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AN IMITATION THEORY OF MANIFOLDS

Dedicated to Professor Jnuzo Tao on his sixtieth birthday

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By an (m, l)-manifold pair (M, L) we mean a smooth m-manifold M and a (regular) l-submanifold L such that L is a closed set in M and $\partial L = L \cap \partial M$. Here, M and L may be non-compact or disconnected and $-1 \leq l < m$ ($l = -1$ means $L = \emptyset$). Let $I = [-1, 1]$. A *reflection* in $(M, L) \times I$ is a smooth involution α on $(M, L) \times I$ such that $\alpha(M \times 1) = M \times (-1)$ and $\mathrm{Fix}(\alpha, M \times I)$ is an m-manifold and α acts non-freely on each component of $M \times I$. Then we can see that $\mathrm{Fix}(\alpha, (M, L) \times I)$ is an (m, l)-manifold pair (cf. Property I). A smooth imbedding ϕ from an (m, l)-manifold pair (M^*, L^*) to $(M, L) \times I$ with $\phi(M^*, L^*) = \mathrm{Fix}(\alpha, (M, L) \times I)$ for a reflection α in $(M, L) \times I$ is called a *reflector* (of the reflection α).

DEFINITION. An (m, l)-manifold pair (M^*, L^*) is an *imitation* of an (m, l)-manifold pair (M, L) *with imitation map* $q: (M^*, L^*) \to (M, L)$, if there is a reflector $\phi: (M^*, L^*) \to (M, L) \times I$ with $q = p_1 \phi$, where p_1 denotes the projection from $(M, L) \times I$ to (M, L).

In Section 1, we shall give six general properties of imitations, meaning that any imitation map $q: (M^*, L^*) \to (M, L)$ has properties close to a diffeomorphism and hence the distinguishment between (M^*, L^*) and (M, L) is not so easy.

DEFINITION. An imitation (M^*, L^*) of (M, L) with imitation map q is *pure* if $q = p_1 \phi$ for a reflector $\phi: (M^*, L^*) \to (M, L) \times I$ of a reflection α such that $\alpha(x, 1) = (x, -1)$ for all $x \in M$.

We also say that such α, ϕ and q are pure. This subtle notion is needed when we ask whether an imitation of an imitation is an imitation of the original manifold pair (See Proposition 2.1). Let $\mathrm{Diff}\, X$ be the diffeomorphism group of a smooth manifold X, which is a topological group (with respect to the compact-open topology). For subspaces A_i, $i = 1, 2, \cdots, s$, and Y of X, we denote the subgroup of $\mathrm{Diff}\, X$ consisting of all $f \in \mathrm{Diff}\, X$ with $f(A_i) = A_i$ (($i = 1, 2, \cdots, s$, and $f | Y = \mathrm{id}_Y$ by $\mathrm{Diff}(X, A_1, A_2, \cdots, A_s, \mathrm{rel}\, Y)$ (or $\mathrm{Diff}(X, A_1, A_2, \cdots, A_s)$ if

$Y=\emptyset$). By $\mathrm{Diff}_0(X, A_1, A_2, \cdots, A_s, \text{rel } Y)$ we denote the path connected component containing $\mathrm{id}_X \in \mathrm{Diff}(X, A_1, A_2, \cdots, A_s, \text{rel } Y)$.

DEFINITION. Two imitations (M^*, L^*), (M^{**}, L^{**}) of (M, L) with imitation maps q, q' are *conjugate* if $f\phi(M^*, L^*)=\phi'(M^{**}, L^{**})$ for some reflectors $\phi\colon (M^*, L^*)\to(M, L)\times I$, $\phi'\colon (M^{**}, L^{**})\to(M, L)\times I$ with $p_1\phi=q$, $p_1\phi'=q'$ and some $f\in\mathrm{Diff}(M\times I, L\times I, M\times 1, M\times(-1))$.

An imitation (M^*, L^*) of (M, L) with imitation map q is said to be *inessential* or *essential* according to whether q is conjugate to a diffeomorphic imitation map $q'\colon (M, L)\cong(M, L)$ or not. It is shown in Section 2 that all imitations of all (m, l)-manifold pairs with $m\leq 2$ are pure and inessential.

DEFINITION. An imitation (M^*, L^*) of (M, L) with imitation map q is *normal* if $q=p_1\phi$ for a reflector $\phi\colon (M^*, L^*)\to(M, L)\times I$ of a reflection α in $(M, L)\times I$ such that $\alpha(x, t)=(x, -t)$ for all $(x, t)\in\partial(M\times I)\cup N_L\times I$, where N_L denotes a neighborhood of L in M.

In Section 3 we show that for each (m, l)-manifold pair (M, L) with $m\geq 3$ there are infinitely many (up to conjugations) essential normal imitations of (M, L), by using the fact that the 11-crossing Kinoshita/Terasaka knot is a knot imitation of a trivial knot. In Section 4 some remarks on the imitations of 4-manifolds are given. In Section 5 we discuss the Whitehead torsion invariant of an imitation map. In fact, we observe that when M is a compact connected oriented m-manifold, the Whitehead torsion $\tau(q)\in\mathrm{Wh}\,\pi_1(M)$ is defined for any imitation map $q\colon M^*\to M$. Further, when q is conjugate to a ∂-diffeomorphic imitation map, we have $\tau(q)=-2\tau$ for some $\tau\in\mathrm{Wh}\,\pi_1(M)$ with $\bar{\tau}=(-1)^{m+1}\tau$. When q is inessential, $\tau(q)=0$. Under the assumption that $m\geq 5$ and $\mathrm{Wh}\,\pi_1(M)$ has no 2-torsion, this invariant enables us to classify homotopy equivalent ∂-diffeomorphic imitation maps $q\colon M^*\to M$ up to conjugations (See Theorem 5.5).

This paper grew out of some parts of the unpublished paper[Ka, 1]. We also note that an analogous definition of imitation was given in [K/K/S] (cf. Properties I, II, IV and Corollary 2.5). Spaces and maps will be considered in the smooth category.

1. Some general properties of imitations

Lemma 1.1. *Let α be a reflection in $M\times I$ with M connected. Then $M'=\mathrm{Fix}(\alpha, M\times I)$ is connected and splits $M\times I$ into two connected submanifolds W_+, W_- such that*

(1) *There is a diffeomorphism $W_+\cong W_-$ sending M'_+ onto M'_- as the identification map and $M\times 1$ onto $M\times(-1)$,*

(2) *The inclusions* i': $M'_+ \to W_+$ *and* i: $M\times 1 \to W_+$ *induce an epimorphism* $i'_\sharp$: $\pi_1(M'_+) \to \pi_1(W_+)$ *and an isomorphism* $i_\sharp$: $\pi_1(M\times 1) \to \pi_1(W_+)$, *and*
(3) *The inclusions* i', i *induce isomorphisms on homology, cohomology and cohomology with compact support, where* M'_+ *and* M'_- *denote the copies of* M' *in* W_+ *and* W_-, *respectively.*

Proof. First note that $\partial M' = \mathrm{Fix}(\alpha, (\partial M)\times I)$ and $\mathrm{Int}\, M' = \mathrm{Fix}(\alpha, (\mathrm{Int}\, M)\times I)$. Let M'_1 be any connected component of M'. Since $\mathrm{Int}\, M'$ and hence $\mathrm{Int}\, M'_1$ are closed sets in $\mathrm{Int}(M\times I)$, we have

$$H_1(M\times I, M\times I - M'_1; Z_2) \cong H_1(\mathrm{Int}(M\times I), \mathrm{Int}(M\times I) - \mathrm{Int}\, M'_1; Z_2)$$
$$\cong H^m_c(\mathrm{Int}\, M'; Z_2)$$

by the Alexander/Spanier duality (cf. [Sp]). Since the natural homomorphism $H_1(M\times I - M'_1; Z_2) \to H_1(M\times I; Z_2)$ is onto and $M\times I$ is connected, it follows that

$$Z_2 \cong H_1(M\times I, M\times I - M'_1; Z_2) \overset{\partial}{\cong} \tilde{H}_0(M\times I - M'_1; Z_2)\,.$$

This implies that M'_1 splits $M\times I$ into two connected submanifolds W_+, W_-. Since $\alpha(W_+) = W_-$, we see that $M'_1 = M'$ and α defines a desired diffeomorphism in (1). To prove (2), (3), we use the fact that α difines a retraction from $M\times I$ to W_+. This means that the inclusion j: $W_+ \to M\times I$ induces monomorphisms

$$j_\sharp : \pi_1(W_+) \to \pi_1(M\times I)\,,$$
$$j_*: H_*(W_+) \to H_*(M\times I)\,,$$

and epimorphisms

$$j^*: H^*(M\times I) \to H^*(W_+)\,,$$
$$j^*_c: H^*_c(M\times I) \to H^*_c(W_+)\,.$$

But the composite ji: $M\times 1 \to M\times 1$ is a (proper) homotopy equivalence. Hence $j_\sharp, j_*, j^*, j^*_c$ and

$$i_\sharp : \pi_1(M\times 1) \to \pi_1(W_+)\,,$$
$$i_* : H_*(M\times 1) \to H_*(W_+)\,,$$
$$i^*: H^*(W_+) \to H^*(M\times 1)\,,$$
$$i^*_c: H^*_c(W_+) \to H^*_c(M\times 1)$$

are all isomorphisms. To complete the proof of (2), let $\tilde{W}_+$, $\tilde{W}_-$ and $\tilde{M}'$ be the preimages of W_+, W_- and M', respectively, under the universal covering $\tilde{M}\times I \to M\times I$. Then $\tilde{W}_+$ and $\tilde{W}_-$ are connected (because $\tilde{M}\times 1$ and $\tilde{M}\times(-1)$ are connected). By the Mayer/Vietoris sequence, we see that $\tilde{M}'$ is connected. Since $j_\sharp$ is an isomorphism, we also see that $\tilde{W}_+$ is simply connected.

