism $\tilde{s}=(s\ 0\ 0)$: $G\to A'\oplus A''\oplus G$ satisfies the equality $\eta_*^2\tilde{s}=\eta_*\tilde{\kappa}(h_1)$. This means that there exists a map k: $SG\to KO\wedge X$ with $(\eta^2{}_\wedge 1)\,k=(\eta_\wedge 1)\,h$. Consequently we get a map h_Q: $\Sigma^1Q\wedge SG\to KO\wedge X$ such that $(\varepsilon_{U\wedge}1)\,h_Q=f_Q$, because $\varepsilon_O\pi\bar{v}^1 f_Q=0$.

Set $h=h_Y\vee h_Q$: $Y\vee(\Sigma^1Q\wedge SG)\to KO\wedge X$. It is obvious that $(\varepsilon_{U\wedge}1)\,h$: $Y\vee(\Sigma^1Q\wedge SG)\to KU\wedge X$ is a quasi KU_*-equivalence. So we can apply Proposition 1.1 to show that the map h is a quasi KO_*-equivalence.

4. Some elementary spectra with a few cells

4.1. We first study KU and KO homologies of some elementary spectra with three cells. The Moore spectrum $SZ/2m$ is obtained by the cofiber sequence $\Sigma^0\xrightarrow{2m}\Sigma^0\xrightarrow{i}SZ/2m\xrightarrow{j}\Sigma^1$. Denote by M_{2m}, N_{2m}, P_{2m}, Q_{2m} and R_{2m} respectively the finite CW-spectra constructed by the following cofiber sequences:

$$(4.1)\qquad \begin{array}{ll} \Sigma^1\xrightarrow{i\eta}SZ/2m\to M_{2m}\to\Sigma^2, & \Sigma^2\xrightarrow{i\eta^2}SZ/2m\to N_{2m}\to\Sigma^3 \\ \Sigma^2\xrightarrow{\tilde{\eta}}SZ/2m\to P_{2m}\to\Sigma^3, & \Sigma^3\xrightarrow{\tilde{\eta}\eta}SZ/2m\to Q_{2m}\to\Sigma^4 \\ \Sigma^4\xrightarrow{\tilde{\eta}\eta^2}SZ/2m\to R_{2m}\to\Sigma^5 & \end{array}$$

where $\tilde{\eta}$: $\Sigma^2\to SZ/2m$ is a coextension of η satisfying $j\tilde{\eta}=\eta$.

Dually we denote by M'_{2m}, N'_{2m}, P'_{2m}, Q'_{2m} and R'_{2m} respectively the finite CW-spectra constructed by the following cofiber sequences:

$$(4.2)\qquad \begin{array}{ll} SZ/2m\xrightarrow{\eta j}\Sigma^0\to M'_{2m}\to\Sigma^1SZ/2m, & \Sigma^1SZ/2m\xrightarrow{\eta^2 j}\Sigma^0\to N'_{2m}\to\Sigma^2SZ/2m \\ \Sigma^1SZ/2m\xrightarrow{\bar{\eta}}\Sigma^0\to P'_{2m}\to\Sigma^2SZ/2m, & \Sigma^2SZ/2m\xrightarrow{\eta\bar{\eta}}\Sigma^0\to Q'_{2m}\to\Sigma^3SZ/2m \\ \Sigma^3SZ/2m\xrightarrow{\eta^2\bar{\eta}}\Sigma^0\to R'_{2m}\to\Sigma^4SZ/2m & \end{array}$$

where $\bar{\eta}$: $\Sigma^1SZ/2m\to\Sigma^0$ is an extension of η satisfying $\bar{\eta}i=\eta$.

The Moore spectrum $SZ/2m$ is self-dual in the sense that $DSZ/2m\cong\Sigma^{-1}SZ/2m$ where DX stands for the Spanier-Whitehead dual of X. By means of [17, Theorem 6.10] we obtain that

(4.3) $M'_{2m}=\Sigma^2DM_{2m}$, $N'_{2m}=\Sigma^3DN_{2m}$, $P'_{2m}=\Sigma^3DP_{2m}$, $Q'_{2m}=\Sigma^4DQ_{2m}$ and

$$R'_{2m}=\Sigma^5 DR_{2m}\,.$$

We will first compute the KU homologies of the elementary spectra mentioned above.

Proposition 4.1. *The KU homologies KU_0X, KU_1X and the conjugation t_* on $KU_0X\oplus KU_1X$ are tabled as follows:*

X =	M_{2m}	N_{2m}	P_{2m}	Q_{2m}	R_{2m}
$KU_0X\cong$	$Z\oplus Z/2m$	$Z/2m$	Z/m	$Z\oplus Z/2m$	$Z/2m$
$KU_1X\cong$	0	Z	Z	0	Z
t_* =	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\m&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$
X =	M'_{2m}	N'_{2m}	P'_{2m}	Q'_{2m}	R'_{2m}
$KU_0X\cong$	Z	$Z\oplus Z/2m$	$Z\oplus Z/m$	Z	$Z\oplus Z/2m$
$KU_1X\cong$	$Z/2m$	0	0	$Z/2m$	0
t_* =	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

where the matrices behave as left action on abelian groups.

Proof. We will investigate the behaviour of the conjugation t_* on $KU_0X\oplus KU_1X$ only in the cases when $X=P'_{2m}$ and Q_{2m}. The other cases are easy.

i) The $X=P'_{2m}$ case: Consider the commutative diagram

$$\begin{tikzcd}
 & & \Sigma^2 \arrow[r, equal] \arrow[d, "h_P"'] & \Sigma^2 \arrow[d, "2m"] \\
\Sigma^1 \arrow[r, "\eta"] \arrow[d, "i"'] & \Sigma^0 \arrow[r, "i_P"] \arrow[d, equal] & P \arrow[r] \arrow[d, "k_P"'] & \Sigma^2 \arrow[d, "i"] \\
\Sigma^1 SZ/2m \arrow[r, "\bar\eta"'] & \Sigma^0 \arrow[r] & P'_{2m} \arrow[r] & \Sigma^2 SZ/2m\,.
\end{tikzcd}$$

Recall (2.3) that $KU_0P\cong KU_0\Sigma^2\oplus KU_0\Sigma^0\cong Z\oplus Z$ on which $t_*=\begin{pmatrix}-1&0\\1&1\end{pmatrix}$. The induced homomorphism h_{P*}: $KU_0\Sigma^2\to KU_0P$ is given by $h_{P*}(1)=(2m,-m)$ because $t_*h_{P*}(1)=-h_{P*}(1)$. Hence an easy computation shows that $KU_0P'_{2m}\cong Z\oplus Z/m$, $KU_1P'_{2m}=0$ and the induced homomorphism k_{P*}: $KU_0P\to KU_0P'_{2m}$ is given by $k_{P*}(x,y)=(x+2y,y)$. So we obtain that $t_*=\begin{pmatrix}1&0\\1&-1\end{pmatrix}$ on $KU_0P'_{2m}\cong Z\oplus Z/m$.

ii) The $X=Q_{2m}$ case: We next consider the commutative diagram

$$\begin{tikzcd}
\Sigma^3 \arrow[r, "\eta"] \arrow[d, equal] & \Sigma^2 \arrow[r] \arrow[d, "\tilde\eta"] & \Sigma^2P \arrow[r] \arrow[d, "h_Q"] & \Sigma^4 \arrow[d, equal] \\
\Sigma^3 \arrow[r, "\tilde\eta\eta"'] & SZ/2m \arrow[r, "i_Q"'] \arrow[d] & Q_{2m} \arrow[r] \arrow[d] & \Sigma^4 \\
 & P_{2m} \arrow[r, equal] & P_{2m} &
\end{tikzcd}$$

Evidently $KU_0Q_{2m}\cong KU_0\Sigma^4\oplus KU_0SZ/2m\cong Z\oplus Z/2m$ and $KU_1Q_{2m}=0$. We will use the induced homomorphism $h_{Q*}\colon KU_{-2}P\to KU_0Q_{2m}$ to determine the behavior of t_* on KU_0Q_{2m}. By means of (4.3) we see that $KU_0P_{2m}\cong KU^3P'_{2m}\cong Z/m$. This implies that $\tilde{\eta}_*\colon KU_0\Sigma^2\to KU_0SZ/2m$ is given by $\tilde{\eta}_*(1)=m$. So the induced homomorphism $h_{Q*}\colon KU_{-2}P\to KU_0Q_{2m}$ is expressed as $h_{Q*}(1,0)=(1,n)$ and $h_{Q*}(0,1)=(0,m)$ for some integer n, where $KU_{-2}P\cong KU_0\Sigma^4\oplus KU_0\Sigma^2\cong Z\oplus Z$. Since $t_*i_{Q*}=i_{Q*}$ on $KU_0SZ/2m$ and $t_*h_{Q*}=h_{Q*}\begin{pmatrix}1 & 0\\ -1 & -1\end{pmatrix}$ on $KU_{-2}P$, an easy computation shows that $t_*=\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}$ on $KU_0Q_{2m}\cong Z\oplus Z/2m$.

