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ON THE ADMISSIBILITY OF SINGULAR PERTURBATIONS IN CAUCHY PROBLEMS

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1. Introduction

In this paper, we shall mainly study sufficient conditions for divergence of solutions of a family of singulary perturbed equations as the positive parameter ε tends to zero. In [4], J. Chaillou studied singular perturbations in Cauchy porblems for hyperbolic operators with constant coefficients, but we shall study singular perturbations in non-characteristic Cauchy problems for kowalewskian operators.

Let $P_1(D)$ and $P_2(D)$ be linear differential operators of kowalewskian with constant coefficients. Put ord $P_1=m$ and ord $P_2=m'$. Assume that $m>m'$. Let us consider the following one-parameter family of Cauchy problems:

$$(1.1)\qquad \begin{cases} (\varepsilon\cdot P_1(D)+P_2(D))\,u(x)=0\,, & \text{in}\quad \boldsymbol{R}^n\,;\\ D_1^{j-1}\,u(x)|_{x_1=0}=\phi_j(x'),\ j=1,\cdots,m\,. \end{cases}$$

When the Cauchy problems (1.1) are uniquely solvable, we can set a problem of the convergence of solutions.

Denote by $\mathcal{A}$ the Cauchy data space and by $\Phi=(\phi_1,\cdots,\phi_m)$ an element of $\mathcal{A}$. Denote by $\mathcal{O}(\boldsymbol{C}^{n-1})$ the set of entire functions defined in $\boldsymbol{C}^{n-1}$. If $\mathcal{A}=\mathcal{O}(\boldsymbol{C}^{n-1})^m$, then the Cauchy-Kowalewski theorem implies that the Cauchy problems (1.1) are globally uniquely solvable. If $\mathcal{A}=\mathrm{F}^{-1}(C_0^\infty(\boldsymbol{R}^{n-1}))^m$, where F^{-1} denotes the inverse Fourier transformation, then the Cauchy problems (1.1) can be solved by the Fourier transformation.

Devide the equation of (1.1) by ε and put $\varepsilon^{-1}=\lambda^{m-m'}$, where $\lambda>0$. Then

$$(1.2)\qquad \begin{cases} (P_1(D)+\lambda^{m-m'}\,P_2(D))\,u=0\,, & \text{in}\quad \boldsymbol{R}^n\,;\\ D_1^{j-1}\,u|_{x_1=0}=\phi_j,\ j=1,\cdots,m\,. \end{cases}$$

Since the convergence or divergence of solutions of (1.1) when $\varepsilon\downarrow 0$ is equivalent to that of (1.2) when $\lambda\uparrow\infty$, we shall deal with (1.2) instead of (1.1).

The reduced problem of (1.2) is

$$(1.3)\qquad \begin{cases} P_2(D)\,u=0\,, & \text{in}\quad \boldsymbol{R}^n\,;\\ D_1^{j-1}\,u|_{x_1=0}=\phi_j,\ j=1,\cdots,m'\,. \end{cases}$$

We shall introduce the notion of "admissibility" of singular perturbations in Cauchy problems. Let U' be a domain of $\boldsymbol{R}_{x'}^{n-1}$ and δ be a positive number.

Definition 1.1. The Cauchy problems (1.1) in $[-\delta, \delta]\times U'$ with the Cauchy data space $\mathcal{A}$ are said to be *admissible* as a singular perturbation with respect to a given Cauchy problem (1.3) if for every Cauchy data Φ in $\mathcal{A}$, the solutions of (1.1) converge to that of (1.3) in $C((-\delta, \delta)\times U')$. The unilateral Cauchy problems (1.1) in $[0, \delta]\times U'$ (resp. in $[-\delta, 0]\times U'$) with the Cauchy data space $\mathcal{A}$ are said to be *admissible* as a singular perturbation with respect to a given unilateral Cauchy problem (1.3) if for every Cauchy data Φ in $\mathcal{A}$, the solutions of (1.1) converge to that of (1.3) in $C((0, \delta)\times U')$ (resp. in $C((-\delta, 0)\times U')$).

Denote the characteristic roots of $P_2(\xi)=0$ with respect to ξ_1 by $\sigma_j(\xi')$, $j=1, \cdots, m'$.

Assumption 1.2. There exists a point ξ_0' in $\boldsymbol{R}^{n-1}$ such that for $1\leqq j< k\leqq m'$

$$\sigma_j(\xi_0') \neq \sigma_k(\xi_0') .$$

Remark. If Assumption 1.2 is satisfied, then there exists an open ball $B_0=B(r_0; \xi_0')$ of radius r_0 with the centre ξ_0' such that all $\sigma_j(\xi')$ are simple on the closure of B_0.

Let $p_{1,0}$ be the coefficient of ξ_1^m in P_1 and $p_{2,0}$ be that of $\xi_1^{m'}$ in P_2. Put $p=p_{2,0}/p_{1,0}$.

Condition 1.3. $(m-m'=2$ and $p<0)$ or $(m-m'=1$ and p is real).

