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EQUIVALENCE THEOREMS AND COINCIDENCE DEGREE FOR MULTIVALUED MAPPINGS

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Introduction

Let X and Y be normed spaces. The inclusion

$$L(x) \in N(x) \tag{I}$$

where $L: D(L) \subset X \rightarrow Y$ is a linear mapping and $N: D(N) \subset X \rightarrow CK(Y)$ is a multivalued mapping, has been studied by many authors such as Mawhin (1972); Gaines and Mawhin (1977); Tarafdar and Teo (1979) and others.

Mawhin (1972) and Gaines and Mawhin (1977) consider L a linear Fredholm mapping with index of Fredholm p and N a possibly nonlinear mapping; Tarafdar and Teo (1979) consider L a linear Fredholm mapping with index of Fredholm zero and N a multivalued mapping possibly noncompact. Using an equivalence theorem which reduces the problem of existence of solution of (I) to that of fixed points of an auxiliary mapping and topological degree, they developed a degree called the coincidence degree for the pair (L, N). This coincidence degree has been applied to nonlinear differential inclusions.

The purpose of this work is to develop a coincidence degree for the pair (L, N) where L is a linear Fredholm mapping with index of Fredholm not necessarily zero and N a multivalued mapping that satisfies a weaker condition than used by Tarafdar and Teo.

Using the equivalence theorem of Tarafdar and Teo ([3]-Theorem 3.1) we prove a new equivalence theorem and we build our coincidence degree such that, even when the index of L is strictly positive, this coincidence degree isn't necessarily zero.

The organization of the paper is as follows: in Section 1 we introduce some basic definitions, propositions necessary to the comprehension of the paper; in Section 2 we present algebraic preliminaries and an equivalence theorem; in Sention 3 we present basic assumptions and main results; in Section 4 we present some basic properties of the coincidence degree and in Section 5 we present an application.

1. Degree theory for multivalued ultimately compact vector fields

Let X be a separated locally convex topological vector space which has the additional property that for each compact subset A there is a retraction of X onto $\overline{\text{co}}\, A$, where by $\overline{\text{co}}\, A$ we denote the closed convex hull of A. If X is metrizable then, by a theorem of Dugundji ([4]), X has the latter property.

If $A \subset X$, we let $K(A)$ and $CK(A)$ denote the families of closed convex and compact convex subsets of A, respectively. A multivalued mapping $T: D(T) \subset X \rightarrow 2^X$ is called upper semicontinuous (u.s.c.) provided that, whenever $x \in D(T)$ and V is an open set containing $T(x)$, there is an open set U such that $x \in U$ and, if $y \in D(T) \cap U$, then $T(y) \subset V$. A u.s.c. multivalued mapping $F: D(F) \subset X \rightarrow K(X)$ is called a compact vector field if $(I-F)(D(F))$ is relatively compact.

Following [4], we are now to define the class of ultimately compact vector field multivalued mapping and after, the topologiacl degree of such multivalued mappings.

Let $D \subset X$ be closed and $T: D \rightarrow K(X)$ be u.s.c. We define a transfinite sequence $\langle K_\alpha \rangle$ by induction as follows. Let $K_0 = \overline{\text{co}}\, T(D)$. Suppose α is an ordinal such that K_β has been defined for $\beta < \alpha$. If α is an ordinal of the first kind, let $K_\alpha = \overline{\text{co}}\, T(D \cap K_{\alpha-1})$; if α is an ordinal of the second kind, we let $K_\alpha = \bigcap_{\beta<\alpha} K_\beta$.

It is easily verified that the following properties hold for $\langle K_\alpha \rangle$:

(i) each K_α is closed and convex with $K_\alpha \subseteq K_\beta$ for $\alpha \geq \beta$;

(ii) $T(K_\alpha \cap D) \subset K_\alpha$ for each ordinal α.