We will moreover compute the KO homologies of the elementary spectra treated in the above proposition.

Proposition 4.2. *The KO homologies KO_iX are tabled as follows:*

i =	0	1	2	3	4	5	6	7
M_{2m}	$Z/2m$	0	$Z\oplus Z/2$	$Z/2$	$Z/4m$	0	Z	0
N_{2m}	$Z/2m$	$Z/2$	$Z/2$	$Z\oplus Z/2$	$Z/4m$	$Z/2$	0	Z
P_{2m}	$Z/2m$	$Z/2$	$Z/2\otimes Z/m$	Z	Z/m	0	0	Z
Q_{2m}	$Z\oplus Z/2m$	$Z/2$	$(*)_m$	0	$Z\oplus Z/m$	0	$Z/2$	0
R_{2m}	$Z/2m$	$Z\oplus Z/2$	$(*)_m$	$Z/2$	Z/m	Z	$Z/2$	$Z/2$
M'_{2m}	Z	$Z/4m$	$Z/2$	$Z/2$	Z	$Z/2m$	0	0
N'_{2m}	Z	$Z/2$	$Z/4m$	$Z/2$	$Z\oplus Z/2$	$Z/2$	$Z/2m$	0
P'_{2m}	Z	0	Z/m	0	$Z\oplus(Z/2\otimes Z/m)$	$Z/2$	$Z/2m$	0
Q'_{2m}	Z	$Z/2$	0	Z/m	Z	$(*)_m$	$Z/2$	$Z/2m$
R'_{2m}	$Z\oplus Z/2m$	$Z/2$	$Z/2$	0	$Z\oplus Z/m$	$Z/2$	$(*)_m$	$Z/2$

in which $(*)_m$ stands for $Z/4$ if m is odd, but $Z/2\oplus Z/2$ if m is even.

Proof. Use the long exact sequences of KO homologies induced by the cofiber sequences (4.1), (4.2). In computing KO_*X for the latter five spectra X we may apply the universal coefficient sequence $0\to\mathrm{Ext}(KO_{3-*}DX, Z)\to KO_*X\to\mathrm{Hom}(KO_{4-*}DX, Z)\to 0$ combined with (4.3) if necessary.

4.2. We next study the KU and KO homologies of some elementary spectra with four cells. Denote by $S_{2m,2n}$, $T_{2m,2n}$, $V_{2m,2n}$, $V'_{2m,2n}$ and $W_{2m,2n}$ respectively the finite CW-spectra constructed by the following cofiber sequneces:

$$
\begin{aligned}
&SZ/2n \xrightarrow{i\eta j} SZ/2m \to S_{2m,2n} \to \Sigma^1 SZ/2n\\
&\Sigma^1SZ/2n \xrightarrow{i\eta^2 j} SZ/2m \to T_{2m,2n} \to \Sigma^2 SZ/2n\\
(4.4)\qquad &\Sigma^1SZ/2n \xrightarrow{i\tilde{\eta}} SZ/2m \to V_{2m,2n} \to \Sigma^2 SZ/2n\\
&\Sigma^1SZ/2n \xrightarrow{\tilde{\eta} j} SZ/2m \to V'_{2m,2n} \to \Sigma^2 SZ/2n
\end{aligned}
$$

$$\Sigma^1 SZ/2n \xrightarrow{i\bar{\eta}+\tilde{\eta}j} SZ/2m \to W_{2m,2n} \to \Sigma^2 SZ/2n\,.$$

Note that

(4.5) $S_{2m,2n} = \Sigma^2 DS_{2n,2m}$, $T_{2m,2n} = \Sigma^3 DT_{2n,2m}$, $V'_{2m,2n} = \Sigma^3 DV_{2n,2m}$ and $W_{2m,2n} = \Sigma^3 DW_{2n,2m}$.

We first consider the commutative diagram

$$\begin{tikzcd}
 & \Sigma^0 \arrow[r, equal] \arrow[d, "2m"] \arrow[dr, "\bar{i}_P"] & \Sigma^0 \arrow[d, "\bar{h}_P"] & \\
\Sigma^1 SZ/2n \arrow[r, "\bar{\eta}"] \arrow[d, equal] & \Sigma^0 \arrow[r, "i_P"] \arrow[d, "i"] & P'_{2n} \arrow[r] \arrow[d, "\bar{k}_P"] & \Sigma^2 SZ/2n \arrow[d, equal] \\
\Sigma^1 SZ/2n \arrow[r, "i\bar{\eta}"'] & SZ/2m \arrow[r] & V_{2m,2n} \arrow[r] & \Sigma^2 SZ/2n\,.
\end{tikzcd}$$

The map $\bar{i}_P$ has a factorization $\bar{i}_P = k_P i_P$ through P where k_P is the map used in the proof of Proposition 4.1 i). So we see that

(4.6) *the induced homomorphism* $\bar{h}_{P*}\colon KU_0\Sigma^0 \to KU_0P'_{2n}$ *is identified with the homomorphism* $f_{2m,n}\colon Z \to Z\oplus Z/n$ *defined by* $f_{2m,n}(1) = (4m, 2m)$.

We also consider the commutative diagram

$$\begin{tikzcd}
 & & \Sigma^2 \arrow[r, equal] \arrow[d, "h_M"'] & \Sigma^2 \arrow[d, "2n"] \\
\Sigma^1 \arrow[r, "i\eta"] \arrow[d, "i"'] & SZ/2m \arrow[r, "i_M"] \arrow[d, equal] & M_{2m} \arrow[r] \arrow[d, "k_M"'] & \Sigma^2 \arrow[d, "i"] \\
\Sigma^1 SZ/2n \arrow[r, "i\bar{\eta}+\tilde{\eta}j"'] & SZ/2m \arrow[r] & W_{2m,2n} \arrow[r] & \Sigma^2 SZ/2n\,.
\end{tikzcd}$$

Lemma 4.3. *The induced homomorphism* $h_{M*}\colon KU_0\Sigma^2 \to KU_0M_{2m}$ *is identified with the homomorphism* $h_{n,m}\colon Z \to Z\oplus Z/2m$ *defined by* $h_{n,m}(1) = (2n, m-n)$.