Let Assumption 1.2 be satisfied and B_0 be the open ball in Remark to Assumption 1.2. In Theorem in §3, we shall show that Condition 1.3 is necessary and sufficient for the admissibility of the Cauchy problems (1.1) in $\boldsymbol{R}^n$ with $\mathrm{F}^{-1}(C_0^\infty(B_0))^m$ and that Condition 1.3 is necessary for the admissibility of the Cauchy problems (1.1) in $[-\delta, \delta]\times U'$ with wider Cauchy data spaces. We shall also study conditions for the admissibility of the unilateral Cauchy problems (1.1) in $\boldsymbol{R}_+^n=\{x\in\boldsymbol{R}^n; x_1>0\}$ or in $\boldsymbol{R}_-^n=\{x\in\boldsymbol{R}^n; x_1<0\}$.

In [2], we have already studied that if the solutions u_ε of (1.1) converge in $C(\boldsymbol{R}_{x_1}; \mathcal{D}'(\boldsymbol{R}_{x'}^{n-1}))$, then the limit satisfies (1.3). On this point, when Condition 1.3 is not satisfied, we shall show that the convergence in $C(\boldsymbol{R}^n)$ of the solutions u_ε of (1.1) in $\boldsymbol{R}^n$ for a data Φ in $\mathrm{F}^{-1}(C_0^\infty(B_0))^m$ implies that $\hat{\phi}_{m'+1}(\xi')$ is represented as a linear combination of $\hat{\phi}_j(\xi')$, $j=1, \cdots, m'$, where $\wedge$ denotes the Fourier transformation.

In case of L^2-theory, K. Uchiyama [6] studied Cauchy problems for the future of hyperbolic equations with variable coefficients when the conditions of

Case 2 or 4 in Theorem in §3 are satisfied under Assumption 1.2 for $B_0=\boldsymbol{R}^{n-1}$.

2. Preliminaries

We shall state asymptotic behaviour of determinants appearing in the expression of solutions of the Cauchy problems. Since these have been proven essentially in [1], the proofs will be omitted.

Let $P_1(\xi)$ and $P_2(\xi)$ be polynomials of $\xi\in\boldsymbol{R}^n$ with constant coefficients as follows:

$$P_1(\xi)=\xi_1^m+\textstyle\sum_{j=0}^{m-1}p_{1,m-j}(\xi')\,\xi_1^j\,, \tag{2.1}$$

$$P_2(\xi)=p\cdot\xi_1^{m'}+\textstyle\sum_{j=0}^{m'-1}p_{2,m'-j}(\xi')\,\xi_1^j\,. \tag{2.2}$$

Here $p_{1,j}(\xi')$ and $p_{2,j}(\xi')$ are polynomials of ξ' with their orders not higher than j and $p\neq0$. For a large positive parameter λ, we put

$$P_\lambda(\xi)=P_1(\xi)+\lambda^{m-m'}\cdot P_2(\xi)\,. \tag{2.3}$$

By replacing $\lambda\cdot|p|^{-1/(m-m')}$ for λ in (2.3), we may assume that

$$p=-\exp i\theta,\ 0\leqq\theta<2\pi\,.$$

Denote the characteristic roots of $P_\lambda(\xi)=0$ with respect to ξ_1 by $\tau_j(\lambda,\xi'),j=1,\cdots,m$ and those of $P_2(\xi)=0$ with respect to ξ_1 by $\sigma_j(\xi'),j=1,\cdots,m'$, respectively. We shall use the following notation:

$$\zeta=\exp\frac{2\pi i}{m-m'}$$

$$\tau_j'=\zeta^{j-m'-1},\ j=m'+1,\cdots,m$$

$$\Theta=\exp\frac{i\theta}{m-m'}\,,\quad \text{where } \theta \text{ is the argument of } -p.$$

By the same method as in Lemma 3.2, [1], we have the following:

Lemma 2.1. *Let Assumption* 1.2 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 1.2. *If the suffixes* $\{j\}$ *of the characteristic roots* $\tau_j(\lambda,\xi')$, $j=1,\cdots,m$ *are properly chosen, then there exists a positive number* $\lambda(B_0)$ *such that if* $\lambda>\lambda(B_0)$, *then* $\tau_j(\lambda,\xi'),j=1,\cdots,m$ *satisfy the following asymptotic properties on the closure of* B_0:
for $j=1,\cdots,m'$

$$\tau_j(\lambda,\xi')=\sigma_j(\xi')+\lambda^{-1}\cdot\tau_{j,1}(\xi')+\lambda^{-2}\cdot\tau_{j,2}(\lambda,\xi')\,, \tag{2.4}$$

and for $j=m'+1,\cdots,m$

$$\tau_j(\lambda,\xi')/\lambda=\tau_j'\cdot\Theta+\lambda^{-1}\cdot\tau_{j,1}(\xi')+\lambda^{-2}\cdot\tau_{j,2}(\lambda,\xi')\,. \tag{2.5}$$

Here $\tau_{j,1}(\xi')$ are continuous on the closure of B_0 and $\tau_{j,2}(\lambda, \xi')$ remain bounded on the closure of B_0 when $\lambda \uparrow \infty$.

