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GRAPHS WITH GIVEN GROUP

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1. Introduction

In 1948 Frucht [2] proved that for any finite group G there is a connected 3-regular graph Γ with $\mathrm{Aut}\,\Gamma\cong G$. Sabidussi [3] extended this result: For any finite group G and any integer $n\geqq 3$ there are infinitely many connected n-regular graphs Γ with $\mathrm{Aut}\,\Gamma\cong G$. Moreover Vogler [5] extended this one of Sabidussi: For any finite group G and any link graph Δ (cf. [5]) with at least one isolated vertex and at least three vertices there are infinitely many connected graphs Γ with constant link Δ and $\mathrm{Aut}\,\Gamma\cong G$. We note that $\mathrm{Aut}\,\Gamma$ acts semiregularly (cf. [6]) on $V\Gamma$ in every above result. In this paper we shall study the following probelm: When can a given abstract finite group be represented as the automorphism group of a cnnected regular graph in which some vertex is left invariant by the automorphism group?

First we remark

Lemma 1. *Let p be a prime and Γ be a connected n-regular graph. If there is a subgroup G of $\mathrm{Aut}\,\Gamma$ with $|G|=p$ such that some vertex v of Γ is fixed by G, then $n\geqq p$ holds.*

Proof. Let $v_1, v_2, \cdots, v_n$ be the vertices adjacent to v. Let us suppose $n<p$. Since G fixes $\{v_1, v_2, \cdots, v_n\}$ as a set, G must fix all v_i's ($i=1, 2, \cdots, n$) (cf. Lemma 2). Similarly, all vertices adjacent to v_i ($1\leqq i\leqq n$) are fixed by G. Because of the connectedness of Γ, G fixes all vertices of Γ by continuing the above argument. Hence we have $G=1$ (the identity group), a contradiction.

The following theorem is the main result of this paper and the condition $n\geqq |G|$ is necessary by Lemma 1.

Theorem 1. *Let G be a finite group and t be a positive integer. Then for any integer $n\geqq \max\{|G|, 3\}$ there exist infinitely many connected n-regular graphs Γ such that $\mathrm{Aut}\,\Gamma\cong G$ and Γ has t vertices $\alpha_1, \alpha_2, \cdots, \alpha_t$ with* $\mathrm{Aut}\,\Gamma=(\mathrm{Aut}\,\Gamma)_{\alpha_1\alpha_2\cdots\alpha_t}$, *that is, $\mathrm{Aut}\,\Gamma$ is isomorphic to G and fixes some t vertices $\alpha_1, \alpha_2, \cdots$ and α_t.*

2. Preliminaries

First let us fix some conventions. By a graph we mean an undirected graph without loops and multiple edges. All graphs considered are finite in this paper except Remark 1. $V\Gamma$ and $E\Gamma$ denote the set of vertices and edges of a graph Γ respectively. If an edge e joins two vertices u and v, we write $e=[u, v]=[v, u]$. By $\mathrm{Aut}\,\Gamma$ we denote the automorphism group of Γ. If $\mathrm{Aut}\,\Gamma=1$, Γ is called asymmetric. For a vertex v in Γ, $N(v)$ denotes the subgraph of Γ induced by the vertices adjacent to v. If Γ is connected and if for some vertex v the subgraph induced by $V\Gamma-\{v\}$ is disconnected, then v is called a cut-vertex of Γ. For vertices u and v, $\partial(u, v)$ denotes the distance between u and v. We define the distance between subsets A and B of $V\Gamma$ by $\partial(A, B)=\min\{\partial(u, v)\colon u\in A, v\in B\}$.

Now we introduce a notion of the type (cf. [2]) $(a_1, a_2, \cdots, a_r)$ $(r=\binom{m}{2})$ of a vertex v of valency m in Γ. Let $u_1, u_2, \cdots, u_m$ be the adjacent vertices of v. We define the number α_{ij} $(i<j)$ as follows:

$\alpha_{ij}=$ the minimum length of circuits which contain the two edges $[u_i, v]$ and $[v, u_j]$ if there exists such a circuit,
$\quad = \infty$ otherwise.

By ranging $\binom{m}{2}$ numbers of α_{ij}'s in increasing order, we get the type $(a_1, a_2, \cdots, a_r)$ of v, where $r=\binom{m}{2}$, $a_1\leqq a_2\leqq\cdots\leqq a_r$ and $\{a_1, a_2, \cdots, a_r\}=\{\alpha_{ij}\colon 1\leqq i<j\leqq m\}$.

We shall make substantial use of methods of Sabidussi [3, 4]: For graphs $\Gamma_1, \Gamma_2, \cdots$ and Γ_k we define the product $\prod_{i=1}^{k}\Gamma_i=\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_k$ by

$$V\left(\prod_{i=1}^{k}\Gamma_i\right)=\prod_{i=1}^{k}V\Gamma_i \text{ (the cartesian product of the sets } V\Gamma_i),$$

$$E\left(\prod_{i=1}^{k}\Gamma_i\right)=\{[(u_1, u_2, \cdots, u_k), (v_1, v_2, \cdots, v_k)]\colon \{i\colon u_i\neq v_i, 1\leqq i\leqq k\} \text{ is one-element set } \{j\} \text{ satisfying } [u_j, v_j]\in E\Gamma_j\}.$$

It is obvious that the product of connected graphs is connected. A graph Γ is called prime if Γ is non-trivial and if $\Gamma\cong\Lambda\times\Pi$ implies that Λ or Π is trivial, where a trivial graph is a vertex-graph. Two graphs Γ and Δ are called relatively prime if $\Gamma\cong\Gamma'\times\Pi$ and $\Delta\cong\Delta'\times\Pi$ imply that Π is a trivial graph. We say that a connected graph Γ can be decomposed into prime factors if there exist connected prime graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_r$ satisfying $\Gamma\cong\prod_{i=1}^{r}\Gamma_i$.

A graph Γ which is attached a graph isomorphic with a graph Δ to a graph Π so as to correspond vertices $u_1, u_2, \cdots, u_m$ of Δ to vertices $v_1, v_2, \cdots, v_m$ of Π respectively is a following graph:

$$V\Gamma=V\Pi\cup V\Lambda,\ V\Pi\cap V\Lambda=\{v_1, v_2, \cdots, v_m\},$$
$$E\Gamma=E\Pi\cup E\Lambda,$$

where Λ is a graph isomorphic with Δ such that there is an isomorphism f from

Λ to Δ with $f(v_i)=u_i (i=1, 2, \cdots, m)$.

For groups H and G, $H \leqq G$ denotes that H is a subgroup of G. For a subset T of a group G, $\langle T \rangle$ is a subgroup of G generated by T. Let G be a permutation group on a set Ω. For elements $\alpha_1, \alpha_2, \cdots, \alpha_t$ in Ω, $G_{\alpha_1\alpha_2\cdots\alpha_t}$ denotes a subgroup $\{g \in G: g(\alpha_i)=\alpha_i \ (i=1, 2, \cdots, t)\}$ of G. G is said to act semiregularly on Ω if $G_\alpha=1$ holds for any $\alpha \in \Omega$.

Lemma 2 [6]. *Let G be a permutation group on Ω. Then for any $\alpha \in \Omega$, $|G|=|G_\alpha| \cdot |G(\alpha)|$ holds.*

Lemma 3. *Let u, v be vertices of graphs Γ, Δ respectively. Then the valency of (u, v) in $\Gamma \times \Delta$ is the sum of the valencies of u and v.*

Lemma 4 [3]. *If in a connected graph Γ there is an edge which is not contained in a 4-cycle, then Γ is prime.*

Theorem 2 [4]. *Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_k$ be connected relatively prime graphs. Then*

$$\mathrm{Aut}\,(\prod_{i=1}^{k} \Gamma_i) \cong \prod_{i=1}^{k} \mathrm{Aut}\,\Gamma_i\,.$$

Theorem 3 [4], *If a connected graph Γ has a prime factor decomposition, then the prime factor decomposition of Γ is unique up to isomorphisms.*

Corollary 1. *Any connected graph has the unique prime factor decomposition up to isomorphisms.*

REMARK 1. The above corollary does not necessarily hold for infinite graphs (cf. [4]).

Lemma 5. *Let Γ, Δ be hamiltonian graphs with $|V\Gamma|$ even. Then $\Gamma \times \Delta$ is a hamiltonian graph.*

Proof. Let $u_1-u_2-\cdots-u_q-u_1$ be a hamiltonian circuit of Γ and $v_1-v_2-\cdots-v_r-v_1$ be that of Δ, where q is even. Then we get the following hamiltonian circuit of $\Gamma \times \Delta$: $(u_1, v_1)-(u_1, v_2)-\cdots-(u_1, v_r)-(u_2, v_r)-(u_2, v_{r-1})-\cdots-(u_2, v_1)-(u_3, v_1)-(u_3, v_2)-\cdots-(u_3, v_r)-(u_4, v_r)-(u_4, v_{r-1})-\cdots-(u_4, v_1)-(u_5, v_1)-\cdots-(u_{p-1}, v_{r-1})-(u_{q-1}, v_r)-(u_q, v_r)-(u_q, v_{r-1})-\cdots-(u_q, v_1)-(u_1, v_1)$.

We note the following as a particular case of Theorem 2.

