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Osaka J. Math.
26 (1989), 527-534

INTEGRAL FORMULAS FOR HARMONIC FUNCTIONS ASSOCIATED WITH BOUNDARIES OF A BOUNDED SYMMETRIC DOMAIN

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(Received October 20, 1988)

1. Introduction

In the case of the unit disc, or the upper half-plane in the theory of one complex variable, the Poisson kernel can be expressed in terms of the Cauchy kernel in the following simple way; in either case, denoting the Cauchy and the Poisson kernels by $\mathcal{S}(z, w)$, $\mathcal{P}(z, u)$ respectively

$$\mathcal{P}(z, u) = \frac{|\mathcal{S}(z, u)|^2}{\mathcal{S}(z, z)}\,. \tag{1}$$

It is natural, therefore, to extend this definition whenever the Cauchy kernel is defined. Hua [3] did this for four classical types of bounded symmetric domains and established some of its basic properties. For generalized half-planes this was done by Korányi [6] who then used the theory of Cayley transform to determine the Cauchy and Poisson kernels for all the bounded symmetric domains (See also [8], [5]).

It is known that the Poisson kernel has another interpretation; it can be regarded as the Jacobian of an automorphism restricted to the boundary. This way of viewing the Poisson kernel was shown to work on arbitrary non-compact Riemannian symmetric spaces by Furstenberg [1]. For any symmetric domain it turns out that these two possible definitions of the Poisson kernel coincide (See [6], though it is not explicitly stated).

Now let D be an irreducible bounded symmetric domain in the canonical Harish-Chandra realization. If r is the rank of D, then the topological boundary ∂D breaks into r boundaries $B_1, \cdots, B_r$, such that $\bar{B}_i \supset B_{i+1} (1 \leq i \leq r-1)$, and B_r is the Silov boundary. As is shown in [4], for each boundary $B_i (1 \leq i \leq r)$, there is a natural measure σ_i on B_i and a Cauchy type kernel function $\mathcal{S}_i(z, w)$ such that

$$f(z) = \int_{B_i} \mathcal{S}_i(z, u) f(u)\, d\sigma_i(u)$$

whenever f is holomorphic in a neighborhood of $\bar{D}$, the closure of D. For the

Silov boundary B_r, the function $\mathcal{S}_r(z, w)$ is the usual Cauchy(-Szegö) kernel of D, from which Hua et al. defined the Poisson kernel by (1). Therefore it is natural to define, for each boundary B_i, the Poisson type kernel $\mathcal{P}_i(z, u)$ by putting

$$\mathcal{P}_i(z, u) = \frac{|\mathcal{S}_i(z, u)|^2}{\mathcal{S}_i(z, z)}, \quad z \in D,\, u \in B_i\,.$$

In this note we show that the kernel $\mathcal{P}_i(z, u)$ represents harmonic functions f in D in terms of the boundary values on B_i, i.e.,

$$f(z) = \int_{B_i} \mathcal{P}_i(z, u) f(u)\, d\sigma_i(u)$$

whenever f is harmonic in D and continuous on its closure $\bar{D}$. We also show that the kernel $\mathcal{P}_i(z, u)$ can be regarded as the Jacobian of an automorphism restricted to the boundary B_i, i.e., if g is an automorphism of D,

$$\mathcal{P}_i(g\cdot o, u) = \frac{d\sigma_i(g^{-1}\cdot u)}{d\sigma_i(u)}\,,$$

where o is the origin of D.