Since the transfinite sequence $\langle K_\alpha \rangle$ is nonincreasing, there is an ordinal γ such that $K_\gamma = K_{\gamma+1}$, and hence $K_\gamma = K_\beta$ for each $\beta \geqq \gamma$. We define $K = K(T, D) = K_\gamma$. Then, it is clear that $T(D \cap K) \subset K$ and, in fact, that $\overline{\text{co}}\, T(D \cap K) = K$.

Definition 1.1. A u.s.c. multivalued mapping $T: D \rightarrow K(X)$, where $D \subset X$ is closed, is called ultimately compact if either $K \cap D = \Phi$, or if $K \cap D \neq \Phi$, then $T(D \cap K)$ is relatively compact. If T is ultimately compact then $I-T$ is called ultimately compact vector field.

Definition 1.2. Let $\Omega \subset X$ be open and let $T: \bar{\Omega} \rightarrow K(X)$ be an ultimately compact multivalued mapping with $0 \notin x - T(x)$ for each $x \in \partial\Omega$, where $\partial\Omega$ is the boundary of Ω and $\bar{\Omega}$ is the closure of Ω. If $K \cap \Omega = \Phi$, we define the degree of $I-T$ on Ω with respect to zero, denoted by $d(I-T, \Omega, 0)$, to be zero. If $K \cap \Omega \neq \Phi$, let ρ be a retraction of X on K and we define

$$d(I-T, \Omega, 0) = d_c(I-T\rho, \rho(\Omega), 0),$$

where the right-hand term is the topological degree for compact vector field multivalued mapping given by Ma, Tsoy-Wo (1972).

To see that this topological degree is well defined and has all the usual properties of the topological degree studied by Ma, Tsoy-Wo, see Petryshyn and Fitzpatrick [4].

DEFINITION 1.3 (measure of noncompactness). Let C be a lattice with a minimal element which we denote by 0 (zero). A mapping $\phi: 2^X \to C$ is called a measure of noncompactness if for any $D \subset X$ and $B \subset X$ it satisfies the following properties:

(i) $\phi(\overline{\text{co}}\, D) = \phi(D)$;

(ii) $\phi(D) = 0$ if only if D is compact;

(iii) $\phi(D \cup B) = \max\{\phi(D), \phi(B)\}$.

It follows immediately that if $D \subset B$, then $\phi(D) \leqq \phi(B)$.

The following definition is due to [4].

DEFINITION 1.4. Let ϕ be a measure of noncompactness in X and let $T: D \subset X \to CK(X)$ be a u.s.c. multivalued mapping.

(i) Then T is said to be ϕ-condensing if $\phi(T(\Omega)) \not\geqq \phi(\Omega)$ for all $\Omega \subset D$ such that Ω is not relatively compact. In case C is also linearly ordered, the above condition reduces to the requirement that $\phi(T(\Omega)) < \phi(\Omega)$ for each $\Omega \subset D$ which is not relatively compact;

(ii) If we additionally assume that C is such that for each $c \in C$ and $\lambda \in \boldsymbol{R}$ with $\lambda > 0$ there is defined an element $\lambda_c \in C$, then T is said to be k-ϕ-contraction if $\phi(T(\Omega)) \leqq k\phi(\Omega)$ for each $\Omega \subset D$ and some $k > 0$.

Proposition 1.1 ([4]). *Let $D \subset X$ be closed and let $T: D \to CK(X)$ be ϕ-condensing. Then T is ultimately compact.*

Proposition 1.2 ([4]). *Let $\phi: 2^X \to \boldsymbol{R}^+ = \{t \in \boldsymbol{R};\ t \geqq 0\} \cup \{\infty\}$ be a measure of noncompactness and suppose that $T: D \subset X \to CK(X)$ is a k-ϕ-contraction, $0 < k < 1$, with $\phi(T(D)) \in \boldsymbol{R}$. Then T is ϕ-condensing if either X is quasi-complete or D is complete.*

2. Algebraic preliminaries and an equivalence theorem

Let X and Y be two vector spaces and let $L: D(L) \subset X \to Y$ be a linear mapping, where $D(L)$ is the domain of L. We shall denote the kernel of L by $\ker L$; the range of L by $R(L)$ and the quotient space $Y/R(L)$, the cokernel of L, by coker L.