Lemma 6. *Let Γ, Δ be connected relatively prime graphs with* $\mathrm{Aut}\,\Gamma=1$ *and* $\mathrm{Aut}\,\Delta \neq 1$. *Then* $\mathrm{Aut}\,\Delta \cong \mathrm{Aut}(\Gamma \times \Delta)=\{\bar{\phi}: \phi \in \mathrm{Aut}\,\Delta\}$ *holds, where $\bar{\phi}$ is a permutation on $V\Gamma \times V\Delta$ defined by $\bar{\phi}(u, v)=(u, \phi(v))$ for $u \in V\Gamma$ and $v \in V\Delta$.*

Theorem 4 [1]. *Let Γ be a connected 3-regular graph which is not isomorphic to K_4. Then there exists a hamiltonian 3-regular graph Γ' such that Aut Γ'*

$\cong Aut\Gamma$, $|V\Gamma'|=6|V\Gamma|$, *the girth of* $\Gamma'\geqq 4$ *and that* Γ' *has an edge which is not contained in a* 4*-cycle.*

Lemma 7. *Let m be an even integer with $m\geqq 12$. Then there exists a connected* 3*-regular asymmetric graph* Γ *with* $|V\Gamma|=m$.

Proof. Let Γ be a graph defined by

$$V\Gamma = \{1, 2, \cdots, m\}\ ,$$
$$E\Gamma = \{[1, 2], [1, m], [1, m-1], [2, 3], [2, 6], [3, 4], [3, 5], [4, 5], [4, 6]\} \cup \{[i, i+2], [i+1, i+3], [i+2, i+3]: i = 5, 7, 9, \cdots, m-3\}\ .$$

Then Γ is a connected 3-regular graph with $|V\Gamma|=m$. Let σ be an automorphism of Γ. We want to show $\sigma=1$. Now the types of vertices 1, 2, 3, 4, 5, 6, 7, 8, $m-1$, and m are $(3, m/2, m/2+1)$, $(4, m/2, m/2+1)$, $(3, 4, 5)$, $(3, 4, 5)$, $(3, 5, 6)$, $(4, 5, 6)$, $(4, 5, 7)$, $(4, 5, 7)$, $(3, 4, 5)$ and $(3, 4, 5)$ respectively and the type of every other vertex i $(i=9, 10, \cdots, m-2)$ is $(4, 4, 6)$. Hence σ fixes 1, 2, 5 and 6. Since $V(N(2))\cap V(N(5))=\{3\}$ and $V(N(5))\cap V(N(6))=\{4\}$, σ fixes 3 and 4. By noticing $N(5)$ and $N(6)$, we find that σ fixes 7 and 8 respectively. So by noticing $N(7)$ and $N(8)$, we find that σ fixes 9 and 10 respectively. Similarly, σ fixes every other vertex of Γ.

Lemma 8. *Let m be a positive integer. Then there exist infinitely many connected* 3*-regular graphs* Γ *which are asymmetric, hamiltonian and prime such that* $|V\Gamma|$ *is divisible by m and the girth of* $\Gamma\geqq 4$.

Proof. Let m_1 be an even multiple of m with $m_1\geqq 12$, where there are infinitely many choices of m_1. By Lemma 7 there is a connected 3-regular asymmetric graph Γ_1 with $|V\Gamma_1|=m_1$. Then by Theorem 4 there exists a hamiltonian 3-regular asymmetric graph Γ' such that $|V\Gamma'|=6|V\Gamma_1|$ and the girth of $\Gamma'\geqq 4$ and that Γ' has an edge which is not contained in a 4-cycle. By Lemma 4 Γ' is also prime.

Lemma 9. *There is a connected asymmetric graph* Σ *such that some vertex of* Σ *has valency* 1 *and every other vertex has valency* 5.

Proof. Let Σ be a graph defined by

$$V\Sigma = \{1, 2, \cdots, 14\}\ ,$$
$$E\Sigma = \{[1, 5], [1, 7], [1, 10], [1, 12], [1, 13], [2, 3], [2, 5], [2, 6], [2, 9], [2, 13], [3, 5], [3, 7], [3, 8], [3, 13], [4, 7], [4, 8], [4, 11], [4, 12], [4, 13], [5, 6], [5, 11], [6, 7], [6, 9], [6, 10], [7, 9], [8, 10], [8, 11], [8, 12], [9, 10], [9, 12], [10, 11], [11, 12], [13, 14]\}.$$

Then Σ is a connected graph in which the vertex 14 has valency one and every other vertex has valency five. Let σ be an automorphism of Σ. Then σ fixes and $(4, g)$. Since $(3, g)$ is the unique vertex which is adjacent to $(2, g)$ and $(1, g)$, it is fixed by τ. By noticing $N((3, g))$ and $N((4, g))$, we find that τ fixes $(6, g)$ and $(2s+5, g)$ respectively. By our choice of S, the circuit $(2s+5, g)-(4, g)-(1, g)-(3, g)-(6, g)-(7, g)-\cdots-(2s+4, g)-(2s+5, g)$ is the unique shortest one which contains a path $(2s+5, g)-(4, g)-(1, g)-(3, g)-(6, g)$ but does not contain a vertex of type $(3, *, *)$ except $(1, g)$ and $(3, g)$. Hence τ fixes all vertices of the circuit, and $\tau(v)=v$ holds for any vertex v of $\Delta(g)$. Therefore τ fixes $(2i+5, x_i g)$ $(i=1, 2, \cdots, s)$. Hence we have $\tau(5, x_i g)=(5, x_i g)$ $(i=1, 2, \cdots, s)$, because if $\tau(5, x_j g)=(5, y)$, then by the similar argument to the above we have $\tau\,(2j+5, x_j g)=\sigma_{(x_j g)^{-1}y}(2j+5, x_j g)=(2j+5, y)$. Thus again by the similar argument we have that $\tau(v)=v$ holds for any vertex v of $\Delta(x_i g)$ $i=1, 2, \cdots, s)$. Hence we have $\tau=1$ because of $\langle x_1, x_2, \cdots, x_s\rangle=G$.

Lemma 11. *The following graph Γ_4 is a connected prime graph satisfying that $Aut\,\Gamma_4$ is isomorphic to G and acts semiregularly on $V\Gamma_4$ and that just $|G|$ vertices have valency* 3 *with type* $(4, 4, 4)$, *just $|G|$ vertices have valency* 3 *with type* $(4, 4, 6)$ *and every other vertex has valency* 4:

$$
\begin{aligned}
V\Gamma_4 &= V\Gamma_3 \cup \{(i', g)\colon i=2, 3, \cdots, 2s+5, g\in G\}\ ,\\
E\Gamma_4 &= E\Gamma_3 \cup \{[(i, g), (i', g)]\colon 2\leqq i\leqq 2s+5, g\in G\} \cup \{[(1, g), (5', g)],\\
&\qquad [(2', g), (3', g)], [(2', x_1 g), (3', g)], [(2', g), (5', g)],\\
&\qquad [(3', g), (6', g)], [(4', g), (5', g)], [(4', g), ((2s+5)', g)]\colon g\in G\}\\
&\quad \cup \{[(i', g), ((i+1)', g)]\colon 6\leqq i\leqq 2s+4, g\in G\} \cup\\
&\qquad \{[((2i+4)', g), ((2i+5)', x_i g)]\colon 1\leqq i\leqq s, g\in G\},
\end{aligned}
$$

where Γ_3 is the graph defined in Lemma 10.

Proof. Since the proof is similar to that of Lemma 10, we describe it briefly. Let g be any element in G and $\Delta(g)$ be a subgraph of Γ_4 induced by $\{(i, g), (j', g)\colon 1\leqq i\leqq 2s+5, 2\leqq j\leqq 2s+5\}$. Since $\Delta(g)$ is connected and $\langle x_1, x_2, \cdots, x_s\rangle=G$, Γ_4 is connected. Since $[(1, g), (3, g)]$ is not contained in a 4-cycle, Γ_4 is prime by Lemma 4. It is obvious that $(5, g)$ has valency 3 with type $(4, 4, 4)$, $(4', g)$ has valency 3 with type $(4, 4, 6)$, and any (i, g) $(i\neq 5)$ and any (j', g) $(j\neq 4)$ have valency 4. Now for any $h\in G$, let us define a bijection $\sigma_h\colon V\Gamma_4\to V\Gamma_4$ by $\sigma_h(i, g)=(i, gh)$ and $\sigma_h(i', g)=(i', gh)$. Then σ_h is an automorphism of Γ_4, and we have $\mathrm{Aut}\,\Gamma_4\geqq\{\sigma_h\colon h\in G\}\cong G$.