Notation 2.2. For complex numbers or functions τ_j and ϕ_j, $j=1, \cdots, m$

$$D_0 = D_0(\tau_1, \cdots, \tau_m) = \begin{vmatrix} (\tau_1)^0 & \cdots & (\tau_m)^0 \\ \vdots & & \vdots \\ (\tau_1)^{m-1} & \cdots & (\tau_m)^{m-1} \end{vmatrix}$$

$$D_k = D_k(\tau_1, \cdots, \tau_m; \phi_1, \cdots, \phi_m)$$
$$= \begin{vmatrix} (\tau_1)^0 & \cdots & (\tau_{k-1})^0 & \phi_1 & (\tau_{k+1})^0 & \cdots & (\tau_m)^0 \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ (\tau_1)^{m-1} & \cdots & (\tau_{k-1})^{m-1} & \phi_m & (\tau_{k+1})^{m-1} & \cdots & (\tau_m)^{m-1} \end{vmatrix},$$

where $k=1, \cdots, m$.

$$J = m' + \cdots + (m-1)$$
$$A_0 = \Theta^J \cdot (\tau'_{m'+1} \cdots \tau'_m)^{m'} \cdot D_0(\tau'_{m'+1}, \cdots, \tau'_m)$$
$$A_k = \Theta^{J-m'} \cdot (-1)^{k-m'-1} \cdot (\tau'_{m'+1} \cdots \tau'_{k-1} \cdot \tau'_{k+1} \cdots \tau'_m)^{m'+1}$$
$$\times D_0(\tau'_{m'+1}, \cdots, \tau'_{k-1}, \tau'_{k+1}, \cdots, \tau'_m),\ k = m'+1, \cdots, m.$$

By the same method as in Lemma 2.4, [1], we have the following:

Lemma 2.3. *Let Φ belong to $\mathrm{F}^{-1}(C_0^\infty(B_0))^m$ and denote by $\hat{\phi}$ the Fourier transform of ϕ with respect to x'. Assume that $\tau_j(\lambda, \xi')$, $j=1, \cdots, m$ satisfy the asymptotic properties* (2.4) *and* (2.5) *in Lemma* 2.1. *Then*

$$\lim_{\lambda\uparrow\infty} D_0(\tau_1, \cdots, \tau_m)/\lambda^J = D_0(\sigma_1, \cdots, \sigma_{m'}) \cdot A_0 . \tag{2.6}$$

For $k=1, \cdots, m'$

$$\lim_{\lambda\uparrow\infty} D_k(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m)/\lambda^J = D_k(\sigma_1, \cdots, \sigma_{m'}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'}) \cdot A_0 . \tag{2.7}$$

For $k=m'+1, \cdots, m$

$$\lim_{\lambda\uparrow\infty} D_k(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m)/\lambda^{J-m'} = D_{m'+1}(\sigma_1, \cdots, \sigma_{m'+1}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'+1}) \cdot A_k , \tag{2.8}$$

where $\sigma_{m'+1}$ is a dummy variable, that is, by the definition, $D_{m'+1}(\sigma_1, \cdots, \sigma_{m'+1}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'+1})$ is independent of $\sigma_{m'+1}$.

The convergences are uniform on the closure of B_0.

Since all τ'_j are distinct, it implies that $A_k \neq 0$ for $k=0, m'+1, \cdots, m$. We shall use the exact values of A_k/A_{k+1}.

Lemma 2.4.

(2.9)
$$A_0 = \Theta^J \cdot (-1)^{mm'} \cdot \prod_{0\leqq l<l'\leqq m-m'-1}(\zeta^{l'}-\zeta^l).$$

For $k=m'+1, \cdots, m$

(2.10)
$$A_k = \Theta^{J-m'} \cdot (-1)^{mm'} \cdot \zeta^{m'^2+m'+1-km'} \cdot \prod_{0\leqq l<l'\leqq m-m'-2}(\zeta^{l'}-\zeta^l).$$

(2.11)
$$A_k/A_{k+1} = \zeta^{m'},\ k=m'+1, \cdots, m-1.$$

Proof. We have

(2.12)
$$\begin{aligned}(\tau'_{m'+1}\cdots\tau'_m)^{m'} &= \zeta^{(1+\cdots+(m-m'-1))m'} \\ &= \exp\left(\frac{2\pi i}{m-m'}\cdot m'(m-m'-1)(m-m')/2\right) \\ &= \exp(\pi i \cdot m'(m-m'-1)).\end{aligned}$$

Since $m'(m-m'-1)\equiv mm'$ (mod 2), then (2.12)$=(-1)^{mm'}$. The Vandermonde determinant $D_0(\tau'_{m'+1}, \cdots, \tau'_m)=D_0(1, \zeta, \cdots, \zeta^{m-m'-1})$ equals the difference product $\prod_{0\leqq l<l'\leqq m-m'-1}(\zeta^{l'}-\zeta^l)$. This implies (2.9).