Proof. Consider the induced homomorphism $h_{M*} = h_2\colon KO_2\Sigma^2 \to KO_2M_{2m}$. An easy computation shows that $h_2\colon Z \to Z\oplus Z/2$ is expressed as $h_2(1) = (n, q_0)$ for some $q_0 \in Z/2$. We will verify that $q_0 \in Z/2$ is non-trivial. In order to observe the complexification $\varepsilon_{U*} = \varepsilon_2\colon KO_2M_{2m} \to KU_2M_{2m}$ and the realification $\varepsilon_{O*} = e_2\colon KU_2M_{2m} \to KO_2M_{2m}$ we recall that $t\varepsilon_U = \varepsilon_U$, $\varepsilon_U\varepsilon_O = 1+t$ and $t_* = \begin{pmatrix} 1 & 0 \\ -1 & -1 \end{pmatrix}$ on $KU_2M_{2m} \cong Z\oplus Z/2m$. As is easily checked, $\varepsilon_2\colon Z\oplus Z/2 \to Z\oplus Z/2m$ and $e_2\colon Z\oplus Z/2m \to Z\oplus Z/2$ are respectively given by $\varepsilon_2(x, y) = (2x, my-x)$ and $e_2(z, w) = (z, 0)$. We here choose a map $\rho\colon M_{2m} \to \Sigma^1$ satisfying $\rho i_M = j$. Then the composite ρh_M is just the Hopf map $\eta\colon \Sigma^2 \to \Sigma^1$, and hence $\rho_* h_2(1) = 1 \in KO_2\Sigma^1 \cong Z/2$. On the other hand, the composite homomorphism $\rho_* e_2\colon KU_2M_{2m} \to KO_2M_{2m} \to KO_2\Sigma^1$ is evidently trivial. So we see that $\rho_*(0, q_0) = 1$, which means that $q_0 = 1$.

This implies that $\varepsilon_2 h_2(1)=(2n, m-n)$, and hence the result follows immediately.

We will here discuss the homomorphisms $f_{m,n}\colon Z\to Z\oplus Z/n$ and $h_{m,n}\colon Z\to Z\oplus Z/2n$ defined by $f_{m,n}(1)=(2m, m)$ and $h_{m,n}(1)=(2m, n-m)$ respectively. The results (4.7)–(4.15) obtained below will be needed in studying the KU homologies of $V_{2m,2n}$ and $W_{2m,2n}$ later. Let $C_{m,n}$ denote the cokernel of $f_{m,n}$. Thus the sequence

$$0 \to Z \xrightarrow{f_{m,n}} Z\oplus Z/n \xrightarrow{g_{m,n}} C_{m,n} \to 0$$

is exact. Write $m=2^k m'$ and $n=2^l n'$ with m', n' odd.

In the $k\geqq l$ case it follows that

(4.7) $C_{m,n}\cong Z/2m\oplus Z/2^l\oplus Z/n'$, and

(4.8) $g_{m,n}\colon Z\oplus Z/2^l\oplus Z/n'\to Z/2m\oplus Z/2^l\oplus Z/n'$ is given by $g_{m,n}(x, y_1, y_2)=(x, y_1, x-2y_2)$. In particular, $g_{m,n}(1, 0, \frac{n'+1}{2}) = (1, 0, 0)$, $g_{m,n}(0, 1, 0) = (0, 1, 0)$ and $g_{m,n}(0, 0, \frac{n'-1}{2}) = (0, 0, 1)$.

On the other hand, in the $k\leqq l$ case it follows that

(4.9) $C_{m,n}\cong Z/2n\oplus Z/2^k\oplus Z/m'$, and

(4.10) $g_{m,n}\colon Z\oplus Z/n\to Z/2n\oplus Z/2^k\oplus Z/m'$ is given by $g_{m,n}(x, y) = (2y-x, y, \frac{(1+m')x}{2})$. In particular, $g_{m,n}(-m'a, 2^k b) = (1, 0, 0)$, $g_{m,n}(2m'a, m'a) = (0, 1, 0)$ and $g_{m,n}(2^{k+2}b, 2^{k+1}b) = (0, 0, 1)$ for some integers a, b with $m'a+2^{k+1}b = 1$.

Denote by $D_{m,n}$ the cokernel of $h_{m,n}\colon Z\to Z\oplus Z/2n$. Obviously $2h_{m,n}=s_{2n}f_{2m,2n}$ where $s_{2n}\colon Z\oplus Z/2n\to Z\oplus Z/2n$ denotes the automorphism defined by $s_{2n}(x, y)=(x, -y)$. So there exists a short exact sequence

$$0 \to Z/2 \xrightarrow{c_{m,n}} C_{2m,2n} \xrightarrow{d_{m,n}} D_{m,n} \to 0\,.$$

Here the connecting homomorphism $c_{m,n}$ is obtained as $c_{m,n}(1)=g_{2m,2n}\, s_{2n}\, h_{m,n}(1)$. In place of $c_{m,n}$ we write with emphasis $c'_{m,n}$ when $k\geqq l$ and $c''_{m,n}$ when $k\leqq l$.

The connecting homomorphism $c'_{m,n}\colon Z/2\to Z/4m\oplus Z/2^{l+1}\oplus Z/n'$ is expressed as $c'_{m,n}(1)=(2m, m-n, 0)$. Thus $c'_{m,n}(1)=(2m, n, 0)$ if $k>l$, and $c'_{m,n}(1)=(2m, 0, 0)$ if $k=l$. In the $k>l$ case it follows that

(4.11) $D_{m,n}\cong Z/2^{k+2}\oplus Z/2^l\oplus Z/m'\oplus Z/n'$, and

(4.12) $d_{m,n}\colon Z/4m\oplus Z/2^{l+1}\oplus Z/n'\to Z/2^{k+2}\oplus Z/2^l\oplus Z/m'\oplus Z/n'$ is given by $d_{m,n}(u, v, w) = (u-2^{k+1-l}v, v, u, w)$. In particular, $d_{m,n}(m'a, 0, 0) = (1, 0, 0, 0)$, $d_{m,n}(2^{k+1-l}m'a, m'a, 0) = (0, 1, 0, 0)$, $d_{m,n}(2^{k+2}b, 0, 0) = (0, 0, 1, 0)$ and $d_{m,n}(0, 0, 1) =(0, 0, 0, 1)$ for some integers a, b with $m'a+2^{k+2}b = 1$.

Moreover, in the $k=l$ case it follows that

(4.13) $D_{m,n}\cong Z/2m\oplus Z/2^{l+1}\oplus Z/n'$, and

(4.14) $d_{m,n}\colon Z/4m\oplus Z/2^{l+1}\oplus Z/n'\to Z/2m\oplus Z/2^{l+1}\oplus Z/n'$ is the canonical epimorphism.

On the other hand, the connecting homomorphism $c''_{m,n}\colon Z/2\to Z/4n\oplus Z/2^{k+1}\oplus Z/m'$ is expressed as $c''_{m,n}(1)=(2n, m-n, 0)$. Thus $c''_{m,n}(1)=(2n, m, 0)$ if $k<l$, and $c''_{m,n}(1)=(2n, 0, 0)$ if $k=l$. This means that

(4.15) $$c''_{m,n}=c'_{n,m} \quad \text{in the} \quad k\leqq l \quad \text{case}.$$

4.3. Using the results discussed in 4.2 we will compute the KU homologies of the elementary spectra with four cells given in 4.2.

Proposition 4.4. *Let $m=2^k m'$ and $n=2^l n'$ with m', n' odd. The KU homologies KU_0X, KU_1X and the conjugation t_* on $KU_0X\oplus KU_1X$ are tabled as follows:*

$X\ =$	$S_{2m,2n}$	$T_{2m,2n}$	$V_{2m,2n}$	
			$k+1\geqq l$	$k+1\leqq l$
$KU_0X\cong$	$Z/2m$	$Z/2m\oplus Z/2n$	$Z/4m\oplus Z/n$	$Z/2m\oplus Z/2n$
$KU_1X\cong$	$Z/2n$	0	0	0
$t_*\ =$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\n'&-1\end{pmatrix}$	$\begin{pmatrix}1&m'\\0&-1\end{pmatrix}$

$X\ =$	$V'_{2m,2n}$		$W_{2m,2n}$		
	$k\leqq l+1$	$k\geqq l+1$	$k<l$	$k=l$	$k>l$
$KU_0X\cong$	$Z/m\oplus Z/4n$	$Z/2m\oplus Z/2n$	$Z/m\oplus Z/4n$	$Z/2m\oplus Z/2n$	$Z/4m\oplus Z/n$
$KU_1X\cong$	0	0	0	0	0
$t_*\ =$	$\begin{pmatrix}1&0\\2^{l+2-k}n'&-1\end{pmatrix}$	$\begin{pmatrix}1&2^{k-l}m'\\0&-1\end{pmatrix}$	${}^tA_{l-k}$	$\begin{pmatrix}1&0\\n'&-1\end{pmatrix}$	A_{k-l}