Thus, the natural homomorphism $i'_\#: \pi_1(M'_+) \to \pi_1(W_+)$ is onto, obtaining (2). For (3), note that $H_*(M\times I, W_+)=H^*(M\times I, W_+)=H_c^*(M\times I, W_+)=0$ since j_*, j^* and j_c^* are isomorphisms. By excision, $H_*(W_-, M'_-)=H^*(W_-, M'_-)=H_c^*(W_-, M'_-)=0$. By (1), $H_*(W_+, M'_+)=H^*(W_+, M'_+)=H_c^*(W_+, M'_+)=0$, meaning that

$$i'_*: H_*(M'_+) \to H_*(W_+)\,,$$
$$i'^*: H^*(W_+) \to H^*(M'_+)\,,$$
$$i_c^*: H_c^*(W_+) \to H_c^*(M'_+)$$

are all isomorphisms. This completes the proof.

Let α be a reflection in $(M, L)\times I$. By Lemma 1.1, $\mathrm{Fix}(\alpha, M\times I)$ splits each connected component of $M\times I$, $L\times I$, $(M-L)\times I$, $(\mathrm{Int}\, M)\times I$, $(\mathrm{Int}\, L)\times I$, $(\partial M)\times I$ and $(\partial L)\times I$ into two connected components. Hence we obtain the following:

Property I. *Every imitation map q: $(M^*, L^*)\to(M, L)$ defines imitation maps $M^*\to M$, $L^*\to L$, $M^*-L^*\to M-L$, $(\mathrm{Int}\, M^*, \mathrm{Int}\, L^*)\to(\mathrm{Int}\, M, \mathrm{Int}\, L)$ and $(\partial M^*, \partial L^*)\to(\partial M, \partial L)$.*

We see from Lemma 1.1 that any imitation map q: $M^*\to M$ induces isomorphisms on homology, cohomology and cohomology with compact support. Hence we obtain from Property I and Five Lemma the following:

Property II. *Every imitation map q: $(M^*, L^*)\to(M, L)$ induces isomorphisms on homology, cohomology and cohomology with compact support.*

In Lemma 1.1 (3) Stiefel/Whitney and Pontrjagin classes of M'_+ and $M\times 1$ coincide through the cobordism W_+ (cf. Milnor/Stasheff [M/S]). Hence we have the following:

Property III. *Every imitation map q: $M^*\to M$ preserves Stiefel/Whitney and Pontrjagin classes of M^* and M.*

By Properties I, III, (M, L) is an orientable manifold pair if and only if so is (M^*, L^*). When (M, L) is an oriented manifold pair, we orient (M^*, L^*) so that $q|\mathrm{Int}\, M^*$: $\mathrm{Int}\, M^*\to\mathrm{Int}\, M$ and $q|\mathrm{Int}\, L^*$: $\mathrm{Int}\, L^*\to\mathrm{Int}\, L$ are degree one maps, unless otherwise stated, by using Properties I and II.

Property IV. *Let p: $(\tilde{M}, \tilde{L})\to(M, L)$ be any regular or irregular covering map, where $\tilde{M}$ may be branched along some components of L when $l=m-2$. Consider the pullback diagram of this covering map p by any imitation map q: $(M^*, L^*)\to(M, L)$:*

$$\begin{array}{ccc} (\tilde{M}^*, \tilde{L}^*) & \xrightarrow{\tilde{q}} & (\tilde{M}, \tilde{L}) \\ \tilde{p}\downarrow & & \downarrow p \\ (M^*, L^*) & \xrightarrow{q} & (M, L)\,. \end{array}$$

Then $\tilde{p}$ is a covering map (this is well known) and $\tilde{q}$ is an imitation map.

To obtain Property IV, we use the following lemma:

Lemma 1.2. *Let α be a reflection in $M\times I$ with M connected. For any connectehd unbranched covering $p_I=p\times \mathrm{id}_I: \tilde{M}\times I\to M\times I$, α lifts to a unique reflection $\tilde{\alpha}$ in $\tilde{M}\times I$.*

Proof. By Lemma 1.1 $M'=\mathrm{Fix}(\alpha, M\times I)$ is connected and the natural homomorphism $\pi_1(M', x_0)\to\pi_1(M\times I, x_0)$, $x_0\in M'$, is onto, so that $\tilde{M}'=p_I^{-1}M'$ is connected and α induces the identity automorphism on $\pi_1(M\times I, x_0)$. By the lifting property, α lifts to a unique involution $\tilde{\alpha}$ on $\tilde{M}\times I$ with $\mathrm{Fix}(\tilde{\alpha}, \tilde{M}\times I)$ $=\tilde{M}'$. Since $\tilde{\alpha}(\tilde{M}\times 1)=\tilde{M}\times(-1)$, the proof of Lemma 1.2 is completed.

Proof of Property IV. Let ϕ: $(M^*, L^*)\to(M, L)\times I$ be a reflector of a reflection α in $(M, L)\times I$ with $q=p_1\phi$. Let $p_I=p\times\mathrm{id}_I$: $(\tilde{M}, \tilde{L})\times I\to(M, L)\times I$ be the product covering map. We shall show that α lifts, under p_I, to a unique reflection $\tilde{\alpha}$ in $(\tilde{M}, \tilde{L})\times I$ with $\mathrm{Fix}(\tilde{\alpha}, (\tilde{M}, \tilde{L})\times I)=p_I^{-1}\phi(M^*, L^*)$. When p is unbranched, we apply Lemma 1.2 to each component of $\tilde{M}\times I$ and $\tilde{L}\times I$ and obtain a unique reflection $\tilde{\alpha}$ in $(\tilde{M}, \tilde{L})\times I$ lifting α with $\mathrm{Fix}(\tilde{\alpha}, (\tilde{M}, \tilde{L})\times I)=$ $p_I^{-1}\phi(M^*, L^*)$. When p is branched, the same argument shows that $\alpha\,|\,(M-L)\times I$ and $\alpha\,|\,L\times I$ lift to unique reflections $\tilde{\alpha}_{(M-L)\times I}$ in $(\tilde{M}-\tilde{L})\times I$ and $\tilde{\alpha}_{L\times I}$ in $\tilde{L}\times I$ with $\mathrm{Fix}(\tilde{\alpha}_{(M-L)\times I}, (\tilde{M}-\tilde{L})\times I)=p_I^{-1}\phi(M^*-L^*)$ and $\mathrm{Fix}(\tilde{\alpha}_{L\times I}, \tilde{L}\times I)=p_I^{-1}\phi L^*$, respectively. Since p is a smooth branched covering map and α is a smooth reflection, we see that $\tilde{\alpha}_{(M-L)\times I}$ and $\tilde{\alpha}_{L\times I}$ determine a unique smooth reflection $\tilde{\alpha}$ in $(\tilde{M}, \tilde{L})\times I$ with $\mathrm{Fix}(\tilde{\alpha}, (\tilde{M}, \tilde{L})\times I)=p_I^{-1}\phi(M^*, L^*)$. Let $(\tilde{M}^*, \tilde{L}^*)=$ $\mathrm{Fix}(\tilde{\alpha}, (\tilde{M}, \tilde{L})\times I)$ and $\tilde{\phi}$: $(\tilde{M}^*, \tilde{L}^*)\to(\tilde{M}, \tilde{L})\times I$ be the inclusion, which is a reflector of the reflection $\tilde{\alpha}$ in $(\tilde{M}, \tilde{L})\times I$. Then the imitation map $\tilde{q}=p_1\tilde{\phi}$: $(\tilde{M}^*, \tilde{L}^*)\to(\tilde{M}, \tilde{L})$ and the covering map $\tilde{p}=\phi^{-1}p_I\tilde{\phi}$: $(\tilde{M}^*, \tilde{L}^*)\to(M^*, L^*)$ constitute a desired pullback diagram, for $p\tilde{q}=q\tilde{p}$ and $\tilde{q}\,|\,\tilde{p}^{-1}(x^*)$: $\tilde{p}^{-1}(x^*)\to p^{-1}(x)$ is a bijection for any $x^*\in M^*$ and $x\in M$ with $q(x^*)=x$. This completes the proof.

For a group π, let $\pi=\pi^{(0)}\supset\pi^{(1)}\supset\pi^{(2)}\supset\cdots$ be the derived series of π, i.e., a series with $\pi^{(i+1)}=[\pi^{(i)}, \pi^{(i)}]$, $i=0, 1, 2, \cdots$, and $\bar{\pi}=\pi/\bigcap_{i=0}^{\infty}\pi^{(i)}$. For example, if π is a free group, then $\bigcap_{i=0}^{\infty}\pi^{(i)}=\{1\}$ (cf. [L/S; p. 14]).