Since $(m-m'-1)(m'+1)\equiv(m-1)(m'+1)$ (mod 2), it implies that

(2.13)
$$\begin{aligned}&(\tau'_{m'+1}\cdots\tau'_{k-1}\cdot\tau'_{k+1}\cdots\tau'_m)^{m'+1} \\ &\quad = (-1)^{(m-1)(m'+1)}\cdot\zeta^{(m'+1)(m'+1-k)}.\end{aligned}$$

We have

$$\begin{aligned}&\zeta^{m-k}\cdot(\tau'_{m'+1}, \cdots, \tau'_{k-1}, \tau'_{k+1}, \cdots, \tau'_m) \\ &\quad = \zeta^{m-k}\cdot(1, \cdots, \zeta^{k-m'-2}, \zeta^{k-m'}, \cdots, \zeta^{m-m'-1}) \\ &\quad = (\zeta^{m-k}, \cdots, \zeta^{m-m'-2}, 1, \cdots, \zeta^{m-k-1}) \\ &\quad = (\tau'_{m+m'-k+1}, \cdots, \tau'_{m-1}, \tau'_{m'+1}, \cdots, \tau'_{m+m'-k}).\end{aligned}$$

Multiply the jth row of $D_0(\tau'_{m'+1}, \cdots, \tau'_{k-1}, \tau'_{k+1}, \cdots, \tau'_m)$ by $\zeta^{(m-k)(j-1)}, j=2, \cdots, m-m'-1$. Then

$$\begin{aligned}&\zeta^{(m-k)(m-m'-2)(m-m'-1)/2}\cdot D_0(\tau'_{m'+1}, \cdots, \tau'_{k-1}, \tau'_{k+1}, \cdots, \tau'_m) \\ &\quad = D_0(\tau'_{m+m'-k+1}, \cdots, \tau'_{m-1}, \tau'_{m'+1}, \cdots, \tau'_{m+m'-k}) \\ &\quad = (-1)^{(k-m'-1)(m-k)}\cdot D_0(\tau'_{m'+1}, \cdots, \tau'_{m-1}).\end{aligned}$$

Since

$$\zeta^{(m-k)(m-m'-2)(m-m'-1)/2} = \zeta^{m-k}\cdot(-1)^{(m-k)(m-m'-3)}$$

and

$$\begin{aligned}&(k-m'-1)(m-k)-(m-k)(m-m'-3) = (m-k)(k-m+2) \\ &\quad = -(m-k)(m-k-1)+(m-k) \equiv m-k \text{ (mod 2)},\end{aligned}$$

it implies that

$$(2.14)\qquad \begin{aligned} &D_0(\tau'_{m'+1}, \cdots, \tau'_{k-1}, \tau'_{k+1}, \cdots, \tau'_m) \\ &\quad = \zeta^{k-m}\cdot(-1)^{m-k}\cdot D_0(\tau'_{m'+1}, \cdots, \tau'_{m-1}) . \end{aligned}$$

The power of ζ in A_k is

$$(m'+1)(m'+1-k)+k-m \equiv m'^2+m'+1-km' \pmod{m-m'}$$

and that of (-1) in A_k is

$$k-m'-1+(m-1)(m'+1)+m-k = mm'+2m-2m'-2 \equiv mm' \pmod{2} .$$

The Vandermonde determinant $D_0(\tau'_{m'+1}, \cdots, \tau'_{m-1})$ equals the difference product $\prod_{0\leqq l<l'\leqq m-m'-2}(\zeta^{l'}-\zeta^l)$. Thus (2.10) is proven.

By easy calculation, (2.10) implies (2.11). [Q.E.D.]

3. The admissibility of singular perturbations

First we shall show the global unique solvability of (1.2) with entire functions data. For a polynomial $P(\xi)=\sum_\alpha p_\alpha \xi^\alpha$, denote $M(P)=\max_\alpha |p_\alpha|$. Devide the equation of (1.2) by $p_{1,0}$ and put $p=p_{2,0}/p_{1,0}$ and replace $\lambda\cdot|p|^{-1/(m-m')}$ for λ. Then we may assume that $|p|=1$ and $P_\lambda(\xi)$ is (2.3) satisfying (2.1) and (2.2). Put $M=M(P_1)+M(P_2)$. Then $M(P_\lambda)<\lambda^{m-m'}M$, for $\lambda>1$. For $b(\xi_1, \cdots, \xi_n)=\sum_{|\alpha|=m}\xi^\alpha-\xi_1^m$, denote $b(\rho)=b(\rho, 1, \cdots, 1)$. Then $b(\rho)$ is a polynomial of order $m-1$ with positive coefficients. Put $B=\sup_{\rho\geqq 1} b(\rho)\cdot\rho^{-(m-1)}$. The Cauchy-Kowalewski theorem in [5] implies that if ϕ_j, $j=1, \cdots, m$ are analytic in $|x_i|\leqq r$, $i=1, \cdots, n$, then the formal power series solution of (1.2) converges absolutely for sufficiently large ρ in

$$(3.1)\qquad \sum_{i=2}^n |x_i|+\rho|x_1|<r\{1-\lambda^{m-m'}MB/\rho\} .$$

Put $\rho=2\lambda^{m-m'}MB$ and let $r\uparrow\infty$, then (3.1) sweeps out the whole space for fixed λ. When $r<\infty$, it is difficult to check whether there exists a domain U independent of λ such that every solution u_λ of (1.2) exists in U.