Let σ be any automorphism of Γ_4. From now on we are to show $\sigma=\sigma_h$ for some $h\in G$. For an element g in G there is an element g' in G with $\sigma(5, g)=(5, g')$. Let us set $h=g^{-1}g'$ and $\tau=\sigma_h^{-1}\sigma$, then we have $\sigma_h(5, g)=(5, g')$ and $\tau\in(\mathrm{Aut}\,\Gamma_4)_{(5, g)}$. If $\tau=1$ is shown, then $\mathrm{Aut}\,\Gamma_4=\{\sigma_h\colon h\in G\}$ holds, and $\mathrm{Aut}\,\Gamma_4$ acts semiregularly on $V\Gamma_4$. First we can easily find that τ fixes $(1, g)$, $(2, g)$, $(3, g)$, $(4, g)$, $(5, g)$, $(6, g)$, $(2s+5, g)$, $(2', g)$, $(3', g)$, $(4', g)$, $(5', g)$, $(6', g)$ and $((2s+5)', g)$ because of $\tau(5, g)=(5, g)$. Next by our choice of S, the circuit $(2s+5, g)-$

14, and so σ fixes 13 which is adjacent to 14. Since the adjacent vertices of 13 are 1, 2, 3, 4 and 14 and since $N(1), N(2), N(3)$ and $N(4)$ have just zero, four, two and three edges respectively, σ fixes 1, 2, 3 and 4. Therefore $V(N(1)) \cap V(N(2))=\{5, 13\}$ follows $\sigma(5)=5$ and so $V(N(1)) \cap V(N(3))=\{5, 7, 13\}$ follows $\sigma(7)=7$. Hence σ fixes 8 and 12, because $V(N(3)) \cap V(N(4))=\{7, 8, 13\}$ and $V(N(1)) \cap V(N(4))=\{7, 12, 13\}$ respectively. By noticing $N(1)$ and $N(4)$, we find that σ fixes 10 and 11 respectively. Then $V(N(5))=\{1, 2, 3, 6, 11\}$ follows $\sigma(6)=6$, and so σ fixes the rest 9. Thus we have $\sigma=1$.

REMARK 2. In §3 we often use similar arguments to those in Lemmas 7 and 9.

3. Proof of Theorem 1

Let G be a finite group. If $G=1$, Theorem 1 holds obivously by [2] or [3]. Hence hereafter we assume $G>1$, and we are to complete the proof of Theorem 1 by Propositions 1, 2, $\cdots$ and 8. Let $S=\{x_1, x_2, \cdots, x_s\}$ be a subset of G whose number of elements is minimum in all subsets which generate G.

Lemma 10. *The following graph Γ_3 is a connected prime graph satisfying that $Aut\,\Gamma_3$ is isomorphic to G and acts semiregularly on $V\Gamma_3$ and that just $|G|$ vertices have valency 2 with type (4) and every other vertex has valency 3:*

$$\begin{aligned} V\Gamma_3 = {} & \{(i, g) : i = 1, 2, \cdots, 2s+5, g \in G\}\ , \\ E\Gamma_3 = {} & \{[(1,g), (2,g)], [(1,g), (3,g)], [(1,g), (4,g)], [(2,g), (3,g)], \\ & [(2,g), (5,g)], [(3,g), (6,g)], [(4,g), (5,g)], [(4,g), (2s+5,g)] : \\ & g \in G\} \cup \{[(i,g), (i+1,g)] : 6 \leqq i \leqq 2s+4, g \in G\} \cup \\ & \{[(2i+4,g), (2i+5, x_i g)] : 1 \leqq i \leqq s, g \in G\}\ . \end{aligned}$$

Proof. Let g be any element in G and $\Delta(g)$ be a subgraph of Γ_3 induced by (i, g): $1 \leqq i \leqq 2s+5\}$. Then $\Delta(g)$ is connected. Since $\langle x_1, x_2, \cdots, x_s \rangle = G$ and since there exists an edge of which one end is in $\Delta(g)$ and the other end is in $\Delta(x_i g)$ for $i=1, 2, \cdots, s$, Γ_3 is connected. Since $[(1, g), (3, g)]$ is not contained in a 4-cycle, Γ_3 is prime by Lemma 4. It is obvious that $(5, g)$ has valency 2 with type (4) and (j, g) $(j \neq 5)$ has valency 3. Now for any $h \in G$, let us define a bijection σ_h: $V\Gamma_3 \to V\Gamma_3$ by $\sigma_h(i, g)=(i, gh)$. Then σ_h is an automorphism of Γ_3, and we have $\mathrm{Aut}\,\Gamma_3 \geqq \{\sigma_h : h \in G\} \cong G$.

Let σ be any automorphism of Γ_3. From now on we are to show $\sigma=\sigma_h$ for some $h \in G$. For an element g in G there is an element g' in G with $\sigma(5, g)=(5, g')$. Let us set $h=g^{-1}g'$ and $\tau=\sigma_h^{-1}\sigma$, then we have $\sigma_h(5, g)=(5, g')$ and $\tau \in (\mathrm{Aut}\,\Gamma_3)_{(5,g)}$. If $\tau=1$ is shown, then $\mathrm{Aut}\,\Gamma_3=\{\sigma_h : h \in G\}$ holds, and $\mathrm{Aut}\,\Gamma_3$ acts semiregularly on $V\Gamma_3$. Now the adjacent vertices of $(5, g)$ are $(2, g)$ and $(4, g)$ which have types $(3, *, *)$ and $(4, *, *)$ respectively. Hence τ fixes both of them. Therefore τ fixes $(1, g)$, because it is the unique vertex which is adjacent to $(2, g)$

$(4, g)-(1, g)-(3, g)-(6, g)-(7, g)-(8, g)-\cdots-(2s+4, g)-(2s+5, g)$ is the unique shortest one which contains a path $(2s+5, g)-(4, g)-(1, g)-(3, g)-(6, g)$ but contains neither a vertex of valency 3 nor a vertex of type $(3, *, *, *, *, *)$ except $(1, g)$ and $(3, g)$. Hence τ fixes all vertices of the circuit, and $\tau(v)=v$ holds for any vertex v of $\Delta(g)$. Therefore since τ fixes $(7, x_1 g)$, $(9, x_2 g)$, $(11, x_3 g)$, ... and $(2s+5, x_s g)$, $\tau(v)=v$ holds for any vertex v of $\Delta(x_i g)$ $(i=1, 2, \cdots, s)$ by the similar argument to the above. Hence we have $\tau=1$ because of $\langle x_1, x_2, \cdots, x_s\rangle=G$.

Lemma 12. *The following graph* Γ_5 *is a connected prime graph satisfying that* Aut Γ_5 *is isomorphic to* G *and acts semiregularly on* $V\Gamma_5$ *and that just* $|G|$ *vertices have valency* 4 *with type* (4, 4, 4, 4, 4, 5) *and every other vertex has valency* 5:

$$
\begin{aligned}
V\Gamma_5 = {} & V\Gamma_4 \cup \{(\bar{i}, g), (\overline{i'}, g): i = 2, 3, \cdots, 2s+5, g\in G\},\\
E\Gamma_5 = {} & E\Gamma_4 \cup \{[(i, g), (\bar{i}, g)], [(i', g), (\overline{i'}, g)], [(\bar{i}, g), (\overline{i'}, g)]:\\
& 2\leqq i\leqq 2s+5, g\in G\} \cup \{[(1, g), (\bar{5}, g)], [(\bar{2}, g), (\bar{3}, g)], [\bar{2}, g), (4', g)],\\
& [(\bar{2}, g), (\bar{5}, g)], [(\overline{2'}, g), (\overline{3'}, g)], [(\overline{2'}, x_1 g), (\overline{3'}, g)],\\
& [(\overline{2'}, g), (\overline{5'}, g)], [(\bar{3}, g), (\bar{4}, g)], [\bar{3}, g), (\bar{6}, g)], [(\overline{3'}, g), (\overline{6'}, g)],\\
& [(\bar{4}, g), (\bar{5}, g)], [(\bar{4}, g), (\overline{2s+5}, g)], [(\overline{4'}, g), (5, g)], [(\overline{4'}, g), (\overline{5'}, g)],\\
& [(\overline{4'}, g), (\overline{(2s+5)'}, g]\} \cup \{[(\bar{i}, g), (\overline{i+1}, g)], [(\overline{i'}, g), (\overline{(i+1)'}, g]:\\
& 6\leqq i\leqq 2s+4, g\in G\} \cup \{[(\overline{2i+4}, g), (\overline{2i+5}, x_i g)],\\
& [(\overline{(2i+4)'}, g), (\overline{(2i+5)'}, x_i g)]: 1\leqq i\leqq s, g\in G\},
\end{aligned}
$$

where Γ_4 *is the graph defined in Lemma* 11.

Proof. Since the proof is similar to that of Lemma 10, we describe it briefly. Let g be any element in G and $\Delta(g)$ be a subgraph of Γ_5 induced by $\{(i, g), (j', g), (\bar{j}, g), (\overline{j'}, g): 1\leqq i\leqq 2s+5, 2\leqq j\leqq 2s+5\}$. Since $\Delta(g)$ is connected and $\langle x_1, x_2, \cdots, x_s\rangle=G$, Γ_5 is connected. Since $[(1, g), (3, g)]$ is not contained in a 4-cycle, Γ_5 is prime by Lemma 4. It is clear that $(\overline{5'}, g)$ has valency 4 with type (4, 4, 4, 4, 4, 5) and that any (i, g), any (i', g), any $(\bar{i}, g)$ and any $(\overline{j'}, g)$ $(j\neq 5)$ have valency 5. Now for any $h\in G$, let us define a bijection σ_h: $V\Gamma_5\to V\Gamma_5$ by $\sigma_h(i, g)=(i, gh)$, $\sigma_h(i', g)=(i', gh)$, $\sigma_h(\bar{i}, g)=(\bar{i}, gh)$ and $\sigma_h(\overline{i'}, g)=(\overline{i'}, gh)$. Then σ_h is an automorphism of Γ_5, and we have Aut $\Gamma_5\geqq\{\sigma_h: h\in G\}\cong G$.