2. Preliminaries

We begin by reviewing the general background on bounded symmetric domains (cf. [2], [9]). Every bounded symmetric domain D can be written as $D = G/K$, where G is a connected semisimple linear Lie group and K is a maximal compact subgroup of G, such that G operates holomorphically on D. In this note we assume that G is simple, i.e., that D is irreducible. We further assume that the complexification G_C of G is simply connected. Let $\mathfrak{g}$, $\mathfrak{k}$ be the Lie algebras of G, K and $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ be the corresponding Cartan decomposition. We denote the complexifications of $\mathfrak{g}$, $\mathfrak{k}$, $\mathfrak{p}$ by $\mathfrak{g}_C$, $\mathfrak{k}_C$, $\mathfrak{p}_C$, respectively. Then $\mathfrak{p}_C$ is decomposed into the direct sum of two complex subalgebras $\mathfrak{p}^+$, $\mathfrak{p}^-$, which are $(\pm\sqrt{-1})$-eigenspaces of the complex structure of $\mathfrak{p}$, respectively, and are abelian subalgebras of $\mathfrak{g}_C$ normalized by $\mathfrak{k}_C$. Let $P^{\pm}$, K_C, be the connected subgroups of G_C corresponding to $\mathfrak{p}^{\pm}$, $\mathfrak{k}_C$, respectively. Then the map $\mathfrak{p}^+ \times K_C \times \mathfrak{p}^- \to G_C$, given by $(X^+, k, X^-) \to \exp X^+ \cdot k \cdot \exp X^-$, is a holomorphic diffeomorphism onto a dense open subset $P^+K_CP^-$ of G_C, which contains G. Therefore every element $g \in P^+K_CP^-$ can be written in a unique way as

$$g = \pi_+(g)\cdot\pi_0(g)\cdot\pi_-(g)\,, \quad \pi_0(g) \in K_C,\, \pi_{\pm}(g) \in P^{\pm}\,. \tag{2}$$

Furthermore, the map $\zeta\colon P^+K_CP^- \to \mathfrak{p}^+$, given by $\zeta(g) = \log\,(\pi_+(g))$ induces a holomorphic diffeomorphism of $D = G/K$ onto $\zeta(G)$, and $\zeta(G)$ is a bounded domain in $\mathfrak{p}^+$. Henceforce we assume that D is a bounded symmetric domain in $\mathfrak{p}^+$ realized in this manner. In this realization the action of G on D is given by

$$g\cdot z=\zeta(g \exp z)\,,\quad g\in G,\, z\in D\,,$$

and extends smoothly to $\bar{D}$.

Let $\mathfrak{t}$ be a maximal abelian subalgebra of $\mathfrak{k}$. Then $\mathfrak{t}_C$, the complexification of $\mathfrak{t}$, is a Cartan subalgebra of $\mathfrak{g}_C$. Let Φ be the root system of $\mathfrak{g}_C$ relative to $\mathfrak{t}_C$. For each $\alpha\in\Phi$, let H_α, E_α denote the usual basis elements of $\mathfrak{g}_C$. We can choose a linear order in the dual of the real vector space $\sqrt{-1}\,\mathfrak{t}$ such that $\mathfrak{p}^+$ is spanned by the root spaces for noncompact positive roots. We let Φ^+ be the resulting set of positive roots.

We choose a maximal set $\{\gamma_1, \cdots, \gamma_r\}$ of strongly orthogonal noncompact positive roots as follows. Let γ_1 be the highest root of Φ and for each j, γ_{j+1} be the highest positive noncompact root that is strongly orthogonal to each of $\{\gamma_1, \cdots, \gamma_j\}$. We write H_j, E_j for $H_{\gamma_j}, E_{\gamma_j}$. For each $1\leq i\leq r$, we define the partial Cayley transform $c_i\in G_C$ by

$$c_i=\prod_{j=1}^{i} \exp\frac{\pi}{4}(E_{-j}-E_j)\,.$$

Since $c_i\in P^+K_CP^-$, we can define $o_i=\zeta(c_i)$. Let B_i denote the G-orbit of o_i. Then

$$\bar{D}-D=\bigcup_{1\leq i\leq r} B_i \quad \text{(disjoint union)}\,.$$

Moreover $\bar{B}_i\supset B_{i+1}$ $(1\leq i\leq r-1)$, and B_r is the Silov boundary.

Let $C_i(\subset B_i)$ be the boundary component of D containing o_i, and let $P_i=\{g\in G;\, g\cdot C_i=C_i\}$ and $S_i=\{g\in G; g\cdot o_i=o_i\}$. Then P_i is a maximal parabolic subgroup of G, and we have a Langlands decomposition $P_i=M_i\,A_i\,N_i$ such that if we put $L_i=M_i\cap S_i$ then $S_i=L_i\,A_i\,N_i$ (cf. [4]). Further there exists a semisimple subgroup G_i of G such that $C_i=G_i\cdot o_i$.