Given a vector subspace S of a vector space E, there always exists a projector P of E onto S (P is linear and idempotent) and E is the direct sum of $R(P) = S$ and $\ker P$. If E is a topological vector space and if P is bounded, then E is the topological direct sum of $R(P)$ and $\ker P$ ($E = R(P) \oplus \ker P$).

DEFINITION 2.1. If X, Y, L are as above, let P and Q be projections on X and Y respectively such that $R(P) = \ker L$ and $\ker Q = R(L)$. Such a pair (P, Q)

will be called exact with respect to L.

DEFINITION 2.2. Let L_P be the restriction of L to $\ker P \cap D(L)$. The L_P is an isomorphism from $\ker P \cap D(L)$ to $R(L)$. Let K_P: $R(L) \to \ker P \cap D(L)$ be the inverse of L_P. K_P is then called the pseudo-inverse of L associated with P.

Let π: $Y \to \operatorname{coker} L$ be the canonical surjection. It is well known that the restriction of π to $R(Q)$ is an algebraic isomorphism and if Y is a topological vector space and if coker L is equipped with the quotient topology, then π is continuous.

It is immediate that:

$$PK_P = 0;\ LK_P = L_P K_P = I;\ K_P L = I - P\,, \qquad (2.1)$$
$$Q(y) = 0 \Leftrightarrow y \in R(L) \Leftrightarrow \pi y = 0\,.$$

Proposition 2.1 ([1]). *Let P and P' be projections of X onto* ker L *and let $P''=aP+bP'$ for some real numbers a, b. Then P'' is a projection of X onto* ker L *if and only if $a+b=1$. Furthermore, if $a+b=1$ then the pseudo-inverse of L associated with P'' is given by $K_{P''}=aK_P+bK_{P'}$.*

Proposition 2.2 ([1]). *Let (P, Q) and (P', Q') be pairs of projections exact with respecto to L. Then*

$$K_{P'} = (I-P')\,K_P\,,$$
$$PK_{P'} + P'K_P = 0\,,$$

where K_P, $K_{P'}$ denote the pseudo-inverses of L associated with P and P' respectively.

Theorem 2.1 (equivalence theorem) ([3]). *Let X and Y be two vector spaces. Let L: $D(L) \subset X \to Y$ be a linear mapping and N: $D(N) \subset X \to 2^Y$ be a multivalued mapping such that $D(L) \cap D(N) \neq \phi$. Further assume that there is a linear one-to-one mapping Ψ:* coker $L \to$ ker L. *Then $x \in D(L) \cap D(N)$ is a solution of the inclusion*

$$L(x) \in N(x) \qquad (2.2)$$

if and only if x is a fixed point of M_Ψ, wheer M_Ψ: $D(N) \to 2^X$ is defined by

$$M_\Psi(x) = P(x) + [\Psi\pi + K_P(I-Q)]\,N(x) \qquad (2.3)$$

for every pair (P, Q) of exact projections with respect to L.

3. Basic assumptions and main results

In this section we assume the following assumptions:

(a) X is a Banach space and Y is a real normed space.

(b) L: $D(L) \subset X \to Y$ is a linear Fredholm mapping (i.e., $R(L)$ is closed;

dim ker $L<\infty$ and dim coker $L<\infty$) with nonnegative index of Fredholm (ind $L=\dim\ker L-\dim\operatorname{coker} L\geqq 0$).

(c) Ω is a bounded, open set in X and the multivalued mapping $N: \bar{\Omega}\to CK(Y)$ is u.s.c. and $N(\bar{\Omega})$ is bounded in Y. Furthermore suppose that $D(L)\cap\Omega\neq\Phi$.