Property V. *Every imitation map q: $M^*\to M$ with M connected induces an epimorphism $q_\sharp$: $\pi_1(M^*)\to\pi_1(M)$ whose kernel* $\mathrm{Ker}\, q_\sharp$ *is a perfect group* (i.e., $\mathrm{Ker}\, q_\sharp=[\mathrm{Ker}\, q_\sharp, \mathrm{Ker}\, q_\sharp]$), *so that $q_\sharp$ induces an isomorphism $\bar{\pi}_1(M^*)\cong\bar{\pi}_1(M)$.*

Proof. Let $\tilde{M}$ be the universal covering space of M and $\tilde{q}\colon \tilde{M}^*\to\tilde{M}$ be the lift of q. By Property IV, $\tilde{q}$ is an imitation map. By Property II, $\tilde{M}^*$ is connected and $H_1(\tilde{M}^*)=0$. This means that $q_\sharp$ is an epimorphism and $\operatorname{Ker} q_\sharp=\pi_1(\tilde{M}^*)$ is a perfect group. Since $\operatorname{Ker} q_\sharp\subset \bigcap_{i=0}^{\infty}\pi_1(M^*)^{(i)}$, the proof is completed.

2. Pure imitations and surfaces

The reflection r in $(M, L)\times I$ defined by $r(x, t)=(x, -t)$ for all $(x, t)\in M\times I$ is called the *standard reflection.*

Proposition 2.1. *If (M^*, L^*) is an imitation of a manifold pair (M, L) and (M^{**}, L^{**}) is a pure imitation of (M^*, L^*), then (M^{**}, L^{**}) is an imitation of (M, L). Further, if (M^*, L^*) is a pure imitation of (M, L), then (M^{**}, L^{**}) is a pure imitation of (M, L).*

Proof. Let $\phi\colon (M^*, L^*)\to(M, L)\times I$ be a reflector of a reflection α in $(M, L)\times I$ and $\phi'\colon (M^{**}, L^{**})\to(M^*, L^*)\times I$ a reflector of a pure reflection α' in $(M^*, L^*)\times I$. $(M, L)\times I$ admits an α-invariant bicollar neighborhood N of $\phi(M^*, L^*)$ so that there is a diffeomorphism $f\colon (M^*, L^*)\times I\cong N$ with $f^{-1}\alpha f$ the standard reflection in $(M^*, L^*)\times I$. Let α'' be the reflection in $(M, L)\times I$ obtained from α by replacing $\alpha|N$ with $f\alpha' f^{-1}$. Note that if α is pure, then so is α''. The composite $\phi''=f\phi'\colon (M^{**}, L^{**})\to(M, L)\times I$ is a reflector of α'' and the map $q''=p_1\phi''\colon (M^{**}, L^{**})\to(M, L)$ is a desired imitation map, completing the proof.

The following question is unanswerable:

QUESTION. Is every imitation pure?

For a reflection α in $(M, L)\times I$ we denote by f_α the diffeomorphism of (M, L) given by $r\alpha|(M, L)\times 1\colon (M, L)\times 1\to(M, L)\times 1$. Two $f, g\in\operatorname{Diff}(M, L)$ are *concordant* if there is an $h\in\operatorname{Diff}((M, L)\times I, (M, L)\times 1, (M, L)\times(-1))$ with $h|(M, L)\times 1=f\times 1$ and $h|(M, L)\times(-1)=g\times(-1)$. Note that f_α^2 is always concordant to $\mathrm{id}_{(M,L)}$.

Lemma 2.2. *Let an imitation map $q\colon (M^*, L^*)\to(M, L)$ be given by a reflector $\phi\colon (M^*, L^*)\to(M, L)\times I$ of a reflection α in $(M, L)\times I$. If f_α is concordant to f', then q is given by a reflector $\phi'\colon (M^*, L^*)\to(M, L)\times I$ of a reflection α' in $(M, L)\times I$ with $f_{\alpha'}=f'$.*

Proof. Let $h\colon (M, L)\times[1, 2]\to(M, L)\times[-2, -1]$ be a diffeomorphism with $h(x, 1)=(f_\alpha(x), -1)$ and $h(x, 2)=(f'(x), -2)$ for all $x\in M$. For $I^+=[-2, 2]$ we define $\alpha^+\in\operatorname{Diff}(M, L)\times I^+$ by $\alpha^+|(M, L)\times[1, 2]=h$, $\alpha^+|(M, L)\times I=\alpha$ and $\alpha^+|(M, L)\times[-2, -1]=h^{-1}$. Let $d\colon (M, L)\times I^+\to(M, L)\times I$ be the diffeomorphism given by $d(x, t)=(x, t/2)$ for all $(x, t)\in M\times I^+$. Then $\alpha'=d\alpha^+ d^{-1}$ is a reflection in $(M, L)\times I$ with $f_{\alpha'}=f'$ and the composite ϕ': $(M^*, L^*)\xrightarrow{\phi}(M, L)\times I \subset(M, L)\times I^+\xrightarrow{d}(M, L)\times I$ is a reflector of α' with $p_1\phi'=q$. This completes the proof.

Corollary 2.3. *An imitation map* q: $(M^*, L^*)\rightarrow(M, L)$ *is pure if* q *is given by a reflector of a reflection* α *with* f_α *concordant to* $\mathrm{id}_{(M,L)}$.

For example, all imitations of S^n $(0\leq n\leq 5)$ and R^n $(n\geq 0)$ are pure and hence normal (cf. Cerf [Ce], Milnor [Mi, 1; § 9. Lemma 5.7]).

Theorem 2.4. *Let* (M, L) *be an* (m, l)*-manifold pair with* $m\leq 2$. *Then for every reflection* α *in* $(M, L)\times I$, *there is an* $h\in \mathrm{Diff}_0(M\times I, M\times\partial I, L\times I)$ *with* $h\alpha h^{-1}$ *the standard reflectoin in* $(M, L)\times I$. *Further, if* $\alpha\,|\,(\partial M)\times I$ *is the standard reflection, then we can take* h *so that* $h\in\mathrm{Diff}_0(M\times I, M\times\partial I, L\times I, \mathrm{rel}(\partial M)\times I)$.

The following is direct from Theorem 2.4 and Corollary 2.3:

Corollary 2.5. *Any imitation of any* (m, l)*-manifold pair* (M, L) *with* $m\leq 2$ *is inessential and pure.*

Note that the compactness of M is not needed in Theorem 2.4 and Corollary 2.5, though we assumed it in the first draft of this paper (cf. [Ka, 0]). To prove Theorem 2.4 we use the fact that $\mathrm{Diff}(D^n, \mathrm{rel}\,\partial D^n)=\mathrm{Diff}_0(D^n, \mathrm{rel}\,\partial D^n)$ for $n\leq 3$ (cf. [Ce], Hatcher [Ha, Appendix]).

2.6 Proof of Theorem 2.4 when $m=0$. Note that $L=\phi$ and there is an $h_1\in\mathrm{Diff}_0(M\times I, M\times\partial I)$ with $\mathrm{Fix}(h_1\alpha h_1^{-1}, M\times I)=M\times 0$. Since $\mathrm{Diff}(D^1, \mathrm{rel}\,\partial D^1) =\mathrm{Diff}_0(D^1, \mathrm{rel}\,\partial D^1)$, we obtain a desired h, completing the proof.

2.7 Proof of Theorem 2.4 when $m=1$. By 2.6 and the isotopy extension theorem, we can assume that $\alpha\,|\,(L\cup\partial M)\times I$ is the standard reflection. Further, by cutting M along L if $L\neq\emptyset$, we can assume that $L=\emptyset$. Choose a discrete set Ω in $\mathrm{Int}\,M$ which cuts M into closed intervals. Then we have an $h_1\in\mathrm{Diff}_0(M\times I, M\times\partial I, \mathrm{rel}\,(\partial M)\times I)$ such that $h_1\alpha h_1^{-1}(\Omega\times I)=\Omega\times I$ and $\mathrm{Fix}(h_1\alpha h_1^{-1}, M\times I)=M\times 0$. By 2.6 and the isotopy extension theorem, we can assume that $h_1\alpha h_1^{-1}(x, t)=(x, -t)$ for all $(x, t)\in\Omega\times I\cup\partial(M\times I)$. Since $\mathrm{Diff}(D^2, \mathrm{rel}\,\partial D^2)=\mathrm{Diff}_0(D^2, \mathrm{rel}\,\partial D^2)$, we obtain a desired h, completing the proof.

When $m=2$, the following two lemmas are basic to the proof of Theorem 2.4:

Lemma 2.8. *For any connected surface* M *with* $\partial M=\emptyset$ *and a 2-disk* D^2 *in* M *and any reflection* α *in* $M\times I$, *there is an* $h\in\mathrm{Diff}_0(M\times I, M\times\partial I)$ *such that* $h\alpha h^{-1}\,|\,D^2\times I$ *is the standard reflection.*

Lemma 2.9. *For any connected surface* M *with* $\partial M\neq\emptyset$ *and any reflection*

α in $M\times I$ with $\alpha|(\partial M)\times I$ the standard reflection, there is an $h\in \mathrm{Diff}_0(M\times I, M\times\partial I, \mathrm{rel}\,(\partial M)\times I)$ such that $h\alpha h^{-1}$ is the standard reflection in $M\times I$.