We shall use the same notation as in §2. Let Assumption 1.2 be satisfied and B_0 be the open ball in Remark to Assumption 1.2. Since all $\sigma_j(\xi')$ are simple on the closure of B_0, it implies that $D_0(\sigma_1, \cdots, \sigma_{m'})\neq 0$ on the closure of B_0. For Φ in $\mathrm{F}^{-1}(C_0^\infty(B_0))^m$, denote

$$d(\xi';\Phi) = D_{m'+1}(\sigma_1, \cdots, \sigma_{m'+1}; \hat\phi_1, \cdots, \hat\phi_{m'+1})$$

and

$$g(\xi';\Phi) = d(\xi';\Phi)/D_0(\sigma_1, \cdots, \sigma_{m'}) .$$

Then $d(\xi'; \Phi)\not\equiv 0$ is equivalent to $g(\xi'; \Phi)\not\equiv 0$. Denote for $k=m'+1, \cdots m$

$$G_k(x; \Phi) = \frac{A_k}{A_0}\cdot F^{-1}(g(\xi'; \Phi)\cdot \exp i\tau_{k,1}(\xi')\, x_1)\,,$$

and

$$G(x; \Phi) = \prod_{k=m'+1}^{m} G_k(x; \Phi)\cdot \prod_{k=m'+1}^{m-1}(G_k(x; \Phi)+\zeta^{m'}\cdot G_{k+1}(x; \Phi))\,.$$

Then $G_k(x; \Phi)$ and $G(x; \Phi)$ are entire functions. Denote

$$U(G) = \{x\in \boldsymbol{R}^n;\ G(x)\neq 0\}\ , \quad U(G)^+ = U(G)\cap \boldsymbol{R}^n_+\,, \quad \text{and}$$
$$U(G)^- = U(G)\cap \boldsymbol{R}^n_-\,.$$

Assume that $d(\xi'; \Phi)\not\equiv 0$. Then $G_k(0, x'; \Phi)\not\equiv 0$ and (2.11) implies that

$$G_k(0, x'; \Phi)+\zeta^{m'}\cdot G_{k+1}(0, x'; \Phi) = \frac{2A_k}{A_0}\cdot \mathrm{F}^{-1}(g(\xi'; \Phi))\not\equiv 0\,.$$

Thus $G(x; \Phi)\not\equiv 0$ and $U(G(x; \Phi))$ is a dense subset of $\boldsymbol{R}^n$. There exists a data Φ_0 such that $d(\xi'; \Phi_0)\not\equiv 0$. For example, define Φ_0 by

$$\phi_j(x') = \mathrm{F}^{-1}(\beta(\xi')\cdot\sigma_{m'+1}(\xi')^{j-1}),\ j = 1, \cdots, m\,,$$

where $\sigma_{m'+1}(\xi')$ is a non-zero C^∞-function satisfying $\sigma_{m'+1}(\xi')\neq\sigma_j(\xi'), j=1, \cdots, m'$ on the closure of B_0 and $\beta(\xi')$ is a $C_0^\infty(B_0)$-function. Then $d(\xi'; \Phi_0)\not\equiv 0$.

Theorem. *Let Assumption* 1.2 *be satisfied and* B_0 *be the open ball in Remark to Assumption* 1.2. *Let the Cauchy data space be* $\mathcal{A}=\mathrm{F}^{-1}(C_0^\infty(B_0))^m$. *Put* $p=p_{2,0}/p_{1,0}$.

Case 1. *The case when* $m-m'\geqq 3$ *or when* $m-m'=2$ *and* p *is not real or* $p>0$. *If* $d(\xi'; \Phi)\not\equiv 0$, *then the analytic solutions* u_λ *of* (1.2) *diverge at every point* x *in* $U(G(x; \Phi))$ *when* $\lambda\uparrow\infty$.

Case 2. *The case when* $m-m'=2$ *and* $p<0$ *or when* $m-m'=1$ *and* p *is real, that is, the case when Condition* 1.3 *is satisfied. The Cauchy problems* (1.2) *in* $\boldsymbol{R}^n$ *with* $\mathcal{A}$ *is admissible with respect to* (1.3).

Case 3. *The case when* $m-m'=1$ *and* Im $p>0$. *The unilateral Cauchy problems* (1.2) *in* $\boldsymbol{R}^n_-$ *with* $\mathcal{A}$ *is admissible with respect to* (1.3). *If* $d(\xi'; \Phi)\not\equiv 0$, *then the analytic solutions* u_λ *of* (1,2) *diverege at every point* x *in* $U(G(x; \Phi))^+$ *when* $\lambda\uparrow\infty$.

Case 4. *The case when* $m-m'=1$ *and* Im $p<0$. *The unilateral Cauchy problems* (1.2) *in* $\boldsymbol{R}^n_+$ *with* $\mathcal{A}$ *is admissible with respect to* (1.3). *If* $d(\xi'; \Phi)\not\equiv 0$, *then the analytic solutions* u_λ *of* (1.2) *diverege at every point* x *in* $U(G(x; \Phi))^-$ *when* $\lambda\uparrow\infty$.