(d) Let (P, Q) be an exact pair of projection with respect to L and let ϕ be a measure of noncompactness defined from 2^X into C (linearly ordered lattice) such that ϕ satisfies the subadditivity condition ($A\subset X, B\subset X\Rightarrow\phi(A+B)\leqq\phi(A)+\phi(B)$). We assume that with such a measure of noncompactness ϕ, $K_P(I-Q)N$ is ϕ-condensing and that $\phi[K_P(I-Q)N(\bar{\Omega})]<\infty$. Fuxtheremore, we assume that K_P is bounded.

(e) $0\notin(L-N)(D(L)\cap\partial\Omega)$ where $\partial\Omega$ denotes the boundary of Ω.

REMARK 3.1. Assumption (b) authorizes that we assume the continuity of the exact pair of projections (P, Q). Moreover, with the quotient norm topology, coker L is a normed space and the canonical surjection π is continuous with respect to this topology. Also (b) implies that there exists a continuous linear one-to-one mapping Ψ: coker $L\to$ker L.

Proposition 3.1. *Under the assumptions (a) to (d), the condition (d) is independent of choice of the exact pair (P, Q) of continuous projections with respect to L.*

Proof. We follow the proof of Proposition 3.1 of [3]. Suppose that (P, Q) satisfies (d) and let (P', Q') be another exact pair of continuous projections with respect to L. Then by Proposition 2.2, we have

$$\begin{aligned} K_{P'}(I-Q')N &= (I-P')K_P(I-Q')N \\ &\subset(I-P')K_P(I-Q)N+(I-P')K_P(Q-Q')N \\ &= (I-P')K_P(I-Q)N+(I-P')\tilde{K}_P(\pi_Q^{-1}-\pi_{Q'}^{-1})\pi N\,, \end{aligned}$$

where K_P denotes the restriction of K_P to the finite dimensional subspace $(Q-Q')(Y)$ ($\tilde{K}_P$ is continuous); $\pi_Q=\pi/R(Q)$ and $\pi_{Q'}=\pi/r(Q')$. Since $\pi N(\bar{\Omega})$ is bounded in a finite dimensional subspace of X, it follows that

$$\phi[(I-P')\tilde{K}_P(\pi_Q^{-1}-\pi_{Q'}^{-1})\pi N(\bar{\Omega})]=0\,.$$

Hence, from the subadditivity condition on ϕ, it follows that $K_{P'}(I-Q')N$ is ϕ-condensing and that $\phi[K_{P'}(I-Q')]<\infty$. QED.

The following proposition is basic for our aim.

Proposition 3.2. *If assumptoins (abcd) are satisfied, then for every continuous linear one-to-one mapping* Ψ: coker $L\to$ker L *and any exact pair* (P, Q)

of continuous projections with respect to L, M_Ψ is a ϕ-condensnig multivalued mapping.

Proof. Since P, Q, K_P, π, Ψ are all linear and continuous and $N(x)$ is convex and compact for each $x \in \bar{\Omega}$, it follows that $M_\Psi(x)$ is convex and compact for each $x \in \bar{\Omega}$.
Now, let $A \subset \bar{\Omega}$ such that A is not relatively compact. Then,

$$M_\Psi(A) = [P + [\Psi\pi + K_P(I-Q)]\, N]\,(A) \subset P(A) + \Psi\pi N(A) + K_P(I-Q)\, N(A)$$

and, by the subadditivity of ϕ,

$$\phi\,[M_\Psi(A)] \leqq \phi\,[P(A)] + \phi\,[\Psi\pi N(A)] + \phi\,[K_P(I-Q)\, N(A)]\,.$$

Now, $P(A)$ and $\Psi\pi N(A)$ are bounded subsets of finite-dimensional subspace of X and, therefore,

$$\phi\,[P(A)] = \phi\,[\Psi\pi N(A)] = 0\,.$$

Then,

$$\phi\,[M_\Psi(A)] \leqq \phi\,[K_P(I-Q)\, N(A)]\,.$$

By assumption (d), $K_P(I-Q)\,N$ is ϕ-condensing and, therefore, the above inequality assures that M is ϕ-condensing. QED.