Let σ be any automorphism of Γ_5. From now on we are to show $\sigma=\sigma_h$ for some $h\in G$. For an element g in G there is an element g' in G with $\sigma(\overline{5'}, g)=(\overline{5'}, g')$. Let us set $h=g^{-1}g'$ and $\tau=\sigma_h^{-1}\sigma$, then we have $\sigma_h(\overline{5'}, g)=(\overline{5'}, g')$ and $\tau\in(\mathrm{Aut}\,\Gamma_5)_{(\overline{5'}, g)}$. If $\tau=1$ is shown, then Aut $\Gamma_5=\{\sigma_h: h\in G\}$ holds, and Aut Γ_5 acts semiregularly on $V\Gamma_5$. Let A be the set of vertices of valency 4, B be the set of vertices of type $(3, *, *, \cdots, *)$ and C be the set of vertices which are adjacent to some vertex in B. Then $A=\{(\overline{5'}, g): g\in G\}$, $B=\{(1, g), (2, g), (3, g): g\in G\}$ and $C=\{(1, g), (2, g), (\bar{2}, g), (2', g), (3, g), (\bar{3}, g), (3', g), (4, g), (5, g), (\bar{5}, g), (5', g), (6, g): g\in G\}$ hold. Now the adjacent vertices of $(\overline{5'}, g)$ are $(\overline{2'}, g)$, $(\overline{4'}, g)$, $(\bar{5}, g)$ and $(5', g)$ which are at distances two, three, one and one from B respectively. Moreover the types of $(\bar{5}, g)$ and $(5', g)$ are $(4, 4, \cdots, 4)$ and $(4, 4, \cdots, 4, 5)$ respectively. Hence we can easily find that τ fixes $(1, g)$, (i, g), (i', g), $(\bar{i}, g)$ and $(\overline{i'}, g)$ $(i=2, 3, 4, 5, 6, 2s+5)$. Next by our choice of S, the circuit $(2s+5, g)-(4, g)-(1, g)-(3, g)-(6, g)-(7, g)-(8, g)-\cdots-(2s+4, g)-(2s+5, g)$ is the unique shortest one which contains a path $(2s+5, g)-(4, g)-(1, g)-(3, g)-(6, g)$ but contains neither a vertex in A nor a vertex in B except $(1, g)$ and $(3, g)$ nor a vertex in C except $(6, g)$ and $(4, g)$. Hence τ fixes all vertices of the circuit, and $\tau(v)=v$ holds for any vertex v of $\Delta(g)$. Therefore since τ fixes $(7, x_1 g)$, $(9, x_2 g)$, $(11, x_3 g)$, $\cdots$ and $(2s+5, x_s g)$, $\tau(v)=v$ holds for any vertex v of $\Delta(x_i g)$ $(i=1, 2, \cdots, s)$ by the similar argument to the above. Hence we have $\tau=1$ because of $\langle x_1, x_2, \cdots, x_s\rangle=G$.

Lemma 13. *Let m_0 be max $\{|G|, 3\}$ and i_0 be an integer with $0\leqq i_0\leqq 2$. Suppose that Theorem 1 holds for $n=m_0+i_0$. Then Theorem 1 holds for $n=m_0+i_0+3j$ $(j=0, 1, 2, \cdots)$.*

Proof. Let M be any positive integer. Then there exist different connected (m_0+i_0)-regular graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_M$ each of which has t vertices $\alpha_1, \alpha_2, \cdots, \alpha_t$ with $\mathrm{Aut}\,\Gamma_i=(\mathrm{Aut}\,\Gamma_i)_{\alpha_1\alpha_2\cdots\alpha_t}\cong G$. By Lemma 8 and Corollary 1 there exists a connected 3-regular graph Γ_0 which is asymmetric and prime and which is not a prime factor of Γ_i $(i=1, 2, \cdots, M)$. Hence by Lemma 6 we get different connected (m_0+i_0+3)-regular graphs $\Pi_1, \Pi_2, \cdots, \Pi_M$ each of which has t vertices $\alpha_1, \alpha_2, \cdots, \alpha_t$ with $\mathrm{Aut}\,\Pi_i=(\mathrm{Aut}\,\Pi_i)_{\alpha_1\alpha_2\cdots\alpha_t}\cong G$, where $\Pi_i=\Gamma_0\times\Gamma_i (1\leqq i\leqq M)$. If we continue the above argument, then for each $j=2, 3, 4, \cdots$ we get different connected (m_0+i_0+3j)-regular graphs $\Delta_1, \Delta_2, \cdots, \Delta_M$ each of which has t vertices $\alpha_1, \alpha_2, \cdots, \alpha_t$ with $\mathrm{Aut}\,\Delta_i=(\mathrm{Aut}\,\Delta_i)_{\alpha_1\alpha_2\cdots\alpha_t}\cong G$. Since M is any positive integer, we complete the proof.

Proposition 1. *Theorem 1 holds in the case where $|G|\geqq 4$ and $n\equiv 0 \pmod 3$.*

Proof. Let m be an integer with $|G|\leqq m\leqq |G|+2$ and $m\equiv 0 \pmod 3$. By Lemma 13 in order to prove Proposition 1, it is sufficient to prove it for $n=m$. Let us set $d=m/3$. By Lemma 8 there exist non-isomorphic connected 3-regular graphs $\Gamma_1, \Gamma_2, \cdots, \Gamma_{d-1}$ such that they are asymmetric, hamiltonian and prime and the girth of every Γ_i is at least 4 and $|V\Gamma_{d-1}|$ is divisible by $(2m-1)$, where the number of choices of $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_{d-1}\}$ is infinite. Let us set $\Gamma_0=\Gamma_1\times\Gamma_2\times\cdots\times\Gamma_{d-1}$. Then by Lemmas 3, 5 and Theorem 2, Γ_0 is a hamiltonian $(m-3)$-regular asymmetric graph with the girth of $\Gamma_0\geqq 4$ and $2(m-1)\,\big|\,|V\Gamma_0|$. Let us denote $V\Gamma_0=\{1, 2, \cdots, q\}$, and we may assume that $1-2-3-\cdots-q-1$

is a hamiltonian circuit of Γ_0.

On the other hand by Lemma 10, there exists a connected prime graph Γ_d satisfying that $\mathrm{Aut}\,\Gamma_d$ is isomorphic to G and acts semiregularly on $V\Gamma_d$ and that just $|G|$ vertices have valency 2 with type (4) and every other vertex has valency 3. Let f be a bijection from G to the set of vertices of valency 2 in $V\Gamma_d$. We define a graph Π_0 as follows:

$$\begin{aligned} V\Pi_0 = {} & V(\Gamma_0\times\Gamma_d)\cup\{v_{ig}: i=1,2,\cdots,q/(m-1), g\in G\}\cup \\ & \{\alpha_1,\alpha_2,\cdots,\alpha_{q/(m-1)}\}, \\ E\Pi_0 = {} & E(\Gamma_0\times\Gamma_d)\cup\{[v_{ig},(j,f(g))]: 1\leqq i\leqq q/(m-1), \\ & (i-1)(m-1)+1\leqq j\leqq i(m-1), g\in G, (j,f(g))\in V(\Gamma_0\times\Gamma_d)\} \\ & \cup\{[\alpha_i, v_{ig}]: 1\leqq i\leqq q/(m-1), g\in G\}. \end{aligned}$$

Now we divide our argument into three cases: $m=|G|$, $m=|G|+1$ and $m=|G|+2$.

First let us suppose $m=|G|$. Obviously Π_0 is a connected m-regular graph. We show that $\mathrm{Aut}\,\Pi_0$ is isomorphic to G and fixes $\alpha_1,\alpha_2,\cdots,\alpha_{q/(m-1)}$. By Lemma 6,

$$\mathrm{Aut}(\Gamma_0\times\Gamma_d)=\{\bar{\phi}: \phi\in\mathrm{Aut}\,\Gamma_d\}\cong\mathrm{Aut}\,\Gamma_d\cong G\,,$$

where $\bar{\phi}$ is a permutation on $V(\Gamma_0\times\Gamma_d)$ with $\bar{\phi}(w,y)=(w,\phi(y))$. Now we extend $\bar{\phi}$ to a permutation $\hat{\phi}$ on $V\Pi_0$ as follows:

$$\begin{aligned} &\hat{\phi}(v)=\bar{\phi}(v) \quad \text{for} \quad v\in V(\Gamma_0\times\Gamma_d)\,, \\ &\hat{\phi}(v_{ig})=v_{i\tau(g)}\,(1\leqq i\leqq q/(m-1), g\in G)\,, \quad \text{where} \quad \tau=f^{-1}\phi f\,, \\ &\hat{\phi}(\alpha_j)=\alpha_j(1\leqq j\leqq q/(m-1))\,. \end{aligned}$$

Then $\hat{\phi}$ is an automorphism of Π_0, and we have

$$\mathrm{Aut}\,\Pi_0\geqq\{\hat{\phi}: \phi\in\mathrm{Aut}\,\Gamma_d\}\cong\mathrm{Aut}\,\Gamma_d\cong G\,.$$

Let σ be any automorphism of Π_0. We want to show $\sigma=\hat{\phi}$ for some $\phi\in\mathrm{Aut}\,\Gamma_d$. For any $v_{ig}(1\leqq i\leqq q/(m-1), g\in G)$ there exist just $m-1$ incident edges e such that e is contained in a 3-cycle, because $(1,f(g))-(2,f(g))-(3,f(g))-\cdots-(q,f(g))-(1,f(g))$ is a hamiltonian circuit of the subgraph induced by $\{(j,f(g)): 1\leqq j\leqq q\}$ and α_i is not contained in a 3-cycle. Conversely any vertex other than $v_{ig}(1\leqq i\leqq q/(m-1), g\in G)$ has not the same property, because the girth of Γ_0 is at least 4 and the type of $f(g)$ $(g\in G)$ is (4) in Γ_d. Hence σ fixes $\{v_{ig}: 1\leqq i\leqq q/(m-1), g\in G\}$ as a set. Therefore since σ fixes $V(\Gamma_0\times\Gamma_d)$ as a set, the restriction of σ to $V(\Gamma_0\times\Gamma_d)$ is an automorphism of $\Gamma_0\times\Gamma_d$, that is, $\bar{\phi}$ for some $\phi\in\mathrm{Aut}\,\Gamma_d$. Hence we find $\sigma=\hat{\phi}$ easily. So $\mathrm{Aut}\,\Pi_0=\{\hat{\phi}: \phi\in\mathrm{Aut}\,\Gamma_d\}$ holds. Thus Π_0 is a connected m-regular graph of which the automorphism group is isomorphic to G and fixes $\alpha_1,\alpha_2,\cdots,\alpha_{q/(m-1)}$.