Here $A_i=\begin{pmatrix}a_i&1-a_i^2&0&0\\1&-a_i&0&0\\0&0&1&0\\0&0&0&-1\end{pmatrix}$ *with $a_i=1-2^{i+1}$. The matrix A_{k-l} acts on $Z/2^{k+2}\oplus Z/2^l\oplus Z/m'\oplus Z/n'$ and the transposed matrix ${}^tA_{l-k}$ acts on $Z/2^k\oplus Z/2^{l+2}\oplus Z/m'\oplus Z/n'$.*

Proof. i) The $X=S_{2m,2n}$, $T_{2m,2n}$ cases are easy.

ii) The $X=V_{2m,2n}$ case: From (4.6) it follows that $KU_0V_{2m,2n}\cong C_{2m,n}$ and $KU_1V_{2m,2n}=0$ where $C_{2m,n}$ denotes the cokernel of $f_{2m,n}$. Thus $KU_0V_{2m,2n}\cong Z/4m\oplus Z/2^l\oplus Z/n'$ or $Z/2n\oplus Z/2^{k+1}\oplus Z/m'$ according as $k+1\geqq l$ or $k+1\leqq l$, as is shown by (4.7) and (4.9).

The induced homomorphism $\bar{k}_{P*}\colon KU_0P'_{2m}\to KU_0V_{2m,2n}$ is written as the homomorphism $g_{2m,n}$ given in (4.8) and (4.10). To investigate the behaviour of the conjugation t_* on $KU_0V_{2m,2n}$ we recall that $t_*=\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$ on $KU_0P'_{2n}\cong Z\oplus Z/n$. By making use of (4.8) and (4.10) we can easily observe that $t_*=\begin{pmatrix}1 & 0 & 0\\ 1 & -1 & 0\\ 0 & 0 & -1\end{pmatrix}$ on $KU_0V_{2m,2n}\cong Z/4m\oplus Z/2^l\oplus Z/n'$ if $k+1\geqq l$, and $t_*=\begin{pmatrix}-1 & 0 & 0\\ -1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$ on $KU_0V_{2m,2n}\cong Z/2n\oplus Z/2^{k+1}\oplus Z/m'$ if $k+1\leqq l$. Note that the latter matrix is congruent to $\begin{pmatrix}-1 & 0 & 0\\ 1 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$. Then the result is immediate.

iii) The $X=V'_{2m,2n}$ case: Consider the commutative diagram

$$\begin{tikzcd}
 & & \Sigma^2 \arrow[r, equal] \arrow[d, "h_V"'] & \Sigma^2 \arrow[d, "i"] \\
\Sigma^1 SZ/2n \arrow[r, "\tilde{\eta} j"] \arrow[d, "j"'] & SZ/2m \arrow[r, "i_V"] \arrow[d, equal] & V'_{2m,2n} \arrow[r, "j_V"] \arrow[d, "k_V"] & \Sigma^2 SZ/2n \arrow[d, "j"] \\
\Sigma^2 \arrow[r, "\tilde{\eta}"] & SZ/2m \arrow[r] & P_{2m} \arrow[r] & \Sigma^3
\end{tikzcd}$$

This gives rise to the following commutative diagram

$$\begin{tikzcd}
0 \arrow[dr] & & & & 0 \arrow[dl] \\
 & KU_0\Sigma^2\otimes Z/4n \arrow[dr] \arrow[dd] & & KU_0SZ/2m \arrow[dl] \arrow[dd] & \\
 & & KU_0V'_{2m,2n} \arrow[dl] \arrow[dr] & & \\
 & KU_0\Sigma^2 SZ/2n \arrow[dl] & & KU_0P_{2m} \arrow[dr] & \\
0 & & & & 0
\end{tikzcd}$$

where the diagonal sequences are exact and the vertical arrows are both epimorphism. By means of the duality (4.5) we get that $KU_0V'_{2m,2n}\cong\mathrm{Ext}(KU_0V_{2n,2m}, Z)$, and hence $KU_0V'_{2m,2n}\cong KU_0P_{2m}\oplus(KU_0\Sigma^2\otimes Z/4n)\cong Z/m\oplus Z/4n$ if $k\leqq l+1$, and $KU_0V'_{2m,2n}\cong KU_0\Sigma^2SZ/2n\oplus KU_0SZ/2m\cong Z/2n\oplus Z/2m$ if $k\geqq l+1$.

We next investigate the behaviour of the conjugation t_* on $KU_0V'_{2m,2n}$. In the $k\leqq l+1$ case we use the short exact sequence $0\to KU_0SZ/2m\xrightarrow{i_{V*}}KU_0V'_{2m,2n}\xrightarrow{j_{V*}}KU_0\Sigma^2SZ/2n\to 0$. Here $i_{V*}\colon Z/2m\to Z/m\oplus Z/4n$ is expressed as $i_{V*}(1)=(1, q_1)$ for some integer q_1. Note that $mq_1\equiv 2n \bmod 4n$. As is easily verified, $t_*=\begin{pmatrix}1 & 0\\ 2q_1 & -1\end{pmatrix}$ on $KU_0V'_{2m,2n}\cong Z/m\oplus Z/4n$, which is congruent to the matrix $\begin{pmatrix}1 & 0\\ 2^{l+2-k}n' & -1\end{pmatrix}$. On the other hand, we use the short exact sequence $0\to KU_0\Sigma^2\otimes Z/4n\xrightarrow{h_{V*}}KU_0V'_{2m,2n}\xrightarrow{k_{V*}}KU_0P_{2m}\to 0$ in the $k\geqq l+1$ case. Here $h_{V*}\colon Z/4n\to Z/2n\oplus Z/2m$ is expressed as $h_{V*}(1)=(1, q_2)$ for some integer q_2 satisfying $2nq_2\equiv m \bmod 2m$. Then $t_*=\begin{pmatrix}-1 & 0\\ 2q_2 & 1\end{pmatrix}$ on $KU_0V'_{2m,2n}\cong Z/2n\oplus Z/2m$, which is also congruent to the matrix $\begin{pmatrix} -1 & 0 \\ 2^{k-l}m' & 1 \end{pmatrix}$. The result is now immediate.

iv) The $X=W_{2m,2n}$ case: Lemma 4.3 implies that $KU_0W_{2m,2n}\cong D_{n,m}$ and $KU_1W_{2m,2n}=0$ where $D_{n,m}$ denotes the cokernel of $h_{n,m}$. Thus (4.11), (4.13) and (4.14) show that $KU_0W_{2m,2n}\cong Z/2^{l+2}\oplus Z/2^k\oplus Z/n'\oplus Z/m'$, $Z/2n\oplus Z/2^{k+1}\oplus Z/m'$ or $Z/2^{k+2}\oplus Z/2^l\oplus Z/m'\oplus Z/n'$ according as $k<l$, $k=l$ or $k>l$.

Note that the induced homomorphism $k_{M*}\colon KU_0M_{2m}\to KU_0W_{2m,2n}$ is written as the composite $d_{n,m}g_{2n,2m}s_{2m}\colon Z\oplus Z/2m\to Z\oplus Z/2m\to C_{2n,2m}\to D_{n,m}$. Recall that $t_*=\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$ on $KU_0M_{2m}\cong Z\oplus Z/2m$. The conjugation t_* on KU_0M_{2m} produces a conjugation $t_{n,m}$ on $C_{2n,2m}$ through the epimorphism $g_{2n,2m}s_{2m}$. In place of $t_{n,m}$ we write with emphasis $t'_{n,m}$ when $k\leqq l$ and $t''_{n,m}$ when $k\geqq l$. In ii) we have implicitly observed that $t'_{n,m}=\begin{pmatrix} -1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ on $C_{2n,2m}\cong Z/4n\oplus Z/2^{k+1}\oplus Z/m'$ and $t''_{n,m}=\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ on $C_{2n,2m}\cong Z/4m\oplus Z/2^{l+1}\oplus Z/n'$.