From Proposition 1.2, we see that if assumptions (abcd) are satisfied, then M_Ψ is an ultimately compact multivalued mapping. It follows from the assumption (e) and Theorem 2.1 $0 \notin (I-M_\Psi)\ (D(L) \cap \partial\Omega)$. Thus, the topological degree of the multivalued mapping $I-M_\Psi$ on Ω with respect to zero ($d[I-M_\Psi, \Omega, 0]$) is well defined.

REMARK 3.2. If ind $L=0$, we can consider Ψ: coker $L \rightarrow$ ker L as an isomorphism and if we take the lattice C as $R^+ = \{t \in R;\ t \geq 0\} \cup \{\infty\}$ and we consider the multivalued mapping $K_P(I-Q)\,N$ as a k-set-contraction with $k<1$, then we have the work of Tarafdar and Teo ([3]).

If ind $L>0$, unfortunately we have the following result:

Proposition 3.3. *If* ind $L>0$ *and* $0 \notin (L-N)\,(D(L) \cap \partial\Omega)$ *then, for each linear one-to-one mapping* Ψ: coker $L \rightarrow$ ker L, *one has* $d[I-M_\Psi, \Omega, 0]=0$.

Proof. We follow the proof of Proposition 6.1. of [1] or Proposition XII. 1 of [2] where N is assumed to be single valued. First note that the condition ind $L>0$ implies that there exists a linear one-to-one mapping Ψ: coker $L \rightarrow$ ker L. Also, ind $L>0$ implies that $R(\Psi)$ is a proper subspace of ker L. This and

$$R(I-M_{\Psi})\subset R[I-P-\Psi\pi N-K_P(I-Q)\,N]$$

implies that $R(I-M_{\Psi})$ is necessarily contained in the proper subspace of X given by

$$X' = \ker P\oplus R(\Psi)\,.$$

Then, by the properties of topological degree, there exists a neighbourhood V of the origin such that

$$d[I-M_{\Psi}, \Omega, 0] = d[I-M_{\Psi}, \Omega, y]$$

for all $y\in V$.
If we take y in the nonvoid set $V\cap \boldsymbol{C}_X X'(\boldsymbol{C}_X X'$ is the complement of X' in X), then y does not belong to the $R(I-M_{\Psi})$ and, consequently, $d[I-M_{\Psi}, \Omega, 0]=0$. This complete the proof. QED.

However, this negative result can be overcome by modifying the multivalued mapping M_{Ψ} related to $L-N$ in such a way that the topological degree is no more necessarily equal to zero, as follows (cf. Proposition XII. 3 of [2] in case of a single valued N):

Theorem 3.1. *Under the same notation of Theorem* 2.1, *if* ind $L\geq 0$, *then*:
(i) *every fixed point of the multivalued mapping K_PN is a solution of the inclusion (I) provided L is surjective*;
(ii) *if L is not surjective, the inclusion (I) has a solution if and only if there exists a linear one-to-one mapping* Ψ: coker $L\rightarrow$ker L *such that the multivalued mapping*

$$\tilde{M}_{\Psi} = R_{\Psi}P+[\Psi\pi+K_P(I-Q)]\,N$$

has a fixed point, where R_{Ψ}: ker $L\rightarrow$ker L *is a projector such that* $R(R_{\Psi})=R(\Psi)$.