For each $1\leq i\leq r$, we define a C^∞ function ρ_i on G as follows. Since $P_i=M_i\,A_i\,N_i$ is a parabolic subgroup, each $g\in G$ can be uniquely written in the form $g=kman$ $(k\in K,\, m\in M_i\cap\exp\mathfrak{p},\, a\in A_i,\, n\in N_i)$; so put $\rho_i(g)=(\det(\mathrm{Ad}(a)\,|_{\mathfrak{n}_i}))^{-1}$, where $\mathfrak{n}_i$ is the Lie algebra of N_i. Let dk denote the Haar measure on K such that $\int_K dk=1$. Then we can normalize various left Haar measures in such a way that

$$\int_G f(g)\,dg=\int_{K\times G_i\times S_i} f(kg_is_i)\,\rho_i(s_i)^{-1}\,dk\,dg_i\,ds_i \tag{3}$$

for any integrable f on G (cf. [4], p. 89).

Let $\{\alpha_1, \cdots, \alpha_l\}$ be an enumeration of the set of simple roots for Φ^+ such that α_1 is the unique noncompact simple root for Φ^+, and let λ be the linear form on $\mathfrak{t}_C$ such that

$$2(\lambda, \alpha_1)/(\alpha_1, \alpha_1) = 1 \quad \text{and} \quad (\lambda, \alpha_j) = 0 \quad \text{for} \quad j = 2, \cdots, l$$

where (,) is the inner product induced by Killing form of $\mathfrak{g}_C$. Then λ is the differential of a holomorphic character of K_C.

Let $\mathfrak{t}_C^- = \sum_{i=1}^{r} \boldsymbol{R} H_j$. Then the restrictions of $\mathfrak{t}_C$-roots to $\mathfrak{t}_C^-$ are of the form $\pm\gamma_j$ (each with multiplicity one), $\pm\frac{1}{2}(\gamma_j \pm \gamma_k)$ ($j<k$, each with the same multiplicity $u>0$), $\pm\frac{1}{2}\gamma_j$ (each with the same multiplicity $2v\geq 0$). For each $1\leq i\leq r$, let (as in [4], p. 91)

$$p_i = \frac{1}{2} u(i-1)+u(r-i)+v+1\,, \tag{4}$$

and set $\omega_i = -p_i \lambda$. Note that each p_i is an integer or a half-integer. If p_i is an integer, ω_i is also the differential of a holomorphic character of K_C. For the moment we assume that this is the case and let τ_i be the corresponding character of K_C, i.e., $\tau_i = e^{\omega_i}$. We define J_i: $G\times\bar{D}\to \boldsymbol{C}^\times$ ($\boldsymbol{C}^\times$=the multiplicative group of non-zero compex numbers) by

$$J_i(g, z) = \tau_i(\pi_0(g \exp z))$$

where π_0 is as in (2). Then we have

$$J_i(g_1 g_2, z) = J_i(g_1, g_2\cdot z)\, J_i(g_2, z)\,, \quad g_1, g_2\in G, z\in\bar{D}\,.$$

Let χ: $K_C\to \boldsymbol{C}^\times$ be a holomorphic character of K_C defined by

$$\chi(k) = \det(\mathrm{Ad}(k)|_{\mathfrak{p}^+})\,,$$

and let $\mathcal{K}$: $D\times D\to \boldsymbol{C}^\times$ be a function defined by

$$\mathcal{K}(z, w) = \chi(\pi_0(\exp(-\bar{w})\exp z))$$

where $w\to\bar{w}$ denotes the complex conjugation of $\mathfrak{g}_C$ with respect to $\mathfrak{g}$. Then, up to a constant factor, $\mathcal{K}(z, w)$ is the Bergman kernel functoin of D (cf. [7], [4]). Let $n=\dim_{\boldsymbol{C}} D$, $n_i=\dim_{\boldsymbol{C}} C_i$, and $d_i=\dim_{\boldsymbol{R}} B_i$, and set

$$q_i = \frac{n-n_i}{3n-n_i-d_i}\,.$$

Since $D\times D$ is simply connected, we can define powers $\mathcal{K}(z, w)^{q_i}$ of $\mathcal{K}(z, w)$ with $\mathcal{K}(o, o)^{q_i}=1$. We let

$$\mathcal{S}_i(z, w) = \mathcal{K}(z, w)^{q_i}\,.$$

For a fixed $z \in D$, $\mathcal{S}_i(z, \cdot)$ extends smoothly to $\bar{D}$. If p_i (in (4)) is an integer, then it follows from Lemma 6.24 of [4] that

$$\mathcal{S}_i(z, w) = \tau_i(\pi_0(\exp(-\bar{w}) \exp z)),$$

and we have

$$\mathcal{S}_i(g \cdot z, g \cdot w) = J_i(g, z) \mathcal{S}_i(z, w) \overline{J_i(g, w)}.$$