Use these matrix representations of $t'_{n,m}$ and $t''_{n,m}$, (4.12) and (4.15). Then a routine computation shows that the conjugation t_* on $KU_0W_{2m,2n}$ is represented by the matrix $-A_{l-k}$ or A_{k-l} corresponding to $k<l$ or $k>l$. Here the former matrix $-A_{l-k}$ acts on $Z/2^{l+2}\oplus Z/2^k\oplus Z/n'\oplus Z/m'$ and the latter A_{k-l} acts on $Z/2^{k+2}\oplus Z/2^l\oplus Z/m'\oplus Z/n'$. Since $A_i=\begin{pmatrix} a_i & 1-a_i^2 & 0 & 0 \\ 1 & -a_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ is congruent to $B_i=\begin{pmatrix} a_i & -1+a_i^2 & 0 & 0 \\ -1 & -a_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ with $a_i=1-2^{i+1}$, the result follows in the $k\neq l$ cases. On the other hand, (4.14) says that $d_{n,m}\colon C_{2n,2m}\to D_{n,m}$ is the canonical epimorphism when $k=l$. Therefore the conjugation t_* on $KU_0W_{2m,2n}\cong Z/2m\oplus Z/2^{l+1}\oplus Z/n'$ is represented by the matrix $\begin{pmatrix} 1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, and hence the result is immediate in the $k=l$ case.

4.4. Using the long exact sequences of KO homologies induced by the cofiber sequences (4.4) we can easily compute

Proposition 4.5. *The KO homologies KO_iX are tabled as follows:*

$i\ =$	0	1	2	3	4	5	6	7
$S_{2m,2n}$	$Z/2m$	$Z/4n$	$Z/2\oplus Z/2$	$Z/2\oplus Z/2$	$Z/4m$	$Z/2n$	0	0
$T_{2m,2n}$	$Z/2m$	$Z/2$	$Z/2\oplus Z/4n$	$Z/2\oplus Z/2$	$Z/4m\oplus Z/2$	$Z/2$	$Z/2n$	0
$V_{2m,2n}$	$Z/2m$	0	$Z/2\oplus Z/n$	$Z/2$	$(*)_{m,n}$	$Z/2$	$Z/2n$	0
$V'_{2m,2n}$	$Z/2m$	$Z/2$	$(*)_{n,m}$	$Z/2$	$Z/m\oplus Z/2$	0	$Z/2n$	0
$W_{2m,2n}$	$Z/2m$	0	$Z/2n$	0	$Z/2m$	0	$Z/2n$	0

in which $(*)_{m,n}$ *stands for* $Z/8m$ *if* n *is odd, but* $Z/4m\oplus Z/2$ *if* n *is even.*

For simplicity we denote by V_{2m}, V'_{2m}, W_{8m} and W'_{8m} the cofibers of the following maps

$$i\bar{\eta}\colon \Sigma^1 SZ/2 \to SZ/m\,, \qquad \bar{\eta}j\colon \Sigma^1 SZ/m \to SZ/2$$
$$i\bar{\eta}+\bar{\eta}j\colon \Sigma^1 SZ/2 \to SZ/4m\,, \quad i\bar{\eta}+\bar{\eta}j\colon \Sigma^1 SZ/4m \to SZ/2$$

respectively. Thus

(4.16) $\quad V_{4m}=V_{2m,2},\ V'_{4m}=V'_{2,2m},\ W_{8m}=W_{4m,2}$ and $W'_{8m}=W_{2,4m}$.

But $V_{2m}=SZ/m\vee\Sigma^2 SZ/2$ and $V'_{2m}=SZ/2\vee\Sigma^2 SZ/m$ if m is odd.

As a special case Propositions 4.4 and 4.5 give

Corollary 4.6. i) *The KU homologies* KU_0X, KU_1X *and the conjugation* t_* *on* KU_0X *are tabled as follows:*

X $=$	V_{2m}	V'_{2m}	W_{8m}	W'_{8m}	$W_{2m,2m}$
$KU_0X\cong$	$Z/2m$	$Z/2m$	$Z/8m$	$Z/8m$	$Z/2m\oplus Z/2m$
$KU_1X\cong$	0	0	0	0	0
t_* $=$	1	-1	$4m+1$	$4m-1$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$

ii) *The KO homologies* KO_iX *are tabled as follows:*

i $=$	0	1	2	3	4	5	6	7
V_{2m}	Z/m	0	$Z/2$	$Z/2$	$Z/4m$	$Z/2$	$Z/2$	0
V'_{2m}	$Z/2$	$Z/2$	$Z/4m$	$Z/2$	$Z/2$	0	Z/m	0
W_{8m}	$Z/4m$	0	$Z/2$	0	$Z/4m$	0	$Z/2$	0
W'_{8m}	$Z/2$	0	$Z/4m$	0	$Z/2$	0	$Z/4m$	0
$W_{2m,2m}$	$Z/2m$	0	$Z/2m$	0	$Z/2m$	0	$Z/2m$	0

5. Elementary $Z/2$-actions

5.1. If the cyclic group $Z/2$ of order 2 acts on the abelian group $Z\oplus Z/2^{s+1}$, $s\geq 0$, then its matrix representation is written as one of the following twelve types:

$$\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\ \pm\begin{pmatrix}1&0\\0&-1\end{pmatrix}\ \pm\begin{pmatrix}1&0\\1&-1\end{pmatrix}\ \pm\begin{pmatrix}1&0\\2^s&1\end{pmatrix}\ \pm\begin{pmatrix}1&0\\0&2^s+1\end{pmatrix}\ \pm\begin{pmatrix}1&0\\0&2^s-1\end{pmatrix}$$

where the matrices behave as left action on $Z\oplus Z/2^{s+1}$.

A $Z/2$-action ρ on an abelian group H is said to be *elementary* if the pair (H, ρ) is one of the following kinds of pairs:

(5.1) $(A, 1)\ (B, -1)\ (C\oplus C, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix})\ (Z/8m, 4m\pm 1)\ (Z\oplus Z/2m, \pm\begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix})$
$(Z\oplus Z/2m, \pm\begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix})$

We here deal with a CW-spectrum X such that the conjugation t_* on KU_0X is decomposed into a direct sum of the above elementary $Z/2$-actions, and $KU_1X=0$. Thus

(5.2) KU_0X
$$\cong A\oplus B\oplus(C\oplus C)\oplus A'\oplus B'\oplus(D\oplus D')\oplus(E\oplus E')\oplus(F\oplus F')\oplus(G\oplus G')$$

where each of the summands A' and B' is a direct sum of the forms $Z/8m$ and each of the summands $D\oplus D'$, $E\oplus E'$, $F\oplus F'$ and $G\oplus G'$ is a direct sum of the forms $Z\oplus Z/2m$. Moreover the conjugation t_* acts on each component of KU_0X as follows:

(5.3) $t_* = 1, -1, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ on $A, B, C\oplus C$.
$t_* = 4m+1, 4m-1$ on the component $Z/8m$ of A', B'.
$t_* = \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ m & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ m & -1 \end{pmatrix}$ on the component
$Z\oplus Z/2m$ of $D\oplus D', E\oplus E', F\oplus F', G\oplus G'$.

For any direct sum $H=\bigoplus_i Z/2m_i$ we denote by $H(*)$ the direct sum $\bigoplus_i (*)_{m_i}$ where $(*)_{m_i}\cong Z/4$ or $Z/2\oplus Z/2$ according as m_i odd or even. Besides we write $2H=\bigoplus_i Z/m_i$ and $1/2\,H=\bigoplus_i Z/4m_i$. For any CW-spectrum X satisfying (5.2) with (5.3) we will give a generalization of Lemmas 2.1 and 2.2.