Next when $m=|G|+1$, $|G|+2$, we define graphs Π_1, Π_2 respectively as follows:

$$V\Pi_1 = V\Pi_0,$$
$$E\Pi_1 = E\Pi_0 \cup \{[\alpha_{2i-1}, \alpha_{2i}]: 1 \leqq i \leqq q/2(m-1)\},$$
$$V\Pi_2 = V\Pi_0,$$
$$E\Pi_2 = E\Pi_0 \cup \{[\alpha_{i-1}, \alpha_i]: 2 \leqq i \leqq q/(m-1)\} \cup \{[\alpha_{q/(m-1)}, \alpha_1]\}.$$

Then by the similar argument to that for Π_0, we have that Π_i is a connected m-regular graph of which the automorphism group is isomorphic to G and fixes $\alpha_1, \alpha_2, \cdots, \alpha_{q/(m-1)}$ $(i=1, 2)$.

Since the number of choices of $\{\Gamma_1, \Gamma_2, \cdots, \Gamma_{d-1}\}$ is infinite, the one of choices of Γ_0 is infinite. In particular for an arbitrary positive integer t, the number of choices of $q(=|V\Gamma_0|)$ with $q/(m-1) \geqq t$ is also infinite. Hence Theorem 1 holds in the case where $|G| \geqq 4$ and $n \equiv 0 \pmod 3$.

Proposition 2. *Theorem* 1 *holds in the case where* $|G| \geqq 5$ *and* $n \equiv 1 \pmod 3$.

Proof. Since the proof is similar to that of Proposition 1, we describe it briefly. Let m be an integer with $|G| \leqq m \leqq |G|+2$ and $m \equiv 1 \pmod 3$. By Lemma 13 in order to prove Proposition 2, it is sufficient to prove it for $n=m$. Let us set $d=(m-1)/3$. By the same argument to the first part of the proof of Proposition 1, we get infinitely many hamiltonian $(m-4)$-regular asymmetric graphs Γ_0 with the girth of $\Gamma_0 \geqq 4$ and $2(m-1) \,\|\, |V\Gamma_0|$. Let us denote $V\Gamma_0 = \{1, 2, \cdots, q\}$, and we may assume that $1-2-3-\cdots-q-1$ is a hamiltonian circuit of Γ_0.

On the other hand by Lemma 11, there exists a connected prime graph Γ_d satisfying that $\mathrm{Aut}\,\Gamma_d$ is isomorphic to G and acts semiregularly on $V\Gamma_d$ and that just $|G|$ vertices have valency 3 with type (4, 4, 4), just $|G|$ vertices have valency 3 with type (4, 4, 6) and every other vertex has valency 4. Let f_1 and f_2 be bijections from G to the set of vertices of type (4, 4, 4) and to the set of vertices of type (4, 4, 6) respectively. We define a graph Π_0 as follows:

$$V\Pi_0 = V(\Gamma_0 \times \Gamma_d) \cup \{v_{ig}, u_{ig}: i=1, 2, \cdots, q/(m-1), g \in G\} \cup \{\alpha_i, \beta_i: 1 \leqq i \leqq q/(m-1)\},$$
$$E\Pi_0 = E(\Gamma_0 \times \Gamma_d) \cup \{[v_{ig}, (j, f_1(g))], [u_{ig}, (j, f_2(g))]: 1 \leqq i \leqq q/(m-1), (i-1)(m-1)+1 \leqq j \leqq i(m-1), g \in G\} \cup \{[\alpha_i, v_{ig}], [\beta_i, u_{ig}]: 1 \leqq i \leqq q/(m-1), g \in G\}.$$

Now we divide our argument into three cases: $m=|G|$, $m=|G|+1$ and $m=|G|+2$.

First let us suppose $m=|G|$. Obviously Π_0 is a connected m-regular graph. By Lemma 6,

$$\mathrm{Aut}(\Gamma_0 \times \Gamma_d) = \{\bar{\phi}: \phi \in \mathrm{Aut}\,\Gamma_d\} \cong \mathrm{Aut}\,\Gamma_d \cong G,$$

where $\bar{\phi}$ is a permutation on $V(\Gamma_0 \times \Gamma_d)$ with $\bar{\phi}(w, y) = (w, \phi(y))$. Now we extended $\bar{\phi}$ to a permutation $\hat{\phi}$ on $V\Pi_0$ as follows:

$$\hat{\phi}(v) = \bar{\phi}(v) \quad \text{for} \quad v \in V(\Gamma_0 \times \Gamma_d),$$
$$\hat{\phi}(v_{ig}) = v_{i\tau(g)}\ (1 \leqq i \leqq q/(m-1), g \in G), \quad \text{where} \quad \tau = f_1^{-1}\phi f_1,$$
$$\hat{\phi}(u_{ig}) = u_{i\rho(g)}\ (1 \leqq i \leqq q/(m-1), g \in G), \quad \text{where} \quad \rho = f_2^{-1}\phi f_2,$$
$$\hat{\phi}(\alpha_j) = \alpha_j\ (1 \leqq j \leqq q/(m-1)),$$
$$\hat{\phi}(\beta_j) = \beta_j\ (1 \leqq j \leqq q/(m-1)).$$

By the similar argument to that in the proof of Proposition 1, we have $\mathrm{Aut}\,\Pi_0 = \{\hat{\phi}: \phi \in \mathrm{Aut}\,\Gamma_d\} \cong \mathrm{Aut}\,\Gamma_d \cong G$. Thus Π_0 is a connected m-regular graph of which the automorphism group is isomorphic to G and fixes $\alpha_1, \alpha_2, \cdots, \alpha_{q/(m-1)}, \beta_1, \beta_2, \cdots, \beta_{q/(m-1)}$.

Next when $m = |G|+1$, $|G|+2$, we define graphs Π_1, Π_2 respectively as follows:

$$V\Pi_1 = V\Pi_0,$$
$$E\Pi_1 = E\Pi_0 \cup \{[\alpha_{2i-1}, \alpha_{2i}], [\beta_{2i-1}, \beta_{2i}]: 1 \leqq i \leqq q/2(m-1))\},$$
$$V\Pi_2 = V\Pi_0,$$
$$E\Pi_2 = E\Pi_0 \cup \{[\alpha_{i-1}, \alpha_i], [\beta_{i-1}, \beta_i]: 2 \leqq i \leqq q/(m-1)\} \cup \{[\alpha_{q/(m-1)}, \alpha_1], [\beta_{q/(m-1)}, \beta_1\}.$$

Then by the similar argument to that for Π_0, we have that Π_i is a connected m-regular graph of which the automorphism group is isomorphic to G and fixes $\alpha_1, \alpha_2, \cdots, \alpha_{q/(m-1)}, \beta_1, \beta_2, \cdots, \beta_{q/(m-1)}$ $(i=1, 2)$.

By the similar argument to that in the last part of the proof of Proposition 1, we complete the one of Proposition 2.

Proposition 3. *Theorem* 1 *holds in the case where* $|G| \geqq 6$ *and* $n \equiv 2 \pmod 3$.

Proof. Since the proof is almost same as that of Proposition 1 except using Lemma 12 in place of Lemma 10, we omit it.

REMARK 3. The following Propositions 4, 5, 6, 7 and 8 show that Theorem 1 holds for the cases (i) $|G|=5$, (ii) G is a cyclic group with $|G|=4$, (iii) G is an abelian group of type (2,2), (iv) $|G|=3$ and (v) $|G|=2$ respectively. Though in each proof of the propositions we show for an arbitrary integer t that the existence of a connected n-regular graph Γ satisfying $\mathrm{Aut}\,\Gamma = (\mathrm{Aut}\,\Gamma)_{\alpha_1\alpha_2\cdots\alpha_t} \cong G$ for some t vertices $\alpha_1, \alpha_2, \cdots, \alpha_t$ of Γ, the existence of an infinite number of such graphs is found by the argument of each proof.