Up to now we have assumed that the p_i is an integer. We note that, even if p_i is a half-integer, $J_i(g, z)^2$ is a well defined function on $G \times \bar{D}$, and satisfies the following properties

$$J_i(g_1 g_2, z)^2 = J_i(g_1, g_2 \cdot z)^2 J_i(g_2, z)^2, \tag{5}$$

$$\mathcal{S}_i(g \cdot z, g \cdot w)^2 = J_i(g, z)^2 \mathcal{S}_i(z, w)^2 \overline{J_i(g, w)^2}. \tag{6}$$

Let $d\sigma_i$ be the quasi-invariant measure on $B_i = G/S_i$ defined by

$$\int_{B_i} f(u) \, d\sigma_i(u) = \int_{K \times G_i} |J_i(kg_i, o_i)^{-2}| f(kg_i \cdot o_i) \, dk \, dg_i$$

for all $f \in C_c(B_i)$ (continuous functions with compact support). Then Proposition 4.38 of [4] implies that

$$\int_{B_i} d\sigma_i(u) = \int_{K \times G_i} |J_i(kg_i, o_i)^{-2}| \, dk \, dg_i < \infty.$$

Therefore we can (and do) normalize the Haar measure dg_i on G_i so that

$$\int_{B_i} d\sigma_i(u) = 1. \tag{7}$$

It then follows from formula (6.15) of [4] that

$$f(z) = \int_{B_i} \mathcal{S}_i(z, u) f(u) \, d\sigma_i(u)$$

whenever f is holomorphic in the neighborhood of $\bar{D}$.

3. Integral formulas

As in the introduction we define, for each boundary $B_i (1 \leq i \leq r)$, the Poisson type kernel function $\mathcal{P}_i(z, u)$ by putting

$$\mathcal{P}_i(z, u) = \frac{|\mathcal{S}_i(z, u)|^2}{\mathcal{S}_i(z, z)}, \quad z \in D, \ u \in B_i.$$

Proposition. *For $g \in G$, $u \in B_i$, we have*

$$\mathcal{P}_i(g \cdot o, u) = |J_i(g^{-1}, u)^{-2}| = \frac{d\sigma_i(g^{-1} \cdot u)}{d\sigma_i(u)}.$$

Proof. Since $\mathcal{S}_i(o, w)=1$, (5) and (6) imply

$$\mathcal{S}_i(g\cdot o, u)^2 = J_i(g, o)^2\, \overline{J_i(g^{-1}, u)^{-2}}\,.$$

and

$$\mathcal{S}_i(g\cdot o, g\cdot o) = |J_i(g, o)^2|\,.$$

So the first equality follows from the definition of $\mathcal{P}_i(g\cdot o, u)$.

For the second equality, it suffices to show that

$$\int_{B_i} f(g\cdot u)\, d\sigma_i(u) = \int_{B_i} |J_i(g^{-1}, u)^{-2}|\, f(u)\, d\sigma_i(u)$$

for $f\in C_c(B_i)$. We first note that

$$(8) \qquad |J_i(s_i, o_i)^{-2}| = \rho_i(s_i) \quad \text{for} \quad s_i\in S_i\,;$$

this follows from the argument in the proof of Lemma 6.30 of [4]. Now for each $f\in C_c(B_i)$, we can take $\tilde{f}\in C_c(G)$ such that