2.10 Proof of Theorem 2.4 when $m=2$, assuming Lemmas 2.8 and 2.9. If $\partial M=\emptyset$ and $l=-1$, then we have a desired h by Lemmas 2.8 and 2.9. If $\partial M\neq\emptyset$, we can assume by 2.7 and the isotopy extension theorem that $\alpha|(\partial M, \partial M\cap L)\times I$ is the standard reflection. Hence if $\partial M\neq\emptyset$ and $l=-1$, then we have a desired h by Lemma 2.9. If $l=0$ or 1, then we can further assume by 2.6, 2.7, the isotopy extension theorem and the uniqueness of α-invariant tubular neighborhoods that $\alpha|N(L)\times I$ is the standard reflection for a tubular neighborhood $N(L)$ of L in M. Applying Lemma 2.9 to $\alpha|\mathrm{cl}(M-N(L))\times I$, we obtain a desired h, completing the proof.

Proof of Lemma 2.8. Let $p\in D^2$. It suffices to show that there is an $h_1\in\mathrm{Diff}_0(M\times I, M\times\partial I)$ with $h_1\alpha h_1^{-1}(p\times I)=p\times I$, because then we obtain a desired h by 2.6 and the isotopy extension theorem and the uniqueness of α-invariant tubular neighborhoods. By a proper arc in $M\times I$ we mean the image of a smooth proper imbedding $(I, \{1\}, \{-1\})\to(M\times I, M\times 1, M\times(-1))$. For the proof, we need to consider three cases.

Case (1): $M\cong S^2$.

In this case, any proper arc in $M\times I$ connecting $M\times(-1)$ with $M\times 1$ is ambient isotopic to $p\times I$. Hence we obtain a desired h_1 by considering an α-invariant proper arc in $M\times I$.

Case (2): $M\cong R^2$.

In this case, $M'=\mathrm{Fix}(\alpha, M\times I)\cong R^2$, for M' is an acyclic connected open 2-manifold by Lemma 1.1 (3). It suffices to construct an α-invariant proper arc J in $M\times I$ with $\pi_1(M\times I-J)\cong Z$, because then we see from the Dehn's lemma that J is ambient isotopic to $p\times I$ in $M\times I$ by considering the image of J in $D^2\times I$ under an imbedding $g\times\mathrm{id}_I\colon M\times I\to D^2\times I$ with $g\colon M\cong \mathrm{Int}\,D^2$ a diffeomorphism. To obtain such a J, we first choose a proper arc J' in $M\times I$ meeting M' transversally in a single point, x'. Take a 2-sphere Σ in $M\times\mathrm{Int}\,I$ such that $x'\notin\Sigma$ and Σ meets J' transversally in two points and $\pi_1(N-J')\cong Z$ for the non-compact region N of $M\times I$ divided by Σ. Note that $B=\mathrm{cl}(M\times I-N)$ is a 3-disk. We show that there is an $f\in\mathrm{Diff}_0(M\times I, \mathrm{rel}\, M\times\partial I)$ such that J' meets fM' transversally in a single point and $\Sigma\cap fM'=\emptyset$. To see this, we may consider that Σ meets M' transversally in loops. Let c be a loop in $\Sigma\cap M'$ bounding a 2-disk d in Σ such that $\mathrm{Int}\,d\cap M'=\emptyset$ and $|d\cap J'|\leq 1$. Let d' be a 2-disk in M' bounded by c. Note that $|d'\cap J'|=|d\cap J'|$ ($=0$ or 1). Since $d'\cup d$ bounds a 3-disk in $M\times I$, we have an $f_1\in\mathrm{Diff}_0(M\times I, \mathrm{rel}\, M\times\partial I)$ such that J' meets f_1M' transversally in a single point and the component number of $\Sigma\cap f_1M'$ is smaller than that of $\Sigma\cap M'$. By induction on the component number of $\Sigma \cap M'$, we have a desired f. Then we have $f^{-1}(B) \cap M' = \emptyset$. Let W be one of the manifolds obtained from $M \times I$ by splitting along M' such that $W \cap f^{-1}(B) = \emptyset$. Let $J'' = f^{-1}(J') \cap W$. Since the natural homomorphism $\pi_1(M' - f^{-1}(J')) \to \pi_1(M \times I - f^{-1}(B \cup J')) \cong Z$ is an isomorphism, it follows that $\pi_1(W - J'') \cong Z$. Then $J = J'' \cup \alpha J''$ is an α-invariant proper arc in $M \times I$ with $\pi_1(M \times I - J) \cong Z$ and the proof of the case (2) is completed.

Case (3): $M \not\cong S^2, R^2$.

In this case, we have $H_1(M) \neq 0$ and we have a simple loop c and a simple loop or simple proper open curve c^* in M meeting transversally at the point p. Since $M \times I$ is irreducible, we have an $h \in \mathrm{Diff}_0(M \times I, \mathrm{rel}\, M \times \partial I)$ such that $h(c \times I)$ meets $M' = \mathrm{Fix}(\alpha, M \times I)$ transversally in ∂-parallel loops in $h(c \times I)$. Hence we have an α-invariant annulus A in $M \times I$ with $A \cap M \times 1 = c \times 1$. Since any two homotopic simple loops in M are ambient isotopic, we have a reflection $\alpha' = h' \alpha h'^{-1}$ in $M \times I$ with $h' \in \mathrm{Diff}_0(M \times I, M \times \partial I)$ and an α'-invariant annulus A' in $M \times I$ with $A' \cap M \times 1 = c \times 1$ and $A' \cap M \times (-1) = c \times (-1)$. We may consider that A' meets $c^* \times I$ transversely. Then there is just one arc component J in $A' \cap c^* \times I$ with end points $p \times 1$, $p \times (-1)$. J is ambient isotopic to $p \times I$ in $c^* \times I$ and hence in $M \times I$. This means that any α'-invariant proper arc J' in A' is ambient isotopic to $p \times I$ in $M \times I$ and we have a desired h_1. This completes the proof of Lemma 2.8.

Proof of Lemma 2.9. Consider a division of M into a family of 2-disks $\{B_i \mid 1 \leq i < \nu\}$ such that $\mathrm{Int}\, B_i \cap \mathrm{Int}\, B_j = \emptyset$ for all i, j with $i \neq j$ and $\partial M \cap \partial B_1$ is a compact 1-manifold and for each $k < \nu$, $M_k = \cup_{i=1}^{k} B_i$ is a compact connected surface such that if $k+1 < \nu$, then $\partial M_k \cap \partial B_{k+1}$ is a compact 1-manifold, and for any compact set K in M, $K \cap B_i = \emptyset$ except a finite number of i. We shall construct an $h_1 \in \mathrm{Diff}_0(M \times I, M \times \partial I, \mathrm{rel}\, M \times I)$ such that $h_1 \alpha h_1^{-1} | B_1 \times I$ is the standard reflection. For this purpose we may consider that $M' = \mathrm{Fix}(\alpha, M \times I)$ meets $\mathrm{cl}(\partial B_1 - \partial M) \times I$ transversally in proper arcs and simple loops. Since the natural homomorphism $\pi_1(M') \to \pi_1(M \times I)$ is an isomorphism and $M \times I$ is irreducible, we can eliminate these simple loops by cellular moves. This means that there is an $h_1' \in \mathrm{Diff}_0(M \times I, \mathrm{rel}\, \partial(M \times I))$ with $h_1' M' \cap B_1 \times I = B_1 \times 0$. We may consider that $h_1' \alpha h_1'^{-1} \mathrm{cl}(\partial B_1 - \partial M) \times [-1, 0)$ meets $\mathrm{cl}(\partial B_1 - \partial M) \times (0, 1]$ transversally in proper arcs and simple loops. We can eliminate them by cellular moves, so that we have an $h_1'' \in \mathrm{Diff}_0(M \times I, M \times \partial I, \mathrm{rel}(\partial M) \times I \cup M \times 0)$ with $h_1'' h_1' \alpha h_1'^{-1} h_1''^{-1} | B_1 \times I$ the standard reflection. Thus, $h_1'' h_1'$ gives a desired h_1. Applying the same argument to $(\mathrm{cl}(M - B_1) \times I, B_2 \times I, h_1 \alpha h_1^{-1})$ in place of $(M \times I, B_1 \times I, \alpha)$ we obtain an $h_2 \in \mathrm{Diff}_0(M \times I, M \times \partial I, \mathrm{rel}\, \partial M \times I \cup M_1 \times I)$ with $h_2 h_1 \alpha h_1^{-1} h_2^{-1} | M_2 \times I$ the standard reflection. By continuing this process, we obtain, for each k, an $h_k \in \mathrm{Diff}_0(M \times I, M \times \partial I, \mathrm{rel}\, \partial M \times I \cup M_{k-1} \times I)$ with $h_k \cdots h_2 h_1 \alpha h_1^{-1} h_2^{-1} \cdots h_k^{-1} | M_k \times I$ the standard reflection. When $\nu<+\infty$, $h_{\nu-1}\cdots h_2h_1$ gives a desired h. Let $\nu=+\infty$. Our construction guarantees us that for each $x\in M\times I$, there is a natural number k such that $h_k\cdots h_2h_1(x)\in M_k\times I$. Then we have a $g\in\mathrm{Diff}(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I)$ given by $g|M_k\times I=h_1^{-1}h_2^{-1}\cdots h_k^{-1}|M_k\times I$ for each k. We show that $g\in\mathrm{Diff}_0(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I)$. Then g^{-1} gives a desired h. For each k, we take a path $\tilde{h}_k(t)$, $0\leq t\leq 1$, in $\mathrm{Diff}(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I\cup M_{k-1}\times I)$ with $\tilde{h}_k(t)=\mathrm{id}_{M\times I}$ $(0\leq t\leq 1-1/k)$ and $\tilde{h}_k(1)=h_k^{-1}$. For each t with $0\leq t<1$, there is a natural number k such that $t\leq 1-1/k$. Then we define $\tilde{g}(t)=\tilde{h}_1(t)\tilde{h}_2(t)\cdots\tilde{h}_k(t)$, which is a well-defined continuous function from [0, 1) to $\mathrm{Diff}(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I)$ with $\tilde{g}(0)=\mathrm{id}_{M\times I}$. To see that $g\in\mathrm{Diff}_0(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I)$, it suffices to show that $\lim_{t\to 1-0}\tilde{g}(t)=g$ in $\mathrm{Diff}(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I)$. Take any compact set $K\subset M\times I$ and any open set $U\subset M\times I$ with $g(K)\subset U$. Then $K\subset M_k\times I$ and $g|K=h_1^{-1}h_2^{-1}\cdots h_k^{-1}|K$ for some k. We find a small positive number δ such that $\tilde{h}_1(t)\tilde{h}_2(t)\cdots\tilde{h}_k(t)(K)\subset U$ for all t with $1-\delta<t<1$. Since $\tilde{h}_i(t)|K=\mathrm{id}_K$ for all $i\geq k+1$, we see that $\tilde{g}(t)(K)\subset U$ for all t with $1-\delta<t<1$. Hence $\lim_{t\to 1-0}\tilde{g}(t)=g$ in $\mathrm{Diff}(M\times I, M\times\partial I, \mathrm{rel}\,\partial M\times I)$. This completes the proof of Lemma 2.9.