Proposition 4. *If* $|G|=5$, *Theorem* 1 *holds.*

Proof. First let us suppose $n=5$. Let $\{a_k\}$ be a sequence defined by $a_k=(k+1)k/2$ and m be an integer with $2m\geqq\max\{4, 2t\}$. Let us set

$$A=\{a_{2k-1}\colon 1\leqq k\leqq m\}\cup\{a_{2m}\},\ B=\{a_{2k}\colon 1\leqq k\leqq m-1\}.$$

Let Π be a graph defined by

$$\begin{aligned} V\Pi=&\{\alpha_h\colon h\in A\}\cup\{(i,j)\colon i\in\{1,2,3,\cdots,a_{2m}\}-A,\ j=1,2,\cdots,5\},\\ E\Pi=&\{[\alpha_1,(2,j)]\colon 1\leqq j\leqq 5\}\cup\{[\alpha_h,(h-1,j)\colon h\in A-\{1\},1\leqq j\leqq 5\}\cup\\ &\{[(a_i,j),(a_i-1,j)],[(a_i,j),(a_i+1,j)],[(a_i,j),(a_{i+1}+1,j)],\\ &[(a_i,j),(a_i,j+1)]\colon a_i\in B,1\leqq j\leqq 5\}\ (\text{put } j+1=1 \quad\text{if}\quad j+1=6)\\ &\cup\{[(i,j),(i-1,j)\colon \text{both } i \text{ and } i-1 \text{ are elements in}\\ &\{(1,2,3,\cdots,a_{2m}\}-(A\cup B),1\leqq j\leqq 5\}.\end{aligned}$$

Then Π is a connected graph in which every $\alpha_h(h\in A)$ and every (i,j) ($i\in B$, $1\leqq j\leqq 5$) have valency 5 and every other vertex has valency 2. Let us denote $X=\{\alpha_h\colon h\in A\}$, $Y=\{(i,j)\colon i\in\{1,2,3,\cdots,a_{2m}\}-A,\ 1\leqq j\leqq 5\}$, $Y_1=\{(i,j)\colon i\in B,\ 1\leqq j\leqq 5\}$ and $Y_2=Y-Y_1$.

Let Δ be a graph defined by

$$\begin{aligned} V\Delta=&\{v_0,v_1,v_2,\cdots,v_{10}\},\\ E\Delta=&\{[v_0,v_1],[v_0,v_2],[v_1,v_2],[v_1,v_3],[v_1,v_4],[v_1,v_7],[v_2,v_6],\\ &[v_2,v_7],[v_2,v_8],[v_3,v_4],[v_3,v_7],[v_3,v_8],[v_3,v_9],[v_4,v_5],\\ &[v_4,v_6],[v_4,v_7],[v_5,v_6],[v_5,v_7],[v_5,v_8],[v_5,v_9],[v_6,v_8],\\ &[v_6,v_9],[v_8,v_9],[v_9,v_{10}]\}.\end{aligned}$$

Then Δ is a connected graph in which v_{10} has valency 1, v_0 has valency 2 and every other vertex has valency 5. We remark that $\partial(v_0, v_{10})=4$ and that any vertex of Δ other than v_{10} is contained in a 3-cycle. Furthermore we can easily find $\mathrm{Aut}\,\Delta=1$.

Let Γ be a graph which is attached for each $i\in\{1,2,3,\cdots,a_{2m}\}-(A\cup B)$ and each $j=1,2,\cdots,5$ a graph isomorphic with Δ to Π so as to correspond the vertex v_0 of Δ to the vertex (i,j) of Π and the vertex v_{10} of Δ to the vertex $(i,j+1)$ of Π (put $j+1=1$ if $j+1=6$). Then Γ is a connected 5-regular graph, and we are to show that $\mathrm{Aut}\,\Gamma$ has order 5 and fixes all $\alpha_h\in X$. Let us denote $Z=V\Gamma-(X\cup Y)$. Now let τ be an automorphism of Π defined by

$$\begin{aligned} &\tau(\alpha_h)=\alpha_h \quad \text{for} \quad \alpha_h\in X,\\ &\tau(i,j)=(i,j+1)\ (\text{put } j+1=1 \text{ if } j+1=6) \quad \text{for} \quad (i,j)\in Y.\end{aligned}$$

Then τ has order 5 and is uniquely extended to an automorphism $\bar{\tau}$ of Γ. Of course $|\bar{\tau}|=5$ and $\mathrm{Aut}\,\Gamma\geqq\langle\bar{\tau}\rangle$ hold.

Let σ be any automorphism of Γ. Since in Γ the type of $\alpha_h\in X$ is $(5,5,5,5,5,*,*,*,*,*)$ ($h=1$) or $(6,6,6,6,6,*,*,*,*,*)$ ($h\neq 1$) and the type of any vertex in $Z\cup Y_2$ is $(3,*,*,\cdots,*)$ and the type of any vertex in Y_1 is different from the above types, σ fixes X, $Z \cup Y_2$ and Y_1 as a set respectively. Therefore σ fixes all $\alpha_h \in X$, because $\partial(\alpha_h, Y_1) \neq \partial(\alpha_k, Y_1)$ necessarily holds for $h \neq k \in A$. Hence for each $i \in \{1, 2, 3, \cdots, a_{2m}\} - A$, σ fixes $\{(i, j): 1 \leqq j \leqq 5\}$ as a set. Hence we can find $\sigma \in \langle \bar{\tau} \rangle$. Thus Proposition 4 holds for $n=5$.

On the other hand, Proposition 4 holds for $n=6$, 7 by Propositions 1, 2 respectively. Hence we complete the proof by Lemma 13.

Proposition 5. *If G is a cyclic group with $|G|=4$, Theorem 1 holds.*

Proof. First let us suppose $n=4$. Let $\{a_k\}$ be a sequence defined by $a_k = (k+1)k/2$ and m be an integer with $3m+2 \geqq \max\{5, 2t\}$. Let us set

$$A = \{a_{3k-2}, a_{3k}: 1 \leqq k \leqq m\} \cup \{a_{3m+1}, a_{3m+2}\}, \ B = \{a_{3k-1}: 1 \leqq k \leqq m\} .$$

Let Π be a graph defined by

$$\begin{aligned} V\Pi = \ & \{\alpha_h: h \in A\} \cup \{(i, j): i \in \{1, 2, 3, \cdots, a_{3m+2}\} - A, j = 1, 2, 3, 4\}, \\ E\Pi = \ & \{[\alpha_1, (2, j)]: 1 \leqq j \leqq 4\} \cup \{[\alpha_h, (h-1, j)]: h \in A - \{1\}, 1 \leqq j \leqq 4\} \cup \\ & \{[(a_i, j), (a_i - 1, j)], [(a_i, j), (a_i + 1, j)], [(a_i, j), (a_{i+1} + 1, j)], \\ & [(a_i, j), (a_{i+2} + 1, j)]: a_i \in B, 1 \leqq j \leqq 4\} \cup \{[(i, j), (i-1, j)\text{: both} \\ & i \text{ and } i-1 \text{ are elements in } \{1, 2, 3, \cdots, a_{3m+2}\} - (A \cup B), 1 \leqq j \leqq 4\}. \end{aligned}$$

Then Π is a connected graph in which every α_h $(h \in A)$ and every (i, j) $(i \in B, 1 \leqq j \leqq 4)$ have valency 4 and every other vertex has valency 2. Let us denote $X = \{\alpha_h: h \in A\}$, $Y = \{(i, j): i \in \{1, 2, 3, \cdots, a_{3m+2}\} - A, 1 \leqq j \leqq 4\}$, $Y_1 = \{(i, j): i \in B, 1 \leqq j \leqq 4\}$ and $Y_2 = Y - Y_1$.

Let Δ be a graph defined by

$$\begin{aligned} V\Delta = \ & \{v_0, v_1, v_2, \cdots, v_{12}\}, \\ E\Delta = \ & \{[v_0, v_1], [v_1, v_2], [v_1, v_3], [v_1, v_{10}], [v_2, v_3], [v_2, v_4], [v_2, v_6], \\ & [v_3, v_5], [v_3, v_7], [v_4, v_6], [v_4, v_8], [v_4, v_{10}], [v_5, v_7], [v_5, v_9], \\ & [v_5, v_{11}], [v_6, v_7], [v_6, v_8], [v_7, v_9], [v_8, v_9], [v_8, v_{10}], [v_9, v_{11}], \\ & [v_{10}, v_{11}], [v_{11}, v_{12}]\}. \end{aligned}$$

Then Δ is a connected graph in which v_0 and v_{12} have valency 1 and every other vertex has valency 4. We remark that $\partial(v_0, v_{12})=4$ and that any vertex of Δ other than v_0 and v_{12} is contained in a 3-cycle. Furthermore we can easily find Aut $\Delta = 1$.

Let Γ be a graph which is attached for each $i \in \{1, 2, 3, \cdots, a_{3m+2}\} - (A \cup B)$ and each $j=1, 2, 3, 4$ a graph isomorphic with Δ to Π so as to correspond the vertex v_0 of Δ to the vertex (i, j) of Π and the vertex v_{12} of Δ to the vertex $(i, j+1)$ of Π (put $j+1=1$ if $j+1=5$). Then Γ is a connected 4-regular graph, and we are to show that Aut Γ is a cyclic group of order 4 and fixes all $\alpha_h \in X$. Let us denote $Z = V\Gamma - (X \cup Y)$. Now let τ be an automorphism of Π defined by

$$\tau(\alpha_h)=\alpha_h \quad \text{for} \quad \alpha_h \in X,$$
$$\tau(i,j)=(i,j+1)\ (\text{put } j+1=1 \text{ if } j+1=5) \quad \text{for} \quad (i,j)\in Y.$$

Then τ has order 4 and is uniquely extended to an automorphism $\bar{\tau}$ of Γ. Of course $|\bar{\tau}|=4$ and Aut $\Gamma \geqq \langle\bar{\tau}\rangle$ holds.