Lemma 5.1. *Assume that $KU_1X=0$.*

i) $KC_iX\cong$
$$\begin{cases} A\oplus(B*Z/2)\oplus C\oplus(2A')\oplus(B'*Z/2)\oplus(D\oplus D'*Z/2)\oplus E'\oplus(F\oplus F')\oplus(G'*Z/2) \\ (A\otimes Z/2)\oplus B\oplus C\oplus(A'\otimes Z/2)\oplus(2B')\oplus(1/2\,D')\oplus E'\oplus(F*)\oplus(G\oplus 2G') \\ (A*Z/2)\oplus B\oplus C\oplus(A'*Z/2)\oplus(2B')\oplus D'\oplus(E\oplus E'*Z/2)\oplus(F'*Z/2)\oplus(G\oplus G') \\ A\oplus(B\otimes Z/2)\oplus C\oplus(2A')\oplus(B'\otimes Z/2)\oplus D\oplus(1/2\,E')\oplus(F\oplus 2F')\oplus G'(*) \end{cases}$$
corresponding to $i\equiv 0, 1, 2, 3 \bmod 4$.

ii) $KO_{2i}X\otimes Z[1/2]\cong(A\oplus C\oplus D\oplus F)\otimes Z[1/2]$ *or* $(B\oplus C\oplus E\oplus G)\otimes Z[1/2]$ *according as i even or odd, and* $KO_{2i+1}X\otimes Z[1/2]=0$ *for any i.*

iii) There are short exact sequences
$$0\to KC_3X\to KO_0X\oplus KO_4X\to KC_0X\to 0$$
$$0\to KC_1X\to KO_2X\oplus KO_6X\to KC_2X\to 0$$

and isomorphisms

$$KO_1X\oplus KO_5X\cong(A\otimes Z/2)\oplus(B*Z/2)\oplus(D'*Z/2)\oplus(F'\otimes Z/2)$$
$$KO_3X\oplus KO_7X\cong(A*Z/2)\oplus(B\otimes Z/2)\oplus(E'*Z/2)\oplus(G'\otimes Z/2)\,.$$

Proof. i) Use the exact sequences

$$0\to KC_4X\to KU_4X\xrightarrow{(\pi_U^{-1}(1-t))_*}KU_2X\xrightarrow{(\gamma\pi_U)_*}KC_3X\to 0$$
$$0\to KC_2X\to KU_2X\xrightarrow{((1+t)\pi_U^{-1})_*}KU_0X\xrightarrow{(\gamma\pi_U)_*}KC_1X\to 0$$

and compute the kernels and cokernels of $1\pm t_*$: $KU_0X\to KU_0X$.

ii) First notice that $KO_{2i+1}X\otimes Z[1/2]=0$ because $\varepsilon_O\varepsilon_U=2$. Then it follows that ε_{C*}: $KO_{2i}X\otimes Z[1/2]\to KC_{2i}X\otimes Z[1/2]$ is an isomorphism. The result is now immediate from i).

iii) The cofiber sequence (1.6) gives rise to two exact sequences

$$0\to KO_3X\oplus KO_7X\to KC_3X\xrightarrow{\varphi_0}KU_0X\to KO_2X\oplus KO_6X\to KC_2X\to 0$$

$$0\to KO_1X\oplus KO_5X\to KC_1X\xrightarrow{\varphi_2}KU_{-2}X\to KO_0X\oplus KO_4X\to KC_0X\to 0$$

where $\varphi_i(i=0,2)$ are induced by the composite $\varepsilon_U\tau\pi_C^{-1}$. Note that $\varepsilon_U\tau\pi_C^{-1}\gamma\pi_U=(1+t)\pi_U^{-1}$. Then the kernels and cokernels of $\varphi_i(i=0,2)$ are easily obtained, since $(\gamma\pi_U)_*$: $KU_{i+2}X\to KC_{i+3}X$ has already computed in i).

5.2. By observing Proposition 4.1 and Corollary 4.6 we here list up some of finite CW-spectra X with a few cells such that the conjugation t_* on KU_0X is elementary and $KU_1X=0$.

(5.4)

X =	V_{2m}	V'_{2m}	W_{8m}	W'_{8m}	$W_{2m,2m}$
$KU_0X\cong$	$Z/2m$	$Z/2m$	$Z/8m$	$Z/8m$	$Z/2m\oplus Z/2m$
t_* =	1	-1	$4m+1$	$4m-1$	$\begin{pmatrix}0&1\\1&0\end{pmatrix}$
X =	M_{2m}	Q_{2m}	N'_{2m}	P'_{2m}	R'_{2m}
$KU_0X\cong$	$Z\oplus Z/2m$	$Z\oplus Z/2m$	$Z\oplus Z/2m$	$Z\oplus Z/m$	$Z\oplus Z/2m$
t_* =	$\begin{pmatrix}-1&0\\1&1\end{pmatrix}$	$\begin{pmatrix}1&0\\m&1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\1&-1\end{pmatrix}$	$\begin{pmatrix}1&0\\0&1\end{pmatrix}$

We write $Y_H=\bigvee_i Y_{2m_i}$ for any direct sum $H=\bigoplus_i Z/2m_i$ when $Y=V$, W, M, Q and so on. We will here determine the quasi KO_*-type of a CW-spectrum X satisfying (5.2) with (5.3) under certain restrictions.

Theorem 5.2. *Let X be a CW-spectrum such that KU_0X has a direct sum* *decomposition as* (5.2), $KU_1X=0$ *and* t_* *acts on* KU_0X *as* (5.3). *Assume that* $A\cong A_0\oplus A_1$ *where* A_0 *is 2-torsion free and* A_1 *is a direct sum of cyclic 2-groups. If* $KO_1X=0=KO_7X$, *then* X *is quasi* KO_**-equivalent to the wedge sum* $\Sigma^4 SA_0\vee \Sigma^2 SB\vee(P\wedge SC)\vee V_{A_1}\vee W_{A'}\vee\Sigma^2 W_{B'}\vee\Sigma^2 M_{D'}\vee M_{E'}\vee\Sigma^4 Q_{F'}\vee\Sigma^2 Q_{G'}$. (Cf. [20, Theorem 2.5].)

Proof. Abbreviate by Y the desired wedge sum of elementary spectra with a few cells. From (5.4) it is obvious that $KU_0Y\cong KU_0X$ on both of which the conjugations t_* behave as the same action. Moreover we note that $KO_1Y=0=KO_7Y$ by means of Proposition 4.2 and Corollary 4.6. For each component Y_H of the wedge sum Y we can choose a unique map f_H: $Y_H\to KU\wedge X$ whose induced homomorphism $\kappa_{KU}(f_H)_*$: $KU_0Y_H\to KU_0X$ is the canonical inclusion, because of (1.8). Here H is taken to be $A_0, A_1, B, \cdots, F'$ or G'. Notice that there exists a map g_H: $Y_H\to KC\wedge X$ satisfying $(\zeta_\wedge 1)g_H=f_H$ for each H since $(t_\wedge 1)f_H=f_H$. We will find a map h_H: $Y_H\to KO\wedge X$ such that $(\varepsilon_{U\wedge}1)h_H=f_H$ for each H, and then apply Proposition 1.1 to show that the map $h=\bigvee_H h_H$: $Y=\bigvee_H Y_H\to KO\wedge X$ is a quasi KO_*-equivalence.