3. The Kinoshita/Terasaka 11-crossing knot and the existence of essential imitations

Let K be an $(m-2)$-knot in S^m. If (S^m, K^*) is an imitation of (S^m, K), then K^* is called a *knot imitation* of K (More generally, when K is a link, K^* is called a *link imitation* of K). The first example of an essential imitation has been suggested by a property of the Kinoshita/Terasaka 11-crossing knot, k_{KT}, in [K/T], which we draw in Fig. 1. Fig. 2 shows a 2-knot K in $R^4=S^4-\{\infty\}$ with an involution α_K on (S^4, K) such that $\mathrm{Fix}(\alpha_K, (S^4, K))\cong(S^3, k_{KT})$. It is known that this 2-knot K is trivial, i.e., bounds a 3-disk in S^4 [For example, this follows from a result of Marumoto [Mar], because K is a ribbon 2-knot of 1-fusion and $\pi_1(S^4-K)\cong Z$]. Note that K bounds an α_K-invariant 3-manifold V in S^4. Take an α_K-invariant normal disk bundle $T(K)$ of K in S^4 so that there is a diffeomorphism f: $(K\times[0, 1], K\times 0)\cong(V\cap T(K), K)$. Then $f(K\times 1)$ bounds a 3-disk in $S^4-\mathrm{Int}\,T(K)$ by an argument of Gluck [G, 1]. This enables

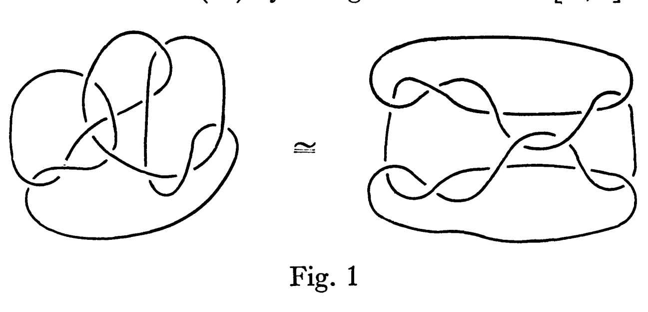

Fig. 1

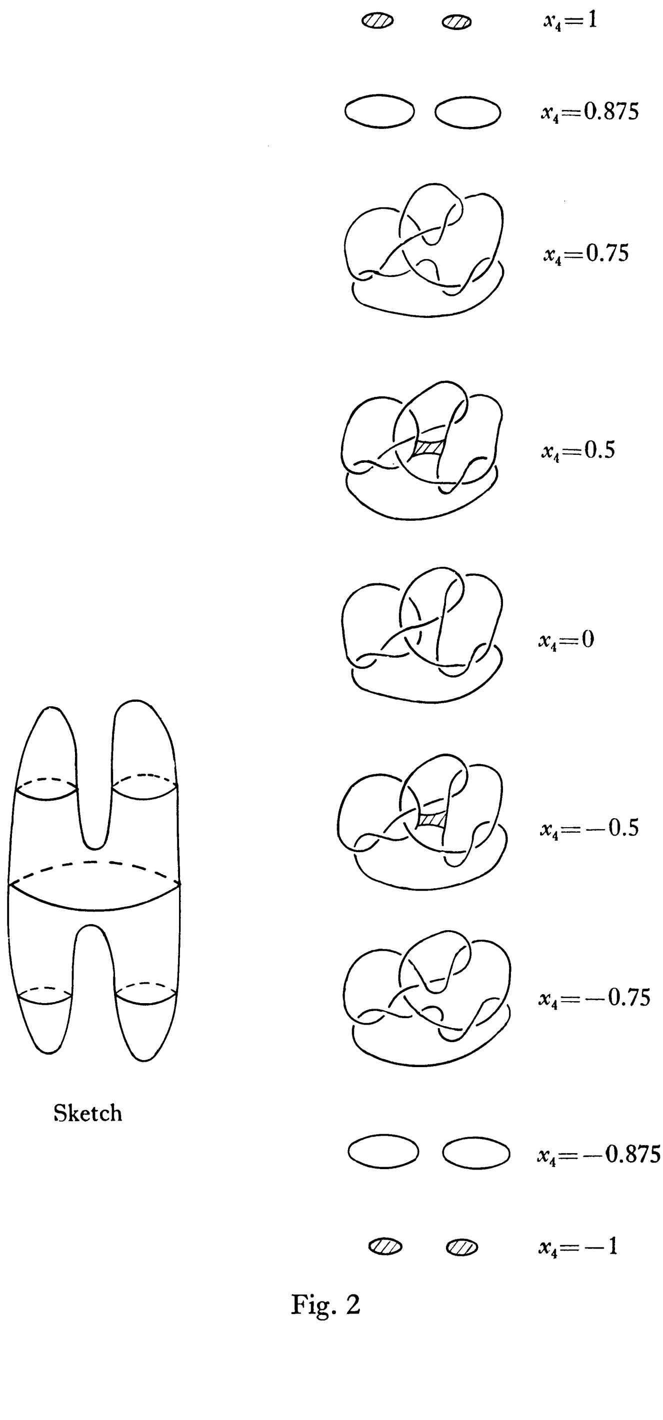

Fig. 2

us to find two disjoint trivial (4,2)-disk pairs (D_i^4, D_i^2), $i=\pm 1$, in (S^4, K) with $\alpha_K(D_1^4, D_1^2)=(D_{-1}^4, D_{-1}^2)$ such that $(S^4-(\mathrm{Int}\,D_1^4\cup \mathrm{Int}\,D_{-1}^4), K-(\mathrm{Int}\,D_1^2\subset \mathrm{Int}\,D_{-1}^2))$ is diffeomorphic to $(S^3, k_0)\times I$ with k_0 a trivial knot. Then α_K defines a reflection α in $(S^3, k_0)\times I$ with $\mathrm{Fix}(\alpha, (S^3, k_0)\times I)\cong(S^3, k_{KT})$ and we see that k_{KT} is a knot imitation of k_0 . By Properties of I, II, IV, the Alexander ploynomial of any knot imitation of a trivial knot must be trivial. Hence any non-trivial knot with up to 10 crossings is no knot imitation of a trivial knot (cf. [B/Z]). That is, k_{KT} is a knot with the smallest crossing number in the class of all knot imitations of a trivial knot. Using a tangle version of the fact that k_{KT} is a knot imitation of a trivial knot, Nakanishi [N] proved, in our terminology, that every link in S^3 has, as a normal link imitation, a prime link (and a hyperbolic link by [So], [Kan]). In a forthcoming paper [Ka, 2], we shall propose a notion finer than a normal imitation, which we call an almost identical imitation, and show the existence of almost identical imitations with hyperbolic exteriors for any (3, 1)-manifold pair in a reasonable large class including all links in S^3. In this section, by making use of an imitation map q: $(S^3, k_{KT})\to(S^3, k_0)$, we shall observe the following weak but general assertion (which contrasts with Corollary 2.5):

Proposition 3.1. *For any* (m, l)*-manifold pair* (M, L) *with* $m\geq 3$, *there are infinitely many* (*up to conjugations*) *essential normal imitations* (M^*, L^*) *of* (M, L).

