Suetake, C.
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ON FINITE POINT TRANSITIVE AFFINE PLANES WITH TWO ORBITS ON l_∞

CHIHIRO SUETAKE

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1. Introduction

Kallaher [3] proposed the following conjecture.

Conjecture. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π. If G has two orbits on the line at infinity, then one of the following statements holds:*

(i) *The plane π is a translation plane, and the group G contains the group of translations of π.*
(ii) *The plane π is a dual translation plane, and the group G contains the group of dual translations of π.*

The purpose of this paper is to study this conjecture. When G_A has two orbits of length 1 and n on the line at infinity, where A is an affine point of π, some work has been done on this conjecture. (See Johnson and Kallaher [2].)

Our notation is largely standard and taken from [3]. Let $\mathcal{P}=\pi\cup l_\infty$ be the projective extention of an affine plane π, and G a collineation group of $\mathcal{P}$. If P is a point of $\mathcal{P}$ and l is a line of $\mathcal{P}$, then $G(P, l)$ is the subgroup of G consisting of all perspectivities in G with center P and axis l. If m is a line of $\mathcal{P}$, then $G(m, m)$ is the subgroup consisting of all elations in G with axis m.

In § 2 we prove the following theorem.

Theorem 1. *Let π be a finite affine plane of order n with a collineation group G and let Δ be a subset of l_∞ such that $|\Delta|=t\geq 2$, $(n, t)=1$ and $(n, t-1)=1$. If there is an integer $k_1>1$ such that $|G(P, l_\infty)|=k_1$ for all $P\in\Delta$ and there is an integer $k_2>1$ sucht that $|G(Q, l_\infty)|=k_2$ for all $Q\in l_\infty-\Delta$, then π is a translation plane, and G contains the group T of translations of π.*

In § 3 and § 4, we prove the following theorem by using Theorem 1.

Theorem 2. *Let π be a finite affine plane of order n with a collineation group G which is transitive on the affine points of π. If G has two orbits of length 2 and $n-1$ on l_∞, then one of the following statements holds:*

(i) *The plane π is a translation plane, and the group G contains the group T of translations of π.*

(ii) $|G(\ell_\infty, \ell_\infty)|=n=2^m$ *for some* $m\geq 1$, $G(P_1, \ell_\infty)=G(P_2, \ell_\infty)=1$ *and* $|G(P, \ell_\infty)|=2$ *for all* $P\in\ell_\infty-\{P_1, P_2\}$.

The planes which are not André planes, satisfying the hypothesis of Theorem 2, include a class of translation planes of order q^3, where q is an odd prime power. (See Suetake [4] and Hiramine [1].)

2. The proof of Theorem 1

In this section, we prove Theorem 1.

Let π be a finite affine plane of order n with a collineation group G, satisfying the hypothesis of Theorem 1. By Theorem 4.5 of [3], $G(\ell_\infty, \ell_\infty)$ is an elementary abelian r-group for some prime r dividing n. Hence there exist positive integers m and s such that $k_1=r^m$ and $k_2=r^s$. Let P be a point of π such that $P\in\Delta$. Let ℓ be an affine line of π such that $\ell\ni P$. Since $G(P, \ell_\infty)$ is semi-regular on $\ell-\{P\}$, $r^m|n$. Similarly, $r^s|n$. By definition, $G(\ell_\infty, \ell_\infty)=\bigcup_{P\in\ell_\infty} G(P, \ell_\infty)$ and $G(P, \ell_\infty)\cap G(Q, \ell_\infty)=1$ for distinct points $P, Q\in\ell_\infty$. Thus

$$\begin{aligned}|G(\ell_\infty, \ell_\infty)| &= 1+\sum_{P\in\Delta}(|G(P, \ell_\infty)|-1)+\sum_{Q\in\ell_\infty-\Delta}(|G(Q, \ell_\infty)|-1)\\ &= 1+t(r^m-1)+(n+1-t)(r^s-1)\,.\end{aligned}$$

Since $r^m \mid |G(\ell_\infty, \ell_\infty)|$, it follows $0\equiv 1-t+(1-t)r^s-1+t \pmod{r^m}$. Therefore $(t-1)r^s\equiv 0 \pmod{r^m}$. Since $(t-1, r)=1$, this imples $r^m|r^s$. Thus $m\leq s$. On the other hand, since $r^s \mid |G(\ell_\infty, \ell_\infty)|$, it follows $0\equiv 1+t(r^m-1)-1+t \pmod{r^s}$. Therefore $tr^m\equiv 0 \pmod{r^s}$. Since $(t, r)=1$, this imples $r^s|r^m$. Thus $m\geq s$. Therefore $m=s$ and $k_1=k_2$. By a result of Gleason (See Theorem 5.2 of [3].), the theorem holds.