Proof. First suppose that L is surjective and that x is a fixed point of K_PN, i.e., $x\in K_PN(x)$. Thus, $L(x)\in LK_PN(x)=N(x)$, i.e., x is a solution of (I).
Now, suppose that L is not surjective and that x is a solution of (I). Then, it follows from Theorem 2.1 that

$$x\in M_{\Psi}(x)$$

with

$$M_{\Psi} = P+[\Psi\pi+K_P(I-Q)]\,N$$

for any linear Ψ: coker $L\rightarrow$ker L which is one-to-one. Now let V be any subspcae of ker L of dimension equal to dim coker L and containing $P(x)$ (such a subspacce necessarily exists) and let R_Ψ be any projector in ker L such that $R(R_\Psi)=V$. Then necessarily

$$P(x) = R_\Psi P(x)$$

and if we take Ψ: coker $L \to$ ker L linear one-to-one such that $R(\Psi)=V$ (such a linear mapping necessarily exists) then

$$\begin{aligned} x \in M_\Psi(x) &= P(x)+[\Psi\pi+K_P(I-Q)]\, N(x) \\ &= R_\Psi P(x)+[\Psi\pi+K_P(I-Q)]\, X(x) \\ &= \tilde{M}_\Psi(x)\,. \end{aligned}$$

Conversely, if $x \in D(L) \cap \bar{\Omega}$ is a fixed point of the multivalued $\tilde{M}_\Psi = R_\Psi P+[\Psi\pi+K_P(I-Q)]\, N$, i.e.,

$$x \in R_\Psi P(x)+[\Psi\pi+K_P(I-Q)]\, N(x)$$

then

$$\begin{aligned} (I-P)\,(x) &= K_P(I-Q)\,(z) \\ P(x) &= R_\Psi P(x)+\Psi\pi(z) \end{aligned}$$

for some $z \in N(x)$. Hence,

$$\begin{aligned} & L(x) = (I-Q)\,(z) \\ & (I-R_\Psi)\, P(x) = \Psi\pi(z) = R_\Psi\, \Psi\pi(z) \end{aligned}$$

which implies

$$\begin{aligned} & L(x) = (I-Q)\,(z) \\ & (I-R_\Psi)\, P(x) = 0,\ R_\Psi\, \Psi\pi(z) = 0\,. \end{aligned}$$

Now, $\Psi\pi(z)=R_\Psi\, \Psi\pi(z)=0$ implies $z \in R(L)$. Hence $(I-Q)\,(z)=z$ and therefore, $L(x) \in N(x)$. QED.

Let V be a vector subspace of ker L such that dim $V=$dim coker L. Then, analogously to Proposition 3.2, we have the following result whose proof is analogous to the proof of that result.

Proposition 3.4. *If assumptions (abcd) are satisfied, then for every continuous linear one-to-one mapping* Ψ: coker $L \to$ ker L *and any exact pair* (P, Q) *of continuous projections with respect to* L*, the multivalued mapping* $\tilde{M}_\Psi$ *stated in Theorem* 3.1 *is* ϕ*-condensing and, for each* $x \in \bar{\Omega}$, $M_\Psi(x) \in CK(X)$.

Then, under the assumption (*abcde*), the topological degree the multivalued mapping $I-\tilde{M}_\Psi$ on Ω with respect to zero is well defined. We shall denote it by

$$d_V[I-\tilde{M}_\Psi, \Omega, 0]\,.$$

For each vector subspace V of $\ker L$ such that $\dim V = \dim \operatorname{coker} L$, $d_V[I-\tilde{M}_\Psi, \Omega, 0]$ is independent of the choice of P, Q and within the same homotopy class (here, the mappings Ψ are such that $R(\Psi)=V$).

DEFINITION 3.1. For each vector subspace V of $\ker L$ such that $\dim V = \dim \operatorname{coker} L$, let $\lfloor_L^V$ be the set of all continuous isomorphism from $\operatorname{coker} L$ into V. Ψ, Ψ' are to be homotopic in $\lfloor_L^V$ if there exists a continuous mapping $\bar{\Psi}$: $\operatorname{coker} L \times [0, 1] \rightarrow V$ such that $\bar{\Psi}(\cdot, 0)=\Psi$, $\bar{\Psi}(\cdot, 1)=\Psi'$ and, for each $\lambda \in [0, 1]$, $\bar{\Psi}(\cdot, \lambda) \in \lfloor_L^V$.