Proof. By the uniqueness of α-invariant tubular neighborhoods of $k_0\times I$ in $S^3\times I$, we may consider that $\alpha\,|\,T(k_0)\times I$ is the standard reflection for a tubular neighborhood $T(k_0)$ of k_0 in S^3. Let $S^3\,(k_{KT};\,1/d)$ be the Dehn surgery manifold of S^3 along k_{KT} with coefficient $1/d$. Then any imitation map q: $(S^3, k_{KT})\to$ (S^3, k_0) associated with this reflection α in $(S^3, k_0)\times I$ induces an imitation map q_d: $S^3(k_{KT};\,1/d)\to S^3$, since the Dehn surgery manifold of S^3 along k_0 with coefficient 1/d is again S^3. By Thurston's hyperbolization theorem [T, 1], k_{KT} is a hyperbolic knot. Then by Thurston's argument on hyperbolic Dehn surgery (cf. [T, 1], [T, 2]), there is a positive integer d^* such that $S^3(k_{KT};\,1/d)$ is hyperbolic with $\mathrm{Vol}\,S^3(k_{KT};\,1/d)<\mathrm{Vol}\,(S^3-k_{KT})$ for all d with $|d|\geq d^*$ and $\sup_{|d|\geq d^*}\mathrm{Vol}\,S^3(k_{KT};\,1/d)=\mathrm{Vol}(S^3-k_{KT})$. Hence we have infinitely many imitation maps $\bar{q}_i$: $\bar{S}_i^3\to S^3$ ($i=1, 2, 3, \cdots$) such that $\bar{S}_i^3$ are hyperbolic manifolds with different volumes. Let $G_i=\pi_i(\bar{S}_i^3)$. By Mostow rigidity (cf. [T, 1]), any two of G_i, $i=1, 2, 3, \cdots$, are not isomorphic. Since $\alpha\,|\,T(k_0)\times I$ was the standard reflection, we may consider that $\bar{q}_i$ induces an imitation map $\bar{D}_i^3\to D^3$ (also denoted by $\bar{q}_i$) for a 3-manifold $\bar{D}_i^3$, obtained from $\bar{S}_i^3$ by removing an open 3-disk. Since $\mathrm{Diff}_0(D^3, \mathrm{rel}\,\partial D^3)=\mathrm{Diff}(D^3, \mathrm{rel}\,\partial D^3)$, we see from Lemma 2.2 that $\bar{q}_i$: $\bar{D}_i^3\to D^3$ is normal for all i. Clearly, $\pi_1(\bar{D}_i^3)\cong G_i$. Let $m\geq 4$. Assume that there is a normal imitation map $\bar{q}_i^{m-1}$: $\bar{D}_i^{m-1}\to D^{m-1}$ with $\pi_1(\bar{D}_i^{m-1})\cong G_i$. Regard S^m as a union $D^{m-1}\times S^1\cup S^{m-2}\times D^2$. Then $\bar{q}_i^{m-1}$: $\bar{D}_i^{m-1}\to D^{m-1}$ induces a normal imitation map $\bar{q}_i^m$: $\bar{D}_i^m = \bar{D}_i^{m-1} \times S^1 \cup S^{m-2} \times D^2 - \mathrm{Int}\, D_0^m \to D^{m-1} \times S^1 \cup S^{m-2} \times D^2 - \mathrm{Int}\, D_0^m \cong D^m$ for an m-disk $D_0^m \subset \mathrm{Int}\, S^{m-2} \times D^2$. Then $\pi_1(\bar{D}_i^m) \cong G_i$. Thus, we have a normal imitation map $\bar{q}_i^m$: $\bar{D}_i^m \to D^m$ with $\pi_1(\bar{D}_i^m) \cong G_i$ for all $m \geq 3$ and all i. To complete the proof, we choose an m-disk D^m in $\mathrm{Int}\,(M-L)$. Replacing D^m by $\bar{D}_i^m$, we obtain from (M, L) a normal imitation (M_i^*, L_i^*) of (M, L) with imitation map q_i^M defined by $\bar{q}_i^m$ and the identity on $M - \mathrm{Int}\, D^m$. Suppose that q_i^M and q_j^M are conjugate for some i, j with $i \neq j$. Take the universal covering space $\widetilde{M-L}$ of $M-L$. By Properties I, IV, q_i^M and q_j^M lift conjugate imitation maps $\tilde{q}_i^{M-L}$: $\widetilde{M_i^*-L_i^*} \to \widetilde{M-L}$ and $\tilde{q}_j^{M-L}$: $\widetilde{M_j^*-L_j^*} \to \widetilde{M-L}$. Note that $\widetilde{M_i^*-L_i^*}$ (or $\widetilde{M_j^*-L_j^*}$, respectively) has just one non-simply connected component, whose fundamental group is isomrophic to a free product of copies of G_i (or G_j, respectively). Thus, a free product of some copies of G_i must be isomorphic to a free product of some copies of G_j. Since G_i and G_j are non-isomorphic indecomposable groups ($\not\cong Z$), it follows from the Kurosh Subgroup Theorem (cf. [L/S]) that G_i is isomorphic to a proper subgroup of G_j and G_j is isomorphic to a proper subgroup of G_i. Thus, G_i is isomorphic to a subgroup N_i of G_i of index $r_i \geq 2$. Let $\bar{\bar{S}}_i^3$ be a covering space of $\bar{S}_i^3$ with $\pi_1(\bar{\bar{S}}_i^3) = N_i$. Since $\bar{S}_i^3$ and $\bar{\bar{S}}_i^3$ are $K(\pi, 1)$-spaces and $G_i \cong N_i$, $\bar{S}_i^3$ is homotopy equivalent to $\bar{\bar{S}}_i^3$. In particular, $H_3(\bar{\bar{S}}_i^3) \cong H_3(\bar{S}_i^3) \cong Z$. This means that $r_i < +\infty$ and $\bar{\bar{S}}_i^2$ is a hyperbolic 3-manifold with $\mathrm{Vol}\, \bar{\bar{S}}_i^3 = r_i \,\mathrm{Vol}\, \bar{S}_i^3$. By Mostow rigidity (cf. [T, 1]), $\mathrm{Vol}\, \bar{\bar{S}}_i^3 = \mathrm{Vol}\, \bar{S}_i^3$. Hence $r_i = 1$, a contradiction. Therefore, any two of q_i^M, $i = 1, 2, 3, \cdots$, are not conjugate. This completes the proof.

4. Remarks on imitations of 4-manifolds

In a forthcoming paper[Ka, 2], we shall show that every closed connected oriented 3-manifold has, as a normal imitation, a hyperbolic 3-manifold (cf. [Ka, 0], [Ka, 1]). The following remark answers in part a question asking whether an analogous assertion holds in dimension 4:

Proposition 4.1. *Let M be a closed 4-manifold. If there is an imitation map q: $M^* \to M$ with M^* negatively (or non-positively, respectively) curved, then Euler characteristic $\chi(M)$ of M is posoitive (or non-negative, respectively).*

Proof. By Chern's result [Ch], $\chi(M^*) > 0$ (or ≥ 0, respectively). By Property II, $\chi(M^*) = \chi(M)$. Hence $\chi(M) > 0$ (or ≥ 0, respectively), completing the proof.

For example, $S^1 \times S^3 \# S^1 \times S^3$ can not have as an imitation any non-positively curved 4-manifold. However, the following question is unanswerable:

QUESTION. Does what non-aspherical closed 4-manifold M have an aspherical 4-manifold as an imitation? (Is the condition $\chi(M)\geq 0$ needed here?)

Next, we consider any exotic 4-space $\tilde{R}^4$, i.e., any smooth open 4-manifold, homeomorphic but not diffeomorphic to R^4 (cf. Gompf [G]).

Proposition 4.2. *$\tilde{R}^4$ is a normal imitation of R^4.*

Proof (based on a suggestion by Y. Matsumoto). Note that there is a diffeomorphism $f\colon \tilde{R}^4\times \mathrm{Int}\, I\cong R^4\times \mathrm{Int}\, I$. For a point $x_0\in \tilde{R}^4$, we have a diffeomorphism $g\colon (R^4\times \mathrm{Int}\, I, f(x_0\times \mathrm{Int}\, I))\cong (R^4, 0)\times \mathrm{Int}\, I$. Let r be the standard reflection in $\tilde{R}^4\times I$ and $\alpha=gf(r\mid \tilde{R}^4\times \mathrm{Int}\, I)f^{-1}g^{-1}$. Then α is an involution on $(R^4, 0)\times \mathrm{Int}\, I$ with $\mathrm{Fix}(\alpha, R^4\times \mathrm{Int}\, I)=gf(\tilde{R}^4\times 0)$. For an open 4-ball neighborhood V of 0 in R^4, we have an $h\in \mathrm{Diff}_0(R^4\times \mathrm{Int}\, I)$ such that $\alpha^h=h\alpha h^{-1}$ acts on $V\times \mathrm{Int}\, I$ by $\alpha^h(x, t)=(x, -t)$ for all $(x, t)\in V\times \mathrm{Int}\, I$, by using the uniqueness of tubular neighborhoods. The action α^h on $R^4\times \mathrm{Int}\, I$ extends to a smooth action α^h_+ on the smooth manifold $X=R^4\times \mathrm{Int}\, I\cup V\times I$ with boundary $V\times\partial I$. Since X is diffeomorphic to $R^4\times I$ and $\mathrm{Fix}(\alpha^+_h, X)=hgf(\tilde{R}^4\times 0)$, we have a reflector $\phi\colon \tilde{R}^4\to R^4\times I$. Hence $\tilde{R}^4$ is an imitation of R^4. By Corollary 2.3, all imitations of R^4 are pure and hence normal. This completes the proof.