$$f(h\cdot o_i) = \int_{S_i} \tilde{f}(hs_i)\, ds_i\,, \quad h\in G\,.$$

Hence

$$\begin{aligned}
&\int_{B_i} f(g\cdot u)\, d\sigma_i(u)\\
&\quad = \int_{K\times G_i} |J_i(kg_i, o_i)^{-2}|\, f(gkg_i\cdot o_i)\, dk\, dg_i\\
&\quad = \int_{K\times G_i\times S_i} |J_i(kg_is_i, o_i)^{-2}|\, \tilde{f}(gkg_is_i)\, \rho_i(s_i)^{-1}\, dk\, dg_i\, ds_i \quad \text{(by (5) and (8))}\\
&\quad = \int_G |J_i(h, o_i)^{-2}|\, \tilde{f}(gh)\, dh \quad \text{(by (3))}\\
&\quad = \int_G |J_i(g^{-1}h, o_i)^{-2}|\, \tilde{f}(h)\, dh\\
&\quad = \int_{K\times G_i\times S_i} |J_i(g^{-1}kg_is_i, o_i)^{-2}|\, \tilde{f}(kg_is_i)\, \rho_i(s_i)^{-1}\, dk\, dg_i\, ds_i\\
&\quad = \int_{K\times G_i} |J_i(g^{-1}kg_i, o_i)^{-2}|\, f(kg_i\cdot o_i)\, dk\, dg_i\\
&\quad = \int_{K\times G_i} |J_i(g^{-1}, kg_i\cdot o_j)^{-2}|\; |J_i(kg_i, o_i)^{-2}|\, f(kg_i\cdot o_i)\, dk\, dg_i \quad \text{(by (5))}\\
&\quad = \int_{B_i} |J_i(g^{-1}, u)^{-2}|\, f(u)\, d\sigma_i(u)\,.
\end{aligned}$$

This proves the Proposition.

Theorem. *If f is harmonic on D and continuous on $\bar{D}$, then for all $z\in D$,*

$$f(z) = \int_{B_i} \mathcal{P}_i(z, u)\, f(u)\, d\sigma_i(u)\,.$$

Proof. For each $z\in D$, choose $g\in G$ such that $z=g\cdot o$. Then, by the mean value theorem for harmonic functions (cf. [2]), we have

$$f(z)=\int_K f(gk\cdot w)\,dk$$

for $w\in D$. The continuity of f on $\bar{D}$ implies that this formula is valid for all $w\in\bar{D}$. Therefore

$$\begin{aligned}
f(z) &= \int_{B_i} f(z)\,d\sigma_i(u) \quad \text{(by (7))}\\
&= \int_{B_i} \Big(\int_K f(gk\cdot u)\,dk\Big)\,d\sigma_i(u)\\
&= \int_K \Big(\int_{B_i} f(gk\cdot u)\,d\sigma_i(u)\Big)\,dk\\
&= \int_K \Big(\int_{B_i} f(g\cdot u)\,d\sigma_i(u)\Big)\,dk \quad \text{(by K-invariance of $d\sigma_i$)}\\
&= \int_{B_i} f(g\cdot u)\,d\sigma_i(u)\\
&= \int_{B_i} \mathcal{P}_i(z,u)f(u)\,d\sigma_i(u) \quad \text{(by Proposition)}\,.
\end{aligned}$$

This finished the proof.

REMARK. If $i\neq r$, the maximal compact subgroup K of G does not act transitively on the boundary B_i. Therefore Proposition implies that the Poisson type kernel $\mathcal{P}_i(z,u)$ is not necessarily harmonic in the variable z.

EXAMPLE. Let $p\geq q$ and

$$D=\{z\in M_{p,q}(\boldsymbol{C});\ 1_q-z^*z>0\}\,.$$

Here $M_{p,q}(\boldsymbol{C})$ refers to all p by q complex matrices, 1_q is the identity matrix of size q, z^* is the conjugate transpose of z and ">0" means "is positive definite". Then (cf. [9]) D is the bounded symmetric domain of rank q, and for each $1\leq i\leq q$, the i-th boundary B_i is given by

$$B_i=\{z\in M_{p,q}(\boldsymbol{C});\ 1_q-z^*z\geq 0 \quad\text{and}\quad \operatorname{rank}(1_q-z^*z)=q-i\}\,.$$

On the other hand the Cauchy type kernel function $\mathcal{S}_i(z,w)$ associated with the boundary B_i is given by (cf. [4], p. 129)

$$\mathcal{S}_i(z,w)=\det(1_q-w^*z)^{-(p+q-i)}\,.$$

Therefore the Poisson type kernel function $\mathcal{P}_i(z,u)$ is given by

$$\mathcal{P}_i(z,u)=\frac{\det(1_q-z^*z)^{p+q-i}}{|\det(1_q-u^*z)|^{2(p+q-i)}}\,,\quad (1\leq i\leq q)\,.$$

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