Proof. The partial Fourier transform with respect to x' of (1.2) is

$$\left\{\begin{array}{l} P_\lambda(D_1, \xi')\,\hat{u}(x_1, \xi') = 0\ ; \\ D_1^{j-1}\,\hat{u}(0, \xi') = \hat{\phi}_j(\xi'), j = 1, \cdots, m\,. \end{array}\right. \tag{3.2}$$

For fixed ξ', (3.2) is a one-parameter family of Cauchy problems of ordinary differential equations. For Φ in $\mathrm{F}^{-1}(C_0^\infty(B_0))^m$, put

$$C_k(\lambda, \xi'; \Phi) = D_k(\tau_1, \cdots, \tau_m; \hat{\phi}_1, \cdots, \hat{\phi}_m)/D_0(\tau_1, \cdots, \tau_m) .$$

The solution $v_\lambda(x_1, \xi')$ of (3.2) is represented by

$$v_\lambda(x_1, \xi') = \sum_{k=1}^m C_k(\lambda, \xi'; \Phi)\cdot \exp i\tau_k(\lambda, \xi')\, x_1 . \tag{3.3}$$

Put $u_\lambda(x)=\mathrm{F}^{-1}(v_\lambda(x_1, \xi'))$. Then u_λ is the solution of (1.2). Lemma 2.3 implies that for $k=1, \cdots, m'$

$$\begin{aligned}&\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi'; \Phi)\\ &\quad = D_k(\sigma_1, \cdots, \sigma_{m'}; \hat{\phi}_1, \cdots, \hat{\phi}_{m'})/D_0(\sigma_1, \cdots, \sigma_{m'}) ,\end{aligned} \tag{3.4}$$

and for $k=m'+1, \cdots, m$

$$\lim_{\lambda\uparrow\infty} C_k(\lambda, \xi'; \Phi)\cdot\lambda^{m'} = \frac{A_k}{A_0}\cdot g(\xi'; \Phi) . \tag{3.5}$$

Denote

$$\begin{aligned} M_+ &= \max\,\{\mathrm{Im}\,\Theta\tau'_j;\, j = m'+1, \cdots, m\} ,\\ M_- &= \min\,\{\mathrm{Im}\,\Theta\tau'_j;\, j = m'+1, \cdots, m\} ,\end{aligned}$$

where $\tau'_j=\zeta^{j-m'-1}=\exp\dfrac{2\pi i(j-m'-1)}{m-m'}$, $j=m'+1, \cdots, m$. Both the maximum and the minimum are attained by one j or two j. It is useful for searching the leading term to illustrate the points $\Theta\tau'_j=\Theta\zeta^{j-m'-1}$, $j=m'+1, \cdots, m$ on the com-

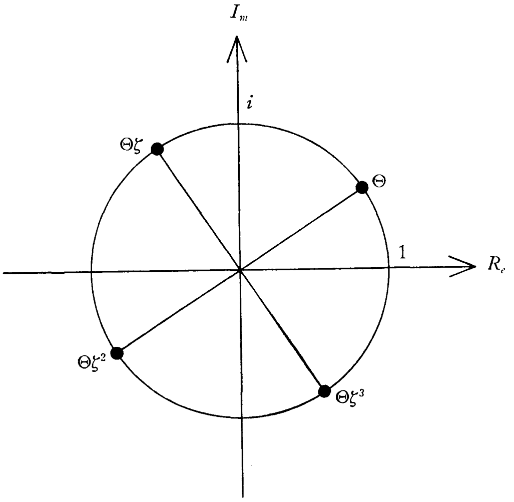

Figure 1a

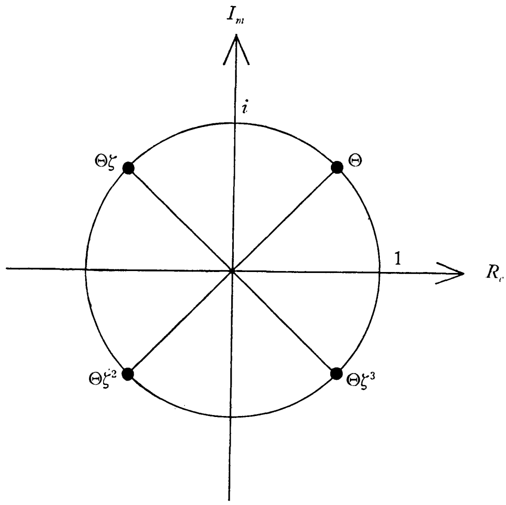

Figure 1b

plex plane. For example, we illustrate the case when $m-m'=4$. In case of Figure 1a ($0<\theta<\pi$), $\Theta\zeta^3$ attains the minimum. In case of Figure 1b ($\theta=\pi$), $\Theta\zeta^2$ and $\Theta\zeta^3$ attain the minimum. Thus, we have the following.