REMARK 3.3. To be homotopic is an equivalence relation which partitions $\lfloor_L^V$ into equivalence classes called homotopy classes.

The following two propositions and corollary are quoted from Gaines and Mawhin (1977):

Proposition 3.5. *Ψ and Ψ' are homotopic in $\lfloor_L^V$ if and only if* $\det(\Psi' \Psi^{-1}) > 0$.

Corollary 3.1. *$\lfloor_L^V$ is partitioned into two homotopy classes.*

DEFINITION 3.2. Ψ: $\operatorname{coker} L \rightarrow V$ is said to be orientation preserving if $\{\Psi a_1, \Psi a_2, \cdots, \Psi a_n\}$ belongs to the orientation chosen in V where $\{a_1, a_2, \cdots, a_n\}$ is a basis for $\operatorname{coker} L$ belonging to a certain chosen orientation. Otherwise, Ψ is said to be orientation reversing.

Proposition 3.6. *If* coker *L and V are oriented then Ψ and Ψ' are homotopic in $\lfloor_L^V$ if and only if they are simultaneously orientation preserving or orientation reversing.*

DEFINITION 3.3. Let V be as given above and suppose that assumptions (abcde) are satisfied and Ψ is an orientation preserving continuous isomorphism from $\operatorname{coker} L$ into V. Then, the coincidence degree of L and N in Ω, denoted by $D[(L, N), \Omega]$, is defined by

$$D[(L, N), \Omega] = \begin{cases} \bigcup_{V \in \aleph_{\ker L}} d_V(I-\tilde{M}_\Psi, \Omega, 0) & \text{if } L \text{ is not surjective} \\ d(I-K_P N, \Omega, 0) & \text{if } L \text{ is surjective} \end{cases}$$

where $\aleph_{\ker L}$ is the family of all vector subspaces V of $\ker L$ such that $\dim V = \dim \operatorname{coker} L$.

REMARK 3.4. Note that if $\operatorname{ind} L = 0$, the projector R_Ψ is the identity mapping and then, Tarafdar and Teo's work ([3]) is a particular case of our work.

4. Basic properties of coincidence degree

In this section, unless otherwise specified, we shall assumed that assumptions (abcde) are satisfied such that the coincidence degree is well defined.

Theorem 4.1. (a) (Existence theorem) *If*

$$D[(L, N), \Omega] \neq \{0\}$$

then (I) has at least one solution in Ω.

(b) (Excision property) *If* $\Omega_0 \subset \Omega$ *is an open set such that*

$$(L-N)^{-1}(0) \subset \Omega_0\,,$$

then,

$$D[(L, N), \Omega] = D[(L, N), \Omega_0]$$

(c) (Addivity property) *If* $\Omega = \Omega_1 \cup \Omega_2$ *where* Ω_1, Ω_2 *are two open sets such that* $\Omega_1 \cap \Omega_2 = \phi$, *then*

$$D[(L, N), \Omega] \subset D[(L, N), \Omega_1] + D[(L, N), \Omega_2]$$

Proof. This theorem follows immediately from Definition 3.3 and corresponding properties of topological degree of ultimately compact multivalued mappings (see, Petryshyn and Fitzpatrick ([4])). QED.

One of the most useful properties of every concept of topological degree is its invariance with respect to some type of homotopy. In the case of coincidence degree we have the following:

Theorem 4.2. *If the assumptions (ab) are satisfied and if the multivalued mapping*

$$\tilde{N}: \bar{\Omega} \times [0, 1] \rightarrow CK(Y)$$

is such that

($\tilde{c}$) $\tilde{N}$ *is u.s.c. on* $\bar{\Omega} \times [0, 1]$,

($\tilde{d}$) $\pi \tilde{N}(\bar{\Omega} \times [0, 1])$ *is bounded,*

($\tilde{e}$) $\phi[K_P(I-Q)\, \tilde{N}(\bar{\Omega} \times [0, 1])] < \infty$ *and* $K_P(I-Q)\, \tilde{N}$ *is* ϕ*-condensing*

(f) *for each* $\lambda \in [0, 1]$,

$$0 \notin [L - \tilde{N}(\cdot, \lambda))\, (D(L) \cap \partial \Omega)\,.$$

Then, $D[(L, \tilde{N}(\cdot, \lambda)), \Omega]$ *is independent of* λ *in* $[0, 1]$.