3. The proof of Theorem 2 when n is odd

In this section, we prove Theorem 2 when n is odd.

Let π be a finite affine plane of odd order n with a collineation group G which is transitive on the affine points of π, satisfying the hypothesis of Theorem 2. Then G has an orbit $\Delta=\{P_1, P_2\}$ of length 2 on ℓ_∞. Let A be an affine point of π. Let Φ be the set of the affine points of π, and let $\Omega=\Phi\cup\ell_\infty$. Then G induces a permutation group on Ω. Φ, Δ and $\ell_\infty-\Delta$ are orbits of G. Since $(|\Phi|, |\Delta|)=(n^2, 2)=1$ and $(|\Phi|, |\ell_\infty-\Delta|)=(n^2, n-1)=1$, by Theorem 3.3 of [3] Δ and $\ell_\infty-\Delta$ are orbits of G_A.

Lemma 3.1. *G_A includes an involutory homology of π.*

Proof. G_A induces a permutation group on $l_\infty-\{P_1, P_2\}$. Since n is odd, $|l_\infty-\{P_1, P_2\}|=n-1$ is even. Let S be a Sylow 2-subgroup of G_A. As G_A is transitive on $l_\infty-\{P_1, P_2\}$, $n-1\,|\,|G_A|$. Hence $S\neq 1$. There exists an involution σ in the center of S. Suppose that σ is a Baer involution. If $P_1\sigma=P_1$, then $P_2\sigma=P_2$ and so $|\{P\in l_\infty-\Delta\,|\,P\sigma=P\}|=\sqrt{n}-1$. This contradicts a result of Lüneburg. (See Corollary 3.6.1 of [3].) If $P_1\sigma\neq P_1$, then $P_2\sigma\neq P_2$ and so $|\{P\in l_\infty-\Delta\,|\,P\sigma=P\}|=\sqrt{n}+1$. This is again a contradiction by Corollary 3.6.1 of [3]. Therefore σ is an involutory homology.

Lemma 3.2. *Let σ be an involutory homology of π such that $\sigma\in G_A$. If $P_1\sigma=P_1$, then π is a translation plane, and G contains the group T of translations of π.*

Proof. Since $P_1\sigma=P_1$, $P_2\sigma=P_2$. Assume that l_∞ is the axis of σ. Then $\sigma\in G(A, l_\infty)$. By a result of André (See Corollary 10.1.3 of [3].), the lemma holds. Assume that l_∞ is not the axis of σ. We a may assume that AP_1 is the axis of σ. Then $\sigma\in G(P_2, AP_1)$. There exists $\tau\in G_A$ such that $P_1\tau=P_2$. Clearly $P_2\tau=P_1$. Since $P_2\tau=P_1$ and $(AP_1)\tau=AP_2$, $\tau^{-1}\sigma\tau\in G(P_1, AP_2)$. Therefore $\sigma(\tau^{-1}\sigma\tau)\in G(A, l_\infty)-\{1\}$, by a result of Ostrom. (See Lemma 4.13 of [3].) Thus the lemma holds by Corollary 10.1.3 of [3].

Lemma 3.3. *If G_A includes an involutory homology of π which does not fix P_1, then the following statements hold:*

(i) *If $P\in l_\infty-\{P_1, P_2\}$, then there exist $Q\in l_\infty-\{P_1, P_2, P\}$ and $\sigma\in G(Q, AP)$ such that $|\sigma|=2$.*

(ii) *If $Q\in l_\infty-\{P_1, P_2\}$, then there exist $P\in l_\infty-\{P_1, P_2, Q\}$ and $\tau\in G(Q, AP)$ such that $|\tau|=2$.*

Proof. By assumption, there exists an involutory homology σ of π such that $\sigma\in G_A$ and $P_1\sigma\neq P_1$. Clearly $P_2\sigma\neq P_2$. There exists $P_0\in l_\infty-\{P_1, P_2\}$ such that AP_0 is the axis of σ. Let Q_0 be the center of σ. Then $Q_0\in l_\infty-\{P_1, P_2, P_0\}$. Let $P\in l_\infty-\{P_1, P_2\}$. Then there exists $\varphi\in G_A$ such that $P=P_0\varphi$. Set $Q=Q_0\varphi$. Clearly $Q\notin\{P_1, P_2\}$. Since $\sigma\in G(Q_0, AP_0)$ and $(AP_0)\varphi=AP$, $\varphi^{-1}\sigma\varphi\in G(Q, AP)$. This yields the statement (i). Similarly, we have the statement (ii).