i) The $H=A_0$ case: Consider the commutative diagram

$$\begin{array}{ccccccccc}
0\to & \mathrm{Ext}(A_0, KO_2X) & \to & [\Sigma^4 SA_0, \Sigma^3 KO\wedge X] & \to & \mathrm{Hom}(A_0, KO_1X) & \to 0\\
& \downarrow \eta_{**} & & \downarrow (\eta_\wedge 1)_* & & \downarrow \eta_{**} & \\
0\to & \mathrm{Ext}(A_0, KO_3X) & \to & [\Sigma^4 SA_0, \Sigma^2 KO\wedge X] & \to & \mathrm{Hom}(A_0, KO_2X) & \to 0
\end{array}$$

with exact rows. Since A_0 is 2-torsion free and KO_3X is a $Z/2$-module by Lemma 5.1 iii), we see that $\mathrm{Ext}(A_0, KO_3X)=0$. So the central arrow $(\eta_\wedge 1)_*$ becomes trivial because $KO_1X=0$. This implies that the composite $(\varepsilon_O\pi\bar{v}^1{}_\wedge 1)f_{A_0}$: $\Sigma^2 SA_0\to KO\wedge X$ is trivial because it coincides with the composite $(\eta_\wedge 1)(\tau\pi\bar{c}^1{}_\wedge 1)g_{A_0}$. Hence there exists a map h_{A_0}: $\Sigma^4 SA_0\to KO\wedge X$ satisfying $(\varepsilon_{U\wedge}1)h_{A_0}=f_{A_0}$.

ii) The $H=B$ case is obtained more simply than the case i), by making use of only the assumption that $KO_7X=0=KO_1X$.

iii) The $H=C$ case: We will find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc}
SC & \xrightarrow{i_{P\wedge}1} & P\wedge SC & \xrightarrow{j_{P\wedge}1} & \Sigma^2 SC\\
h_0\downarrow & & \downarrow g_C & & \downarrow h_1\\
KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X\\
\| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1\\
KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X
\end{array}$$

after replacing the map g_C with $(\zeta_\wedge 1)g_C=f_C$ suitably if necessary. The homomorphism $\kappa_{KO}(g_C(i_P\wedge 1))_*: KO_0SC\to KC_0X$ is just the canonical inclusion $C\subset KC_0X$, and the induced homomorphism $(\tau\pi\bar{c}^1)_*: KC_0X\to KO_5X$ restricted to $C\subset KC_0X$ is trivial by Lemma 5.1 iii). Therefore $\kappa_{KO}((\tau\pi\bar{c}^1\wedge 1)\, g_C(i_P\wedge 1))_*: KO_0SC \to KO_5X$ becomes trivial. As in the case i) we here use the commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & \mathrm{Ext}(C, KO_6X) & \to & [SC, \Sigma^3 KO\wedge X] & \to & \mathrm{Hom}(C, KO_5X) & \to & 0 \\ & & \downarrow \eta_{**} & & \downarrow (\eta\wedge 1)_* & & \downarrow \eta_{**} & & \\ 0 & \to & \mathrm{Ext}(C, KO_7X) & \to & [SC, \Sigma^2 KO\wedge X] & \to & \mathrm{Hom}(C, KO_6X) & \to & 0 \end{array}$$

with exact rows, in which $KO_7X=0$. Then it follows that the composite $(\eta\wedge 1)(\tau\pi\bar{c}^1\wedge 1)\, g_C(i_P\wedge 1)$: $SC\to\Sigma^2 KO\wedge X$ becomes trivial. So we apply Lemma 1.3 to obtain maps h_0: $SC\to KO\wedge X$ and h_1: $SC\to\Sigma^1 KO\wedge X$ as desired where the map g_C might be replaced suitably. However the composite $(\eta\wedge 1)\, h_1$: $SC\to KO\wedge X$ is trivial because $KO_7X=0=KO_1X$. Consequently we get a map h_C: $P\wedge SC\to KO\wedge X$ such that $(\varepsilon_U\wedge 1)\, h_C=f_C$.

iv) The $H=A_1$ case: Setting $A_1=\bigoplus_i Z/2m_i$ we have to find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} \bigvee_i SZ/m_i & \xrightarrow{i_V} & V_{A_1} & \xrightarrow{j_V} & \bigvee_i \Sigma^2 SZ/2 \\ h_0\downarrow & & \downarrow g_{A_1} & & \downarrow h_1 \\ KO\wedge X & \to & KC\wedge X & \to & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta\wedge 1 & & \downarrow \eta\wedge 1 \\ KO\wedge X & \to & KU\wedge X & \to & \Sigma^2 KO\wedge X \end{array}$$

as in the case iii). The complexification ε_{U*}: $KO_0V_{A_1}\to KU_0V_{A_1}$ is the canonical monomorphism $\bigoplus_i Z/m_i\to\bigoplus_i Z/2m_i$, and the realification $(\varepsilon_O\pi\bar{U}^1)_*$: $KU_0X\to KO_6X$ restricted to $A\subset KU_0X$ is factorized through $A\otimes Z/2$ by Lemma 5.1 iii). These facts imply that $\kappa_{KO}((\varepsilon_O\pi\bar{U}^1\wedge 1)f_{A_1})_*$: $KO_0V_{A_1}\to KO_6X$ is trivial. Hence the composite map $(\varepsilon_O\pi\bar{U}^1\wedge 1)f_{A_1}\, i_V$: $\bigvee_i SZ/m_i\to\Sigma^2 KO\wedge X$ becomes trivial because $KO_7X=0$. Applying Lemma 1.3 we get the required maps h_0: $\bigvee_i SZ/m_i\to KO\wedge X$ and h_1: $\bigvee_i SZ/2\to\Sigma^1 KO\wedge X$, after replacing the map g_{A_1} suitably if necessary. Then there exists a map h_{A_1}: $V_{A_1}\to KO\wedge X$ satisfying $(\varepsilon_U\wedge 1)\, h_{A_1}=f_{A_1}$ since $(\eta\wedge 1)\, h_1=0$ as in the case iii).

v) The $H=A'$ case is obtained by a quite similar discussion to the above case iv).

vi) The $H=B'$ case: Set $B'=\bigoplus_i Z/2m_i$ and consider the commutative diagram

$$\begin{array}{ccccc} \bigvee_i \Sigma^2 SZ/m_i & \xrightarrow{i_W} & \Sigma^2 W_{B'} & \xrightarrow{j_W} & \bigvee_i \Sigma^4 SZ/2 \\ h_0 \downarrow & & \downarrow g_{B'} & & \downarrow h_1 \\ KO\wedge X & \rightarrow & KU\wedge X & \rightarrow & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \rightarrow & KU\wedge X & \rightarrow & \Sigma^2 KO\wedge X . \end{array}$$

In this case we can find vertical arrows h_0, h_1 more easily than the case iv), by making use of only the assumption that $KO_7X=0=KO_1X$. The map h_1: $\bigvee_i \Sigma^1 SZ/2 \rightarrow KO\wedge X$ has an extension h_2: $\bigvee_i \Sigma^2 \rightarrow KO\wedge X$, thus $h_1=h_2(\bigvee_i j)$. Hence the composite map $(\eta_\wedge 1) h_1 j_W$: $W_{B'} \rightarrow KO\wedge X$ becomes trivial because $\eta j = j(i\bar{\eta}+\tilde{\eta}j)$. So we get a map $h_{B'}$: $\Sigma^2 W_{B'} \rightarrow KO\wedge X$ satisfying $(\varepsilon_{U\wedge} 1) h_{B'}=f_{B'}$.

vii) The $H=D'$, E' cases are shown by similar discussoins to the case iv). Use the assumption that $KO_7X=0=KO_1X$ in the former case, and Lemma 5.1 iii) and the assumption that $KO_7X=0$ in the latter case.