(1) If $m-m'\geqq 3$ or if $m-m'=2$ and $p=-\Theta^2$ is not real or $p>0$, then $M_+>0$ and $M_-<0$.

(2) If $m-m'=2$ and $p<0$ or if $m-m'=1$ and p is real, then $M_+=M_-=0$.

(3) If $m-m'=1$ and $\operatorname{Im} p=\operatorname{Im} -\Theta>0$, then $M_+=M_-<0$.

(4) If $m-m'=1$ and $\operatorname{Im} p=\operatorname{Im} -\Theta<0$, then $M_+=M_->0$.

We shall show that $M_-<0$ and $d(\xi';\Phi)\not\equiv 0$ imply that u_λ diverge at every point x in $U(G(x;\Phi))^+$. Assume that only l attains the minimum. Fix a point x in $\boldsymbol{R}^n_+$. Denote

$$E_k(\lambda, x; j) = \mathrm{F}^{-1}(C_k(\lambda, \xi'; \Phi)\cdot\lambda^{m'}\cdot\exp i(\tau_k(\lambda, \xi')-\lambda\Theta\tau'_j)\, x_1)\,,$$

for $k=1, \cdots, m$ and $j=m'+1, \cdots, m$. Then

$$u_\lambda(x) = \lambda^{-m'}\cdot\exp(i\lambda\Theta\tau'_l\, x_1)\cdot\textstyle\sum_{k=1}^m E_k(\lambda, x; l)\,. \tag{3.6}$$

Lemma 2.1 implies that for $k=1, \cdots, m'$

$$\operatorname{Im}(\tau_k(\lambda, \xi')-\lambda\Theta\tau'_l) = \operatorname{Im}\sigma_k(\xi')-\lambda M_-+O(\lambda^{-1})\,, \tag{3.7}$$

for $k=m'+1, \cdots, l-1, l+1, \cdots, m$

$$\operatorname{Im}(\tau_k(\lambda, \xi')-\lambda\Theta\tau'_l) = \lambda(\operatorname{Im}\Theta\tau'_k-M_-)+O(1)\,, \tag{3.8}$$

and

$$\operatorname{Im}(\tau_l(\lambda, \xi')-\lambda\Theta\tau'_l) = \operatorname{Im}\tau_{l,1}(\xi')+O(\lambda^{-1})\,. \tag{3.9}$$

Hence

$$\lambda^{m'}\cdot|\exp i(\tau_k(\lambda,\xi')-\lambda\Theta\tau_l')\,x_1|,\ k=1,\cdots,m'$$

and

$$|\exp i(\tau_k(\lambda,\xi')-\lambda\Theta\tau_l')\,x_1|,\ k=m'+1,\cdots,m$$

remain bounded when $\lambda\uparrow\infty$ on the closure of B_0 for fixed $x_1>0$. By (3.4) and (3.5), Lebesgue's bounded convergence theorem implies that for $k=1,\cdots,l-1,l+1,\cdots,m$

$$\lim_{\lambda\uparrow\infty}E_k(\lambda,x;l)=0\,, \tag{3.10}$$

and

$$\lim_{\lambda\uparrow\infty}E_l(\lambda,x;l)=G_l(x;\Phi)\,. \tag{3.11}$$

On the other hand,

$$|\lambda^{-m'}\cdot\exp i\lambda\Theta\tau_l'\,x_1|=\lambda^{-m'}\cdot\exp(-\lambda M_-\,x_1)$$

diverge for fixed $x_1>0$. Since $G_l(x;\Phi)\neq 0$ in $U(G(x;\Phi))^+$, it implies that $u_\lambda(x)$ diverge at every point x in $U(G(x;\Phi))^+$.

Assume that l and $l+1$ attain the minimum. Put $L=\operatorname{Re}\Theta\tau_{l+1}'$. Then $L>0$ and $\operatorname{Re}\Theta\tau_l'=-L$. Denote

$$K(\lambda,x;l)=\sum_{k=1,\cdots,l-1,l+2,\cdots,m}E_k(\lambda,x;l)\,.$$

Then $\lim_{\lambda\uparrow\infty}K(\lambda,x;l)=0$ and

$$u_\lambda(x)=\lambda^{-m'}\cdot\exp(i\lambda\Theta\tau_l'\,x_1)\cdot(E_l(\lambda,x;l)+E_{l+1}(\lambda,x;l)+K(\lambda,x;l))\,.$$

Put

$$\lambda_n(x_1)=\pi\,(n+m'/(m-m'))/Lx_1\,,$$

for fixed $x_1>0$. Then $\exp 2i\lambda_n\,Lx_1=\zeta^{m'}$. Since

$$\lim_{\lambda\uparrow\infty}E_{l+1}(\lambda,x;l+1)=G_{l+1}(x;\Phi)\,,$$

$$E_{l+1}(\lambda,x;l)=E_{l+1}(\lambda,x;l+1)\cdot\exp 2i\lambda\,Lx_1\,,$$

and $A_l/A_{l+1}=\zeta^{m'}$, it implies that

$$\begin{aligned}&\lim_{n\uparrow\infty}(E_l(\lambda_n,x;l)+E_{l+1}(\lambda_n,x;l))\\ &=G_l(x;\Phi)+\zeta^{m'}\cdot G_{l+1}(x;\Phi)\\ &=\frac{A_l}{A_0}\cdot\mathrm{F}^{-1}(g(\xi';\Phi)\cdot(\exp i\tau_{l,1}(\xi')\,x_1+\exp i\tau_{l+1,1}(\xi')\,x_1))\,.\end{aligned} \tag{3.12}$$

Obviously, (3.12)$\neq 0$ in $U(G(x;\Phi))^+$. Thus u_{λ_n} diverge when $n\uparrow\infty$.