Lemma 3.4. *If G_A includes an involutory homology of π which does not fix P_1, then one of the following statements holds:*

(i) *The plane π is a translation plane and G contains the group T of translations of π.*
(ii) *If $P\in l_\infty-\{P_1, P_2\}$, then $G(P, AP)\neq 1$.*

Proof. Let $P\in\ell_\infty-\{P_1, P_2\}$. By Lemma 3.3 (i), there exist $Q\in\ell_\infty-\{P_1, P_2, P\}$ and $\sigma\in G(Q, AP)$ such that $|\sigma|=2$. On the other hand, by Lemma 3.3 (ii) there exist $R\in\ell_\infty-\{P_1, P_2, Q\}$ and $\tau\in G(R, AQ)$ such that $|\sigma|=2$. Assume that $R=P$. Then $\sigma\in G(Q, AP)$ and $\tau\in G(P, AQ)$. By Lemma 4.13 of [3], $\sigma\tau\in G(A, \ell_\infty)-\{1\}$. Thus the statement (i) holds by Corollary 10.1.3 of [3]. Assume that $R\neq P$. Then since $\tau\in G(R, AQ)$ and $(AQ)\sigma=AQ$, $\sigma^{-1}\tau\sigma\in G(R\sigma, AQ)$. As $R\neq R\sigma$, $\tau(\sigma^{-1}\tau\sigma)\in G(Q, AQ)-\{1\}$ by a result of Baer. (See Lemma 4.12 of [3].) Thus $G(Q, AQ)\neq 1$. On the other hand, since G_A acts transitively on $\ell_\infty-\{P_1, P_2\}$, the statement (ii) holds.

Lemma 3.5. *If $G(P, AP)\neq 1$ for all $P\in\ell_\infty-\{P_1, P_2\}$, then there is an integer $k>1$ such that $|G(P, \ell_\infty)|=k$ for all $P\in\ell_\infty-\{P_1, P_2\}$.*

Proof. Let $P\in\ell_\infty-\{P_1, P_2\}$. Let ℓ be an affine line of π such that $\ell\ni P$. By a result of Ostrom and Wagner (See Theorem 4.3 of [3].), there exists $\tau\in G_P$ such that $(AP)\tau=\ell$. Since $G(P, AP)\neq 1$, $\tau^{-1}G(P, AP)\tau=G(P\tau, (AP)\tau)=G(P, \ell)\neq 1$. Therefore by the dual of Corollary 4.6.1 of [3], $G(P, \ell_\infty)\neq 1$. On the other hand, since G_A acts transitively on $\ell_\infty-\{P_1, P_2\}$, the lemma holds.

Lemma 3.6. *If $G(P, AP)\neq 1$ for all $P\in\ell_\infty-\{P_1, P_2\}$, then $|G(P_1, \ell_\infty)|=|G(P_2, \ell_\infty)|>1$.*

Proof. Since the order n of π is odd, by Lemma 3.5 $|G(P, \ell_\infty)|\geq 3$ for all $P\in\ell_\infty-\{P_1, P_2\}$. Therefore

$$\begin{aligned}&\Big|\bigcup_{P\in\ell_\infty-\{P_1,P_2\}} G(P, \ell_\infty)\Big|\\&=1+\sum_{P\in\ell_\infty-\{P_1,P_2\}}(|G(P, \ell_\infty)|-1)\\&\geq 1+2(n-1)\\&=2n-1\\&>n.\end{aligned}$$

Thus $|G(\ell_\infty, \ell_\infty)|>n$. Hence by a result of Ostrom (See Theorem 4.6 of [3].), $G(P, \ell_\infty)\neq 1$ for all $P\in\ell_\infty$. In particular $G(P_1, \ell_\infty)\neq 1$. There exists $\tau\in G_A$ such that $P_2\tau=P_1$. Thus $|G(P_2, \ell_\infty)|=|\tau^{-1}G(P_2, \ell_\infty)\tau|=|G(P_1, \ell_\infty)|>1$. Hence the lemma holds.

Proof of Theorem 2 when n is odd: By Lemmas 3.2, 3.4, 3.5, 3.6 and Theorem 1, the theorem holds.

4. The proof of Theorem 2 when n is even

In this section, we prove Theorem 2 when n is even.

Let π be a finite affine plane of even order n with a collineation group G

which is transitive on the affine points of π satisfying the hypothesis of Theorem 2. Then G has an orbit $\Delta=\{P_1, P_2\}$ of length 2 on l_∞.