viii) The $H=F'$ case: Setting $F'=\bigoplus_i Z/2m_i$, we will find vertical arrows h_0, h_1 making the diagram below commutative

$$\begin{array}{ccccc} \Sigma^4 SF' & \xrightarrow{i_Q} & \Sigma^4 Q_{F'} & \xrightarrow{j_Q} & \Sigma^8 SF \\ h_0 \downarrow & & \downarrow g_{F'} & & \downarrow h_1 \\ KO\wedge X & \rightarrow & KC\wedge X & \rightarrow & \Sigma^3 KO\wedge X \\ \| & & \downarrow \zeta_\wedge 1 & & \downarrow \eta_\wedge 1 \\ KO\wedge X & \rightarrow & KU\wedge X & \rightarrow & \Sigma^2 KO\wedge X . \end{array}$$

where $SF'=\bigvee_i SZ/2m_i$ and $SF=\bigvee_i \Sigma^0$. Since $KO_1X=0$, the composite $(\tau\pi_C^{-1}{}_\wedge 1) g_{F'} i_Q$: $\Sigma^1 SF' \rightarrow KO\wedge X$ has an extension k_0: $\Sigma^2 SF \rightarrow KO\wedge X$. The induced homomorphism $g_{F'*}$: $KO_2Q_{F'} \rightarrow KC_6X$ carries $KO_2Q_{F'}$ onto the component $F\otimes Z/2 \subset KC_6X$. On the other hand, $(\tau\pi_C^{-1})_*$: $KC_6X \rightarrow KO_3X$ restricted to the component $F\otimes Z/2 \subset KC_6X$ is trivial by Lemma 5.1 iii). Combining these facts we see that k_{0*}: $KO_1SF \rightarrow KO_3X$ is trivial. Thus the composite $(\eta_\wedge 1) k_0$: $\Sigma^3 SF \rightarrow KO\wedge X$ becomes trivial, and hence the composite $(\varepsilon_O \pi_U^{-1}{}_\wedge 1) f_{F'} i_Q$: $\Sigma^2 SF' \rightarrow KO\wedge X$ is trivial, too. So we apply Lemma 1.3 to obtain the required maps h_0: $\Sigma^4 SF' \rightarrow KO\wedge X$ and h_1: $\Sigma^5 SF \rightarrow KO\wedge X$.

The coextension $\tilde{\eta}$: $\Sigma^2 \rightarrow SZ/2m$ of η induces an epimorphism $\tilde{\eta}^*$: $[\Sigma^3 SZ/2m, KO\wedge X] \rightarrow [\Sigma^5, KO\wedge X]$ because $j\tilde{\eta}=\eta$. So there exists a map h_2: $\Sigma^3 SF' \rightarrow KO\wedge X$ such that $h_2(\bigvee_i \tilde{\eta})=h_1$. Then the composite map $(\eta_\wedge 1) h_1 j_Q$: $\Sigma^2 Q_{F'} \rightarrow KO\wedge X$ becomes trivial. So we get a map $h_{F'}$: $\Sigma^4 Q_{F'} \rightarrow KO\wedge X$ satisfying $(\varepsilon_{U\wedge} 1) h_{F'}=f_{F'}$ as desired.

ix) The $H=G'$ case is obtained easily by a parallel discussion to the above case viii).

As a special case of Theorem 5.2 we have

Corollary 5.3. *Let X be a CW-spectrum and C, A', B' abelian groups where A' and B' are direct sums of the forms $Z/8m$. Then $X \underset{KO}{\approx} (P\wedge SC)\vee W_{A'}\vee\Sigma^2 W_{B'}$ if and only if $KU_0X\cong C\oplus C\oplus A'\oplus B'$, $KU_1X=0$ and t_* acts on KU_0X as in* (5.3). (Cf. [20, Theorem 1.6].)

Proof. The "only if" part is evident.

The "if" part: In this case it follows from Lemma 5.1 iii) that $KO_{2i+1}X=0$ for any i. So we may apply Theorem 5.2.

As an easy application of Theorem 5.2 combined with Propositions 4.1 and 4.2 and Corollaries 1.6 and 4.6, we obtain

Corollary 5.4. *$P'_{4m} \underset{KO}{\approx} \Sigma^2 M_{2m}$, $P_{4m} \underset{KO}{\approx} \Sigma^{-1}M'_{2m}$, $V_{2m} \underset{KO}{\approx} \Sigma^2 V'_{2m}$, $W_{8m} \underset{KO}{\approx} \Sigma^4 W_{8m} \underset{KO}{\approx} \Sigma^2 W'_{8m}$ and $W_{2m,2m} \underset{KO}{\approx} P\wedge SZ/2m$.*

As a consequence of Theorem 5.2 we can finally show Theorem 3 stated in the introduction.

Proof of Theorem 3. i) The $KU_0X\cong Z/2m$ case: The conjugation t_* on KU_0X behaves as one of the following four types: $t_*=\pm 1$, $4n\pm 1$ $(m=4n)$. Thus the pair (KU_0X, t_*) is itself elementary. So we may apply Theorem 5.2 to show that X is quasi KO_*-equivalent to one of the following four elementary spectra: V_{2m}, $\Sigma^2 SZ/2m$, W_{8n} and $\Sigma^2 W_{8n}$.

ii) The $KU_0X\cong Z\oplus Z/2m$ case: The conjugation t_* on KU_0X behaves as one of the following twelve types: $t_*=\pm\begin{pmatrix}1 & 0\\ 0 & \pm 1\end{pmatrix}$, $\pm\begin{pmatrix}1 & 0\\ 0 & 4n\pm 1\end{pmatrix}$ $(m=4n)$, $\pm\begin{pmatrix}1 & 0\\ 1 & -1\end{pmatrix}$, $\pm\begin{pmatrix}1 & 0\\ m & 1\end{pmatrix}$. Thus the pair (KU_0X, t_*) is itself elementary, too. Hence we can show that X is quasi KO_*-equivalent to one of the twelve elementary spectra given in Theorem 3 ii), by applying Theorem 5.2 again.

References

[1] J.F. Adams: *Vector fields on spheres*, Ann. of Math. **75** (1962), 603–622.

[2] J.F. Adams: Stable homotopy and generalized homology, Univ. of Chicago, 1974.

[3] D.W. Anderson: *A new cohomogogy theory*, Thesis, Univ. of California, Berkeley, 1964.

[4] D.W. Anderson: *Universal coefficient theorems for K-theory*, mimeographed notes, Berkeley.

[5] A.K. Bousfield: *Cohomological localizations of spaces and spectra*, preprint.

[6] A.K. Bousfield: *A classification of K-local spectra*, preprint.

[7] C.W. Curtis and I. Reiner: Methods of representation theory with applica-

tions to finite groups and orders, Pure and Applied Math. A Wiley-Interscience series of texts, monographs and tracts, 1981.

[8] M. Fujii: *K_O-groups of projective spaces*, Osaka J. Math. **4** (1967), 141–149.

[9] K. Johnson: *On compact cohomology theories and Pontrjagin duality*, Trans. Amer. Math. Soc. **279** (1983), 237–247.

[10] M. Karoubi and V. Mudrinski: *K-théorie réelle de fibrés projectifs complexes*, C.R. Acad. Sci. Paris Sér. I Math. **297** (1983), 349–352.

[11] A.G. Kurosh: The theory of groups, I, Chelsea Publ. Co., New York, 1955.

[12] H.R. Margoris: Spectra and the Steenrod algebra, Modules over the Steenrod algebra and the stable homotopy category, North-Holland Mathematical Library 29, North-Holland, 1983.

[13] W. Meier: *Complex and real K-theory and localization*, J. Pure and Applied Algebra **14** (1979), 59–71.

[14] M. Mimura, S. Oka and M. Yasuo: *K-theory and the homotopy types of some function spaces*, in preparation.

[15] D.C. Ravenel: *Localization with respect to certain periodic homology theories*, Amer J. Math **106** (1984), 351–414.

[16] R.M. Seymour: *The real K-theory of Lie groups and homogeneous spaces*, Quart. J. Math. Oxford **24** (1973), 7–30.

[17] E.H. Spanier: *Function spaces and duality*, Ann. of Math. **70** (1959), 338–378.

[18] R. Vogt: Boardman's stable homotpoy category, Lecture notes series 21, Aarhus Univ., 1970.

[19] Z. Yosimura: *Universal coefficient sequences of cohomology theories of CW-spectra, I and II*, Osaka J. Math. **12** (1975), 305–323 and **16** (1979), 201–217.

[20] Z. Yosimura: *The quasi KO-homology types of the real projective spaces*, Proc. Int. Conf. at Kinosaki, Springer-Verlag, **1418** (1990), 156–174.

Department of Mathematics
Osaka City University
Sugimoto, Sumiyoshi-ku
Osaka 558, Japan