By the same argument, $M_+>0$ and $d(\xi';\Phi)\neq 0$ imply that u_λ diverge at every point x in $U(G(x;\Phi))^-$.

We shall show that $M_-\geqq 0$ implies the admissibility of the unilateral Cauchy problems in $\boldsymbol{R}^n_+$. Fix a point x in $\boldsymbol{R}^n_+$. Lemma 2.1 implies that for $k=1,\cdots,m'$

$$|\exp i\tau_k(\lambda,\xi')\,x_1| = \exp(-\mathrm{Im}\,\sigma_k(\xi')+O(\lambda^{-1}))\,x_1 \tag{3.13}$$

and for $k=m'+1,\cdots,m$

$$\begin{aligned}&|\exp i\tau_k(\lambda,\xi')\,x_1|\\ &\quad=\exp(-\lambda\cdot\mathrm{Im}\,\Theta\tau_k'-\mathrm{Im}\,\tau_{k,1}(\xi')+O(\lambda^{-1}))\,x_1\\ &\quad\leqq\exp(-\lambda M_--\mathrm{Im}\,\tau_{k,1}(\xi')+O(\lambda^{-1}))\,x_1\,.\end{aligned} \tag{3.14}$$

Hence (3.13) and (3.14) remain bounded on B_0 when $\lambda\uparrow\infty$. Denote

$$u_0(x)=\sum_{k=1}^{m'}\mathrm{F}^{-1}\left[\frac{D_k(\sigma_1,\cdots,\sigma_{m'};\hat{\phi}_1,\cdots,\hat{\phi}_{m'})}{D_0(\sigma_1,\cdots,\sigma_{m'})}\cdot\exp i\sigma_k(\xi')\,x_1\right].$$

Then u_0 is the solution of (1.3). By (3.4) and (3.5), Lebesgue's bounded convergence theorem implies that

$$\lim_{\lambda\uparrow\infty}u_\lambda(x)=\sum_{k=1}^{m}\mathrm{F}^{-1}(\lim_{\lambda\uparrow\infty}C_k(\lambda,\xi';\Phi)\cdot\exp i\tau_k(\lambda,\xi')\,x_1)=u_0(x)\,.$$

Obviously, this convergence remains true in $C(\boldsymbol{R}^n_+)$.

By the same argument, $M_+\leqq 0$ implies the admissibility of the unilateral Cauchy problems in $\boldsymbol{R}^n_-$ and $M_+=M_-=0$ implies the admissibility of the Cauchy problems in $\boldsymbol{R}^n$. [Q.E.D.]

The divergent property can not be removed by any localization in x-space and this property remains true for wider Cauchy data spaces. Thus we have the following:

Corollary 1. *Let Assumption* 1.2 *be satisfied and* $\mathrm{F}^{-1}(C_0^\infty(B_0))^m$ *be naturally included in the Cauchy data space* $\mathcal{A}$. *Assume that for every* Φ *in* $\mathcal{A}$, *there exists a unique continuous solution* u_ε *of* (1.1) *in* $[-\delta,\delta]\times U'$. *Then Condition* 1.3 *is necessary for the admissibility of the Cauchy problems* (1.1) *in* $[-\delta,\delta]\times U'$ *with* $\mathcal{A}$ *with respect to* (1.3).

Even when Condition 1.3 is not satisfied, there exists a data Φ_1 such that the solutions $u_\lambda(x;\Phi_1)$ converge in $C(\boldsymbol{R}^n)$, for example, $\Phi_1=(0,\cdots,0)$ is a trivial one. The proof of Theorem implies the following:

Corollary 2. *Let Assumption* 1.2 *be satisfied and the Cauchy data space be* $\mathcal{A}=\mathrm{F}^{-1}(C_0^\infty(B_0))^m$. *Assume that Condition* 1.3 *is not satisfied. Then it is neces-*

sary for the convergence of $u_\lambda(x;\Phi_1)$ *in* $C(\boldsymbol{R}^n)$ *that the data* $\Phi_1=(\phi_1,\cdots,\phi_m)$ *satisfies*

$$d(\xi';\Phi_1)=D_{m'+1}(\sigma_1,\cdots,\sigma_{m'+1};\hat{\phi}_1,\cdots,\hat{\phi}_{m'+1})\equiv 0\,.$$

This implies that $\hat{\phi}_{m'+1}$ *is represented as a linear combination of* $\hat{\phi}_j, j=1,\cdots,m'$.

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