Let σ be any automorphism of Γ. Since in Γ the type of any $\alpha_h\in X$ is $(6, 6, 6, 6, *, *)$ and the type of any vertex in Y_1 is not $(6, 6, 6, 6, *, *)$ and since $\{u\in V\Gamma\colon \partial(u, C)\geqq 2$ holds for any 3-cycle $C\}=X\cup Y_1$ holds, σ fixes X, Y_1 and $Z\cup Y_2$ as a set respectively. Therefore σ fixes all $\alpha_h\in X$, because $\partial(\alpha_h, Y_1)\neq \partial(\alpha_k, Y_1)$ necessarily holds for $h\neq k\in A$. Hence for each $i\in\{1, 2, 3, \cdots, a_{3m+2}\}-A$, σ fixes $\{(i,j)\colon 1\leqq j\leqq 4\}$ as a set. Hence we can find $\sigma\in\langle\bar{\tau}\rangle$. Thus Proposition 5 holds for $n=4$.

Next let us suppose $n=5$. Let $\bar{\Gamma}$ be a graph which is attached for each vertex u of the above Γ a graph isomorphic with Σ in Lemma 9 to Γ so as to correspond the vertex of valency one of Σ to u of Γ. So we remark that the set of cut-vertices of $\bar{\Gamma}$ is $V\Gamma$. Hence any automorphism of $\bar{\Gamma}$ induces an automorphism of Γ. Conversely any automorphism of Γ is uniquely extended to an automorphism of $\bar{\Gamma}$. Hence Aut $\bar{\Gamma}$ is a cyclic group of order 4 and fixes all $\alpha_h\in X$. Thus Proposition 5 holds for $n=5$.

On the other hand, Proposition 5 holds for $n=6$ by Proposition 1. Hence we complete the proof by Lemma 13.

Proposition 6. *If G is an abelian group of type* (2,2), *Theorem* 1 *holds.*

Proof. First let us suppose $n=4$. Let $\{a_k\}$, m, A, B, Π, X, Y, Y_1 and Y_2 be the same as in Proposition 5. Let Δ be a garph defined by

$$V\Delta=\{v_1, v_2, \cdots, v_9\},$$
$$E\Delta=\{[v_1, v_5], [v_5, v_6], [v_6, v_2], [v_2, v_4], [v_4, v_8], [v_8, v_7], [v_7, v_3], [v_3, v_1], [v_5, v_7], [v_6, v_8], [v_5, v_9], [v_6, v_9], [v_7, v_9], [v_8, v_9]\}.$$

Then Δ is a connected graph in which v_1, v_2, v_3 and v_4 have valency 2 and every other vertex has valency 4. We can easily find that Aut Δ is an abelian group $\langle\mu\rangle\times\langle\eta\rangle$ of type (2,2), where

$$\mu=(v_1, v_3)(v_5, v_7)(v_6, v_8)(v_2, v_4),$$
$$\eta=(v_1, v_2)(v_3, v_4)(v_5, v_6)(v_7, v_8).$$

Let Γ be a graph which is attached for each $i\in\{1, 2, 3, \cdots, a_{3m+2}\}-(A\cup B)$ a graph isomorphic with Δ to Π so as to correspond vertices v_1, v_2, v_3 and v_4 of Δ to vertices $(i, 1)$, $(i, 2)$, $(i, 3)$ and $(i, 4)$ of Π respectively. Then Γ is a connected 4-regular graph. Let us denote $Z=V\Gamma-(X\cup Y)$. Now μ and η uniquely determine automorphisms $\bar{\mu}$ and $\bar{\eta}$ of Γ respectively, where

$$\bar{\mu}(\alpha_k)=\alpha_k,\ \bar{\eta}(\alpha_k)=\alpha_k \quad \text{for} \quad \alpha_k\in X,$$

$$\bar{\mu}(i, 1) = (i, 3), \bar{\mu}(i, 3) = (i, 1), \bar{\mu}(i, 2) = (i, 4), \bar{\mu}(i, 4) = (i, 2),$$
$$\bar{\eta}(i, 1) = (i, 2), \bar{\eta}(i, 2) = (i, 1), \bar{\eta}(i, 3) = (i, 4), \bar{\eta}(i, 4) = (i, 3)$$

for $i \in \{1, 2, 3, \cdots, a_{3m+2}\} - A$. We want to show that $\mathrm{Aut}\,\Gamma$ is equal to an abelian group $\langle \bar{\mu}, \bar{\eta} \rangle$ of type (2, 2). Let σ be any automorphism of Γ. Since in Γ $\{u \in V\Gamma$: the type of u is $(3, 3, 5, 5, 6, 6)\} = X$ and $\{u \in V\Gamma$: the type of u is $(6, *, *, *, *, *)\} = Y_1$ hold, σ fixes X, Y_1 and $Z \cup Y_2$ as a set respectively. Therefore σ fixes all $\alpha_h \in X$, because $\partial(\alpha_h, Y_1) \neq \partial(\alpha_k, Y_1)$ necessarily holds for $h \neq k \in A$. Hence for each $i \in \{1, 2, 3, \cdots, a_{3m+2}\} - A$, σ fixes $\{(i, j)\colon 1 \leqq j \leqq 4\}$ as a set. Hence we can find $\sigma \in \langle \bar{\mu}, \bar{\eta} \rangle$. Thus Proposition 6 holds for $n=4$.

Now Proposition 6 holds for $n=5$ by the similar argument to that in the rear part of the proof of Proposition 5. Furthermore Proposition 6 holds for $n=6$ by Proposition 1. Hence we complete the proof by Lemma 13.

Proposition 7. *If* $|G| = 3$, *theorem* 1 *holds.*

Proof. First let us suppose $n=3$. Let $\{a_k\}$, m, A and B be the same as in Proposition 4. Let Π be a graph defined by

$$\begin{aligned} V\Pi &= \{\alpha_h\colon h \in A\} \cup \{(i,j)\colon i \in \{1, 2, 3, \cdots, a_{2m}\} - A, j = 1, 2, 3\}, \\ E\Pi &= \{[\alpha_1, (2,j)]\colon 1 \leqq j \leqq 3\} \cup \{[\alpha_h, (h-1, j)]\colon h \in A - \{1\}, 1 \leqq j \leqq 3\} \cup \\ &\quad \{[(a_i, j), (a_i - 1, j)], [(a_i, j), (a_i + 1, j)], [(a_i, j), (a_{i+1} + 1, j)]\colon \\ &\quad a_i \in B, 1 \leqq j \leqq 3\} \cup \{[(i, j), (i-1, j)]\colon \text{both } i \text{ and } i-1 \text{ are elements} \\ &\quad \text{in } \{1, 2, 3, \cdots, a_{2m}\} - (A \cup B), 1 \leqq j \leqq 3\}. \end{aligned}$$

Then Π is a connected graph in which every α_h $(h \in A)$ and every (i, j) $(i \in B, 1 \leqq j \leqq 3)$ have valency 3 and every other vertex has valency 2. Let us denote $X = \{\alpha_h\colon h \in A\}$, $Y = \{(i, j)\colon i \in \{1, 2, 3, \cdots, a_{2m}\} - A, 1 \leqq j \leqq 3\}$, $Y_1 = \{(i, j)\colon i \in B, 1 \leqq j \leqq 3\}$ and $Y_1 = Y - Y_2$.

Let Δ be a graph defined by

$$\begin{aligned} V\Delta &= \{v_1, v_2, v_3, \cdots, v_{18}\}, \\ E\Delta &= \{[v_{1+6k}, v_{2+6k}], [v_{2+6k}, v_{3+6k}], [v_{2+6k}, v_{4+6k}], [v_{3+6k}, v_{5+6k}], \\ &\quad [v_{4+6k}, v_{5+6k}], [v_{4+6k}, v_{6+6k}], [v_{5+6k}, v_{6+6k}]\colon k = 0, 1, 2\} \cup \\ &\quad \{[v_6, v_9], [v_{12}, v_{15}], [v_{18}, v_3]\}. \end{aligned}$$

Then Δ is a connected graph in which v_1, v_7 and v_{13} have valency 1 and every other vertex has valency 3. We can easily find that $\mathrm{Aut}\,\Delta$ is a cyclic group $\langle \eta \rangle$ of order 3, where

$$\eta = \prod_{k=0}^{5} (v_{1+k}, v_{7+k}, v_{13+k}).$$

Let Γ be a graph which is attached for each $i \in \{1, 2, 3, \cdots, a_{2m}\} - (A \cup B)$ a graph isomorphic with Δ to Π so as to correspond vertices v_1, v_7 and v_{13} of Δ to vertices $(i, 1)$, $(i, 2)$ and $(i, 3)$ of Π respectively. Then Γ is a connected 3-regular graph. Let us denote $Z=V\Gamma-(X\cup Y)$. Now η uniquely determines an automorphism $\bar{\eta}$ of Γ, where

$$\bar{\eta}(\alpha_h)=\alpha_h \quad \text{for} \quad \alpha_h\in X\,,$$

$\bar{\eta}(i,1)=(i,2)$, $\bar{\eta}(i,2)=(i,3)$, $\bar{\eta}(i,3)=(i,1)$ for $i\in\{1,2,3,\cdots,a_{2m}\}-A$. We want to show that Aut Γ is equal to a cyclic group $\langle\bar{\eta}\rangle$ of order 3. Let σ be any automorphism of Γ. Since in Γ $\{u\in V\Gamma\colon \partial(u,C)\geqq 3$ holds for any 3-cycle $C\}$ $=X\cup Y_1$ and $\{u\in V\Gamma\colon$ the type of u is $(8,8,8)\}=X$ hold, σ fixes X, Y_1 and $Z\cup Y_2$ as a set respectively. Therefore σ fixes all $\alpha_h\in X$, because $\partial(\alpha_h,Y_1)\neq\partial(\alpha_k,Y_1)$ necessarily holds for $h\neq k\in A$. Hence for each $i\in\{1,2,3,\cdots,a_{2m}\}-A$, σ fixes $\{(i,1),(i,2),(i,3)\}$ as a set. Hence we can find $\sigma\in\langle\bar{\eta}\rangle$. Thus Proposition 7 holds for $n=3$.