REMARK 4.3. Every (smooth) homology 4-sphere $\bar{S}^4$ is a normal imitation of S^4. In fact, it is well-known that $\bar{S}^4$ is the boundary of a smooth contractible 5-manifold W and the double DW is diffeomorphic to S^5. This means that there is a reflector $\bar{S}^4\to S^4\times I$ and $\bar{S}^4$ is an imitation of S^4, which is pure and hence normal by Corollary 2.3.

REMARK 4.4. Every exotic n-sphere $\tilde{S}^n$ $(n\geq 7)$ is no imitation of S^n. In fact, if $\tilde{S}^n$ is an imitation of S^n, then $\tilde{S}^n$ is h-cobordant to S^n by Lemma 1.1. By the h-cobordism theorem [Mi, 1], $\tilde{S}^n$ is diffeomorphic to S^n, a contradiction.

5. Imitations of compact m-manifolds with $m\geq 5$ and the Whitehead torsion invariant

Let M be a compact connected oriented m-manifold, and $\tilde{M}$ be the universal covering space of M. Let $q\colon M^*\to M$ be an imitation map, and $\tilde{q}\colon \tilde{M}^*\to\tilde{M}$ be the lift of q. By Properties IV, II, $\tilde{q}$ induces a homology isomorphism. By Milnor's remark [Mi, 2; Remark 2 (p. 387)], we can define the torsion $\tau(q)\in \mathrm{Wh}\,\pi_1(M)$ to be the torsion $\tau(M_q, M^*)\in \mathrm{Wh}\,\pi_1(M_q)$ for the mapping cylinder M_q of q under the natural identification $\mathrm{Wh}\,\pi_1(M_q)\cong \mathrm{Wh}\,\pi_1(M)$. We call this torsion the *torsion of the imitation map* $q\colon M^*\to M$. Note the fq $(f\in \mathrm{Diff}\, M)$ is also an imitation map.

Lemma 5.1. *If two imitation maps $q\colon M^*\to M$, $q\colon M^{**}\to M$ are conjugate, then we have $\tau(fq)=\tau(q')$ for an $f\in \mathrm{Diff}\, M$.*

Proof. There are reflectors $\phi: M^* \to M\times I$, $\phi': M^{**} \to M\times I$ and an $h \in \mathrm{Diff}(M\times I, M\times 1, M\times(-1))$ with $p_1\phi = q$, $p_1\phi' = q'$ and $h\phi M^* = \phi' M^{**}$. Let $\tilde{q}: \tilde{M}^* \xrightarrow{\tilde{\phi}} \tilde{M}\times I \xrightarrow{\tilde{p}_1} \tilde{M}$ be the lift of $q: M^* \xrightarrow{\phi} M\times I \xrightarrow{p_1} M$. Since $\tilde{\phi}$ and $\tilde{p}_1$ induce homology isomorphisms, we can define the torsions $\tau(\phi)\in \mathrm{Wh}\,\pi_1(M\times I)$ and $\tau(p_1)\in \mathrm{Wh}\,\pi_1(M)$, with the identity $\tau(q) = p_{1*}\tau(\phi)+\tau(p_1)$. But, $\tau(p_1)=0$, so that $\tau(q)=p_{1*}\tau(\phi)=p_{1*}\tau(M\times I, \phi M^*)$. Similarly, $\tau(q')=p_{1*}\tau(M\times I, \phi' M^{**})$. Let $f\in \mathrm{Diff}\, M$ be given by $h|M\times 1 \in \mathrm{Diff}\, M\times 1$. Then $\tau(q')=p_{1*}h_*\tau(M\times I, \phi M^*)=f_* p_{1*}\tau(M\times I, \phi M^*)=f_*\tau(q)=\tau(fq)$. This completes the proof.

The following is direct:

Corollary 5.2. *If an imitation map $q: M^* \to M$ is inessential, then $\tau(q)=0$.*

Let $\phi: M^* \to M\times I$ be a reflector with $p_1\phi = q$. By Lemmas 1.1, 1.2, the lift $(\tilde{W}_+; \tilde{M}\times 1, \phi\tilde{M}^*)$ of the triad $(W_+; M\times 1, \phi M^*)$ to the universal covering space $\tilde{W}_+$ of W_+ gives a homology cobordism. Hence the torsions $\tau(W_+, M\times 1)$, $\tau(W_+, \phi M^*)\in \mathrm{Wh}\,\pi_1(W_+)$ are also defined. Let $p_\pm = p_1 | W_\pm: W_\pm \to M$. By Lemma 1.1, we have $p_{\pm *}: \mathrm{Wh}\,\pi_1(W_\pm) \cong \mathrm{Wh}\,\pi_1(M)$.

Lemma 5.3. *Assume that an imitation $q: M^* \to M$ is ∂-diffeomorphic, that is, $q|\partial M^*: \partial M^* \to \partial M$ is a diffeomorphism if $\partial M \neq \emptyset$. Then for any reflector $\phi: M^* \to M\times I$ with $p_1\phi = q$, we have $\tau(q) = -2p_{+*}\tau(W_+, M\times 1)$ and $\bar{\tau}(W_+, M\times 1) = (-1)^{m+1}\tau(W_+, M\times 1)$, where $\bar{\tau}$ denotes the conjugate of τ.*

The following is direct from Lemmas 5.1 and 5.3:

Corollary 5.4. *If an imitation map $q: M^* \to M$ is conjugate to a ∂-diffeomorphic imitation map $q': M^{**} \to M$, then there is an element $\tau \in \mathrm{Wh}\,\pi_1(M)$ such that $\tau(q) = -2\tau$ and $\bar{\tau} = (-1)^{m+1}\tau$.*

Proof of Lemma 5.3. Using the lift of a collar of ϕM^* in $M\times I$ to $\tilde{M}\times I$, we have $\tau(q)=p_{1*}\tau(M\times I, \phi M^*)=p_{+*}\tau(W_+, \phi M^*)+p_{-*}\tau(W_-, \phi M^*)$. By Lemma 1.1(1), $p_{+*}\tau(W_+, \phi M^*)=p_{-*}\tau(W_-, \phi M^*)$. Hence $\tau(q)=2p_{+*}\tau(W_+, \phi M^*)$. When $\partial M \neq \emptyset$, note that $(\partial M\times I, \phi(\partial M^*))$ is diffeomorphic to $(\partial M\times I, \partial M\times 0)$. Let $(W_+^\sigma; \phi M^{*\sigma}, M_1^\sigma)$ be a triangulation of $(W_+; \phi M^*, M_1)$ with $M_1 = \partial W_+ - \mathrm{Int}\,\phi M^*$ and $(W_+^\delta; \phi M^{*\delta}, M_1^\delta)$ be a dual cell division. The Reidemeister duality between the chain complexes $C_\sharp(\tilde{W}_+^\sigma, \tilde{\phi}\tilde{M}^{*\sigma})$ and $C_\sharp(\tilde{W}_+^\delta, \tilde{M}_1^\delta)$ (cf. [Mi, 3]) implies the identity $\tau(W_+, \phi M^*)=(-1)^m\bar{\tau}(W_+, M_1)=(-1)^m\bar{\tau}(W_+, M\times 1)$ (cf. [Mi, 2]). Hence

$$\tau(q) = (-1)^m 2p_{+*}\bar{\tau}(W_+, M\times 1)$$

and

$$\begin{aligned} p_{1*}\tau(M\times I, W_+) &= p_{-*}\tau(W_-, \phi M^*) = p_{+*}\tau(W_+, \phi M^*) \\ &= (-1)^m p_{+*}\bar{\tau}(W_+, M\times 1)\,. \end{aligned}$$

On the other hand, by the short exact sequence $0\to C_\sharp(\tilde{W}_+, \tilde{M}\times 1)\to C_\sharp(\tilde{M}\times I, \tilde{M}\times 1)\to C_\sharp(\tilde{M}\times I, \tilde{W}_+)\to 0$ under a triangulation of $(M\times I, W_+, M\times 1)$, we have

$$0 = p_{1*}\tau(M\times I, M\times 1) = p_{+*}\tau(W_+, M\times 1)+p_{1*}\tau(M\times I, W_+)\,.$$

That is,

$$p_{1*}\tau(M\times I, W_+) = -p_{+*}\tau(W_+, M\times 1)\,.$$

Therefore, $p_{+*}\tau(W_+, M\times 1)=(-1)^{m+1}p_{+*}\bar{\tau}(W_+, M\times 1)$, that is, $\tau(W_+, M\times 1)=(-1)^{m+1}\bar{\tau}(W_+, M\times 1)$ and $\tau(q)=-2p_{+*}\tau(W_+, M\times 1)$. This completes the proof.

It follows from Properties II, IV that any imitation map $q: M^*\to M$ inducing an isomorphism $q_\sharp: \pi_1(M^*)\cong\pi_1(M)$ is a homotopy equivalence. From now on, we shall consider a homotopy equivalent ∂-diffeomorphic imitation map $q: M^*\to M$ with $m\geq 5$. Our main tool is the (relative) s-cobordism theory due to Barden/Mazur/Stallings (cf. [Mi, 2]).