Lemma 4.1. *G includes a translation of order 2 of π.*

Proof. Since $n^2 \mid |G|$, $2 \mid |G|$. Let S be a Sylow 2-subgroup of G. Then there exists an involution σ in the center of S. By Corollary 3.6.1 of [3] the involution σ is neither a Baer involution, nor an affine elation. It follows that σ is a translation of π.

Lemma 4.2. *$G(l_\infty, l_\infty)$ is an elementary abelian 2-group and $|G(l_\infty, l_\infty)| \geq 2$.*

Proof. If $n=2$, then the lemma holds. Let $n \neq 2$. Considering the action of G on l_∞, by Lemma 4.1 there exist distinct points $Q_1, Q_2 \in l_\infty$ such that $G(Q_1, l_\infty) \neq 1$ and $G(Q_2, l_\infty) \neq 1$. By Theorem 4.5 of [3], the lemma holds.

Lemma 4.3. *If $G(P_1, l_\infty) \neq 1$, then the plane π is a translation plane, and the group G contains the group T of translations of π.*

Proof. There exists an involution σ_i such that $\sigma_i \in G(P_i, l_\infty)$ for $i \in \{1, 2\}$. Then $\sigma_1\sigma_2 \in G(l_\infty, l_\infty)$ and $|\sigma_1\sigma_2|=2$. Let Q be the center of $\sigma_1\sigma_2$. Then $Q \in l_\infty - \{P_1, P_2\}$. Since G acts transitively on $l_\infty - \{P_1, P_2\}$, there exists $r \geq 1$ such that $|G(P, l_\infty)| = 2^r$ for all $P \in l_\infty - \{P_1, P_2\}$. There exists $s \geq 1$ such that $|G(P_1, l_\infty)| = |G(P_2, l_\infty)| = 2^s$. Let $|G(l_\infty, l_\infty)| = 2^t$. Then $t \geq r+s$. Since

$$|G(l_\infty, l_\infty)| = 1 + \sum_{P \in l_\infty - \{P_1, P_2\}} (|G(P, l_\infty)|-1) + \sum_{Q \in \{P_1, P_2\}} (|G(Q, l_\infty)|-1),$$

$$2^t = 1+(n-1)(2^r-1)+2(2^s-1). \qquad (*)$$

By the same argument as in the proof of Theorem 1, $2^r \equiv 0 \pmod{2^s}$ and $2^{s+1} \equiv 0 \pmod{2^r}$. Thus $s \leq r \leq s+1$.

Suppose that $r=s+1$. From $(*)$, $2^t=1+(n-1)(2^{s+1}-1)+2(2^s-1)$ follows. Therefore $n=2^t(2^{s+1}-1)^{-1}$. As n is an integer, this is a contradiction. Hence $r=s$. By Theorem 5.2 of [3], the lemma holds.

Lemma 4.4. *If $G(P_1, l_\infty)=1$, then $|G(l_\infty, l_\infty)|=n=2^m$ for some $m \geq 1$, $G(P_1, l_\infty)=1$ and $|G(P, l_\infty)|=2$ for all $P \in l_\infty - \{P_1, P_2\}$.*

Proof. By assumption, $G(P_2, l_\infty)=1$ follows. If $P \in l_\infty - \{P_1, P_2\}$, then $G(P, l_\infty) \neq 1$. Therefore there exists an integer $r \geq 1$ such that $|G(Q, l_\infty)| = 2^r$ for all $Q \in l_\infty - \{P_1, P_2\}$. Suppose that $r \geq 2$. Then

$$\begin{aligned} &|G(l_\infty, l_\infty)| \\ &= \sum_{Q \in l_\infty - \{P_1, P_2\}} (|G(Q, l_\infty)|-1)+1 \\ &= (2^r-1)(n-1)+1 \end{aligned}$$

$$\begin{aligned} &\geq 3(n-1)+1 \\ &= 3n-2 \\ &> n\,. \end{aligned}$$

By Theorem 4.6 of [3], it follows that $G(Q, l_\infty)\neq 1$ for all $Q\in l_\infty$. In particular $G(P_1, l_\infty)\neq 1$, a contradiction. Hence $r=1$. Therefore $|G(l_\infty, l_\infty)|=(2-1)\cdot(n-1)+1=n$. Therefore there exists an integer $m\geq 1$ such that $n=2^m$. Thus the lemma holds.

Proof of Theorem 2 when n is even: By Lemmas 4.3 and 4.4, the theorem holds.

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Amagasakiminami High School
Amagasaki, Hyogo 660
Japan