Next let us suppose $n=4$. Let $\{a_k\}$, m, A and B be the same things as in Proposition 5. Let Π be a graph defined by

$$\begin{aligned}
V\Pi = {} & \{\alpha_h\colon h\in A\}\cup\{(i,j)\colon i\in\{1,2,3,\cdots,a_{3m+2}\}-A,\ j=1,2,3\}\cup \\
& \{(i,4)\colon i\in B\}, \\
E\Pi = {} & \{[\alpha_1,(2,1)],[\alpha_1,(2,2)],[\alpha_1,(2,3)],[\alpha_1,(3,4)]\}\cup\{[\alpha_h,(h-1,1)], \\
& [\alpha_h,(h-1,2)],[\alpha_h,(h-1,3)],[\alpha_h,(b_h,4)]\colon h\in A-\{1\},\ b_h= \\
& \max\{b\in B\colon b<h\}\}\cup\{[(a_j,4),(a_{j-3},4)]\colon \text{both } a_j \text{ and } a_{j-3} \text{ are} \\
& \text{elements in } B\}\cup\{[(a_i,j),(a_i-1,j)],[(a_i,j),(a_i+1,j)], \\
& [(a_i,j),(a_{i+1}+1,j)],[(a_i,j),(a_{i+2}+1,j)]\colon a_i\in B,\ 1\leqq j\leqq 3\}\cup \\
& \{[(i,j),(i-1,j)]\colon \text{both } i \text{ and } i-1 \text{ are elements in} \\
& \{1,2,3,\cdots,a_{3m+2}\}-(A\cup B),\ 1\leqq j\leqq 3\}.
\end{aligned}$$

Then Π is a connected graph in which every $\alpha_h(h\in A)$ and every (i,j) $(i\in B$, $1\leqq j\leqq 4)$ have valency 4 and every other vertex has valency 2. Let us denote $X=\{\alpha_h\colon h\in A\}$, $Y_1=\{(i,j)\colon i\in B,\ 1\leqq j\leqq 3\}$, $Y_2=\{(i,4)\colon i\in B\}$, $Y_3=\{(i,j)\colon i\in\{1,2,3,\cdots,a_{3m+2}\}-(A\cup B),\ 1\leqq j\leqq 3\}$ and $Y=Y_1\cup Y_2\cup Y_3$.

Let Δ be the same as in Proposition 5. Let Γ be a graph which is attached for each $i\in\{1,2,3,\cdots,a_{3m+2}\}-(A\cup B)$ and each $j=1,2,3$ a graph isomorphic with Δ to Π so as to correspond the vertex v_0 of Δ to the vertex (i,j) of Π and the vertex v_{12} of Δ to the vertex $(i,j+1)$ of Π (put $j+1=1$ if $j+1=4$). Then Γ is a connected 4-regular graph, and we are to show that Aut Γ has order 3 and fixes all $\alpha_h\in X$. Let us denote $Z=V\Gamma-(X\cup Y)$. Now let τ be an automorphism of Π defined by

$$\begin{aligned}
&\tau(\alpha_h)=\alpha_h \quad \text{for} \quad \alpha_h\in X\,, \\
&\tau(i,4)=(i,4) \quad \text{for} \quad (i,4)\in Y_2\,, \\
&\tau(i,j)=(i,j+1)\ (\text{put } j+1=1 \text{ if } j+1=4) \quad \text{for} \quad (i,j)\in Y_1\cup Y_3\,.
\end{aligned}$$

Then τ has order 3 and is uniquely extended to an automorphism $\bar{\tau}$ of Γ. Of course $|\bar{\tau}|=3$ and Aut $\Gamma\geqq\langle\bar{\tau}\rangle$ holds.

Let σ be any automorphism of Γ. Since in Γ

$\{u\in V\Gamma\colon \partial(u, C)\geqq 3$ holds for any 3-cycle $C\} = Y_2$,

$\{u\in V\Gamma\colon \partial(u, C) = 0$ holds for some 3-cycle $C\} = Z$,

$\{u\in V\Gamma\colon \partial(u, C) = 1$ holds for some 3-cycle C and $\partial(u, C')\geqq 1$ holds for any 3-cycle $C'\} = Y_3$,

$\{u\in V\Gamma\colon \partial(u, C) = 2$ holds for some 3-cycle C and $\partial(u, C')\geqq 2$ holds for any 3-cycle $C'\} = X\cup Y_1$ and

$\{u\in V\Gamma\colon \partial(u, C) = 2$ holds for some 3-cycle C, $\partial(u, C')\geqq 2$ holds for any 3-cycle C' and any $w\in V(N(u))$ is adjacent to a vertex on some 3-cycle$\} = Y_1$,

σ fixes X, Y_1, Y_2, Y_3 and Z as a set respectively. Let Π' be a subgraph induced by $X\cup Y_1\cup Y_3$. Then in Π' $\partial(\alpha_h, Y_1)\neq\partial(\alpha_k, Y_1)$ necessarily holds for $h\neq k\in A$. Therefore in Γ σ fixes all $\alpha_h\in X$, and σ fixes all $(i, 4)\in Y_2$. Hence for each $i\in\{1, 2, 3, \cdots, a_{3m+2}\}-A$, σ fixes $\{(i, 1), (i, 2), (i, 3)\}$ as a set. Hence we can find $\sigma\in\langle\tau\rangle$. Thus Proposition 7 holds for $n=4$.

Now Proposition 7 holds for $n=5$ by the similar argument to that in the rear part of the proof of Proposition 5. Hence we complete the proof by Lemma 13.

Proposition 8. *If* $|G|=2$, *Theorem* 1 *holds.*

Proof. First let us suppose $n=3$. Let m be an odd integer with $m\geqq\max\{5, t+1\}$. Let Γ be a graph defined by

$$\begin{aligned} V\Gamma = {} & \{(1, 1), (m, 1), (m-1, 1), (m-1, 2)\}\cup\{(i, j)\colon i = 2, 3, \cdots, m-2,\\ & j = 1, 2, 3\},\\ E\Gamma = {} & \{[(1, 1), (m, 1)], [(1, 1), (2, 1)], [(1, 1), (2, 2)], [(m, 1), (m-1, 1)],\\ & [(m, 1), (m-1, 2)], [(m-1, 1), (m-1, 2)], [(m-2, 1), (m-1, 1)],\\ & [(m-2, 2), (m-1, 2)]\}\cup\{[(i, j), (i+1, j)]\colon 2\leqq i\leqq m-3, j=1, 2\}\cup\\ & \{[(i, 1), (i, 3)], [(i, 2), (i, 3)]\colon 2\leqq i\leqq m-2\}\cup\{[(i, 3), (i+1, 3)]\colon\\ & i = 2, 4, 6, \cdots, m-1, m-3\}.\end{aligned}$$

Then Γ is a connected 3-regular graph. Furthermore we can find that Aut Γ is $\langle\tau\rangle$, where τ is a tranposition $\prod_{k=2}^{m-1}((k, 1), (k, 2))$. Thus Proposition 8 holds for $n=3$.

Next let us suppose $n=4$. In this case we modify the proof with the case $n=4$ of Proposition 6, that is, we alter Δ in the place as follows:

$$\begin{aligned} V\Delta = {} & \{v_1, v_2, \cdots, v_7\},\\ E\Delta = {} & \{[v_1, v_5], [v_5, v_2], [v_2, v_7], [v_7, v_4], [v_4, v_3], [v_3, v_6], [v_6, v_1],\\ & [v_5, v_6], [v_5, v_7], [v_6, v_7]\}.\end{aligned}$$

Then Δ is a connected graph in which v_1, v_2, v_3 and v_4 have valency 2 and every other vertex has valency 4. Furthermore we have $\mathrm{Aut}\,\Delta=\langle\eta\rangle$, where $\eta=(v_1, v_2)(v_3, v_4)(v_6, v_7)$. So by the similar argument to that in the place, we can easily find that $\mathrm{Aut}\,\Gamma$ has order 2 and fixes all $\alpha_h \in X$.

Now Proposition 8 holds for $n=5$ by the similar argument to that in the rear part of the proof of Proposition 5. Hence we complete the proof by Lemma 13.

REMARK 4. By the proofs of the propositions, we may add "$\mathrm{Aut}\,\Gamma$ is semi-regular on $V\Gamma-I(\mathrm{Aut}\,\Gamma)$" to the conclusion of Theorem 1, where $I(\mathrm{Aut}\,\Gamma)$ is the set of vertices v satisfying $\mathrm{Aut}\,\Gamma(v)=v$.

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