Theorem 5.5. *For $m\geq 5$ we have the following:*
(1) *For every element $\tau\in \mathrm{Wh}\,\pi_1(M)$ with $\bar{\tau}=(-1)^{m+1}\tau$, there is a homotopy equivalent ∂-diffeomorphic imitation map $q: M^*\to M$ with $\tau(q)=-2\tau$,*
(2) *Assume that $\mathrm{Wh}\,\pi_1(M)$ is 2-torsion-free. Then two homotopy equivalent ∂-diffeomorphic imitation maps $q: M^*\to M$, $q': M^{**}\to M$ are conjugate if and only if we have $\tau(fq)=\tau(q')$ for an $f\in \mathrm{Diff}\, M$.*

Corollary 5.6. *Assume that $m\geq 5$ and $\mathrm{Wh}\,\pi_1(M)$ is 2-torsion-free. Then a homotopy equivalent ∂-diffeomorphic imitation map $q: M^*\to M$ is inessential if and only if $\tau(q)=0$.*

Proof of Theorem 5.5. To see (1), note that there is a relative h-cobordism $(W; M, M^*)$ with $\tau(W, M)=\tau$. Since $\tau+(-1)^m\bar{\tau}=0$, the double of W pasting two copies of M^* is a product (cf. [Mi, 2]). Hence we obtain a homotopy equivalent ∂-diffeomorphic imitation map $q: M^*\to M$ with $\tau(q)=-2\tau$, proving (1). Next, we show the 'if' part of (2). (The 'only if' part follows from Lemma 5.1.) For this purpose, we may assume that $f=\mathrm{id}_M$. Let $\phi: M^*\to M\times I$, $\phi': M^{**}\to M\times I$ be reflectors with $p_1\phi=q$, $p_1\phi'=q'$. The triads $(W_+; M\times 1, \phi M^*)$ and $(W'_+; M\times 1, \phi' M^{**})$ (obtained from $M\times I$ by splitting along ϕM^* and $\phi' M^{**}$, respectively) are relative h-cobordisms, because q, q' are homotopy equivalent ∂-diffeomorphic imitation maps. By Lemma 5.3, $\tau(q)=-2p_{+*}\tau(W_+, M\times 1)$ and $\tau(q')=-2p'_{+*}\tau(W'_+, M\times 1)$ (where $p'_+=p_1\,|\,W'_+: W'_+\to M$). Since $\mathrm{Wh}\,\pi_1(M)$ is 2-torsion-free and $\tau(q)=\tau(q')$, we have $p_{+*}\tau(W_+, M\times 1)=p'_{+*}\tau(W'_+, M\times 1)$. By [Mi, 2], there is a diffeomorphism $g: W_+\cong W'_+$ such that $g\,|\,M\times 1=\mathrm{id}_{M\times 1}$ and $g(\phi M^*)=\phi' M^{**}$. By Lemma 1.1 (1), we can construct $\bar{g}\in \mathrm{Diff}\,(M\times I, M\times 1, M\times(-1))$ with $\bar{g}(\phi M^*)=\phi' M^{**}$. Thus, q and q' are conjugate. This completes the proof.

EXAMPLE 5.7. Let C_5 be a cyclic group of order 5. Let t be an automorphism of C_5 sending each element to its inverse, and G be the HNN group of C_5 by t. Note that $[G, G]=C_5$ and G is the 2-knot group of the 2-twist spun figure eight knot and hence the group of an $(m-2)$-knot K in S^m for all $m\geq 5$. Wh C_5 is known to be an infinite cyclic group with a generator represented by $\tau=x+\bar{x}-1$ for a generator x of C_5 (cf. [Mi, 2]). Since t induces the identity on Wh C_5, Wh C_5 is imbedded in Wh G by a monomorphism induced from the inclusion $C_5\subset G$ (cf. Farrell/Hsiang[F/H]). Let m be odd ≥ 5. Then $\bar{\tau}=(-1)^{m+1}\tau$. Applying Theorem 5.5 (1) to the compact exterior $E^m=S-\text{Int}\, N(K)$ with $N(K)$ a normal disk bundle of K in S^m, we have a homotopy equivalent ∂-diffeomorphic imitation map $q_n^E: E_n^*\to E$ with $\tau(q_n^E)=-2n\tau$ for all non-negative integers n. Note that the adjunction space $E_n^*\cup N(K)$ identifying ∂E_n^* with $\partial N(K)$ by the diffeomorphism $q_n^E\,|\,\partial E_n^*$: $\partial E_n^*\cong\partial N(K)$ is a homotopy m-sphere $\tilde{S}^m$ and q_n^E extends to an imitation map $\tilde{q}_n$: $(\tilde{S}^m, \tilde{K}_n^*)\to(S^m, K)$. By Lemma 1.1, $\tilde{S}^m$ is h-cobordant to S^m, so that $\tilde{S}^m$ is diffeomorphic to S^m. Thus, we have an imitation map q_n: $(S^m, K_n^*)\to(S^m, K)$ such that $q_n^{-1}N(K)=N(K_n^*)$ is a normal disk bundle of K_n^* in S^m and $q_n|N(K_n^*), K_n^*)$: $(N(K_n^*), K_n^*)\to(N(K), K)$ is a diffeomorphism and $S^m-\text{Int}\, N(K_*^n)=E_n^*$ and $q_n\,|\,E_n^*=q_n^E$: $E_n^*\to E$, which is a homotopy equivalent ∂-diffeomorphic imitation map.

Assertion 5.8. *Any two of* $q_0, q_1, q_2, q_3, \cdots$ *are not conjugate.*

In fact, if q_n and q_s are conjugate, then q_n^E and q_s^E are conjugate. By Lemma 5.1, there is an $f\in\text{Diff}\, E$ with $\tau(q_s^E)=\tau(fq_n^E)$. But, f induces an automorphism f_* of Wh G with f_*Wh C_5=Wh C_5. Since $\tau(q_s^E)=-2s\tau$ and $\tau(fq_n^E)=f_*(\tau(q_n^E))=\pm 2n\tau$ and n, $s\geq 0$, we see that $n=s$, as desired.

References

[B/Z] G. Burde/H. Zieschang: Knots, Walter de Gruyter, 1985.

[C] J. Cerf: *Sur les Difféomorphismes dela Sphère de Dimension Trois* $(\Gamma_4=0)$, Lecture Notes in Math. 53, Springer-Verlag, 1968.

[Ch] S.S. Chern: *On curvature and characteristic classes of a Riemann manifold,* Abh. Math. Sem. Univ. Hamburg **20** (1955), 117–126.

[F/H] F.T. Farell/W.C. Hsiang: *Manifolds with* $\pi=G\times_\alpha T$, Amer. J. Math. **95** (1973), 813–848.

[G] R.E. Gompf: *An infinite set of exotic* R^4*'s*, J. Differential Geometry **21** (1985), 283–300.

[G,1] H. Gluck: *The embedding of two-spheres in the four-sphere,* Trans. Amer. Math. Soc. **104** (1962), 308–333.

[Ha] A.E. Hatcher: *A proof of the Samle conjecture,* Ann. Math. **117** (1983), 553–607.

[Kn] T. Kanenobu: *Hyperbolic links with Brunnian properties,* J. Math. Soc. Japan **38** (1986), 295–308.

[Ka,0] A. Kawauchi: *Hyperbolic imitations of 3-manifolds,* A preliminary version, 1987 (unpublished).

[Ka,1] A. Kawauchi: *Imitations of* (3,1)*-dimensional manifold pairs,* Sūgaku **40** (1988), 193–204 (in Japanese; An English translation will appear).

[Ka,2] A. Kawauchi: *Almost identical imitations of* (3,1)*-dimensional manifold pairs,* Osaka J. Math. (to appear).

[K/K/S] A. Kawauchi/T. Kobayashi/M. Sakuma: *On 3-manifolds with no periodic maps,* Japan. J. Math. **10** (1984), 185–193.

[K/T] S. Kinoshita/H. Terasaka: *On unions of knots,* Osaka Math. J. **9** (1957), 131–153.

[L/S] R.C. Lyndon/P.E. Schupp: *Combinatorial Group Theory,* Ergebnisse der Mathematik und ihrer Grenzgebiete **89**, Springer-Verlag, 1977.

[Ma] Y. Marumoto: *On ribbon 2-knots of 1-fusion,* Math. Sem. Notes, Kobe Univ. **5** (1977), 59–68.

[Mi,1] J.W. Milnor: *Lectures on the h-Cobordism,* Princeton Univ. Press, Princeton, 1965.

[Mi,2] J.W. Milnor: *Whitehead torsion,* Bull. Amer. Math. Soc. **72** (1966), 358–426.

[Mi,3] J.W. Milnor: *A duality theorem for Reidemeister torsion,* Ann. Math. **76** (1962), 137–147.

[M/S] J.W. Milnor/J.D. Stasheff: *Characteristic Classes,* Ann. Math. Studies 76, Princeton Univ. Press, Princeton, 1974.

[N] Y. Nakanishi: *Primeness of links,* Math. Sem. Notes, Kobe Univ. **9** (1981), 415–440.

[So] T. Soma: *Simple links and tangles,* Tokyo J. Math. **6** (1983), 65–73.

[Sp] E.H. Spanier: *Algebraic Topology,* McGraw Hill, 1966.

[T,1] W.P. Thurston: *Three dimensional manifolds, Kleinian groups and hyperbolic geometry,* Bull. Amer. Math. Soc. **6** (1982), 357–381.

[T,2] W.P. Thurston: *The Geometry and Topology of 3-Manifolds* (preprint).

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