Note. Here, ϕ, P, Q, K_P are the same as given in assumption (d).

Proof. It is an easy consequence of Definition 3.3 and the corresponding property of topological degree of an ultimately compact vector field (Petryshyn and Fitzpatrick ([4])). QED.

Theorem 4.3. *If O is a symmetric bounded neighbourhood of the origin and N is odd ($N(-x)=-N(x)$ for all $x\in O$) such that $L(x)\notin N(x)$ for all $x\in\partial O\cap D(L)$, then $D[(L, N), O]\neq\{0\}$.*

Proof. Note that, how P, Q, K_P, Ψ, R_Ψ and are all linear, the condition on N implies that $\tilde{M}_\Psi$ is also odd. Thus, by the corresponding property of topological degree of an ultimately compact vector field (see [4]) and Definition 3.3, it follows that $D[(L, N), O]\neq\{0\}$. QED.

5. Application to multivalued boundary value problem for elliptic partial differential equation

Let $G\subset \boldsymbol{R}^n$ be a bounded domain whose boundary ∂G is a C^∞-manifold. We will consider real-valued functions of the following type: $u: G\to\boldsymbol{R}$. For a multi-index $\alpha=(\alpha_1, \cdots, \alpha_n)$ and a function $u: G\to\boldsymbol{R}$ the symbol

$$D^\alpha u = D^{|\alpha|}u/(\partial^{\alpha_1} x_1\cdots\partial^{\alpha_n} x_n)$$

will denote the partial derivative of u (if it exists) of the order $|\alpha|=\alpha_1+\cdots+\alpha_n$.

Let $C^m(G)$ be a space of all functions u from G into $\boldsymbol{R}$ which are continuous together with derivatives $D^\alpha u$, $|\alpha|\leq m$, and let

$$\tilde{C}^m_p(G) = \{u\in C^m(G): (\sum_{|\alpha|\leq m}\int_G |D^\alpha u(x)|^p dx)^{1/p}<\infty\}$$

for $1<p<\infty$. In the space $\tilde{C}^p_m(G)$ we define the norm as follows:

$$||u||_{m,p} = (\sum_{|\alpha|\leq m}\int_G |D^\alpha u(x)|^p dx)^{1/p}$$

By $H_{m,p}(G)$ we will denote the Sobolev space which is the completion of $\tilde{C}^m_p(G)$ with respect to the norm $||\cdot||_{m,p}$. By $C^\infty_0(G)$ we will denote the space of all functions $u\in C^\infty(G)=\bigcap_{m=0}^{\infty} C^m(G)$, which have a compact support in G.

Let $u, v: G\to\boldsymbol{R}$ be two integrable functions. We say that the function v is the α-th weak derivative of u if, for every $f\in C^\infty_0(G)$,

$$\int_G u(x)\, D^\alpha f(x)\, dx = (-1)^{|\alpha|}\int_G v(x) f(x)\, dx\,.$$

Then we write $D^\alpha(u)=v$.

Let $L^p(G)$, $p>1$, be the Banach space of all measurable functions $y: G\to\boldsymbol{R}$, for which $\int_G |y(u)|^p du<\infty$, with the norm

$$||y||_p = (\int_G |y(u)|^p du)^{1/p}$$

The following two facts are well known (see [5]).

(5.1) $$H_{m,p}(G)=\{u\in L^p(G);\ \tilde{D}^\alpha u\in L^p(G),\ |\alpha|\leq m\}$$

Let α be such that $|\alpha|\leq m$. The mapping $\tilde{D}^\alpha$: $H_{m,p}(G)\to L^p(G)$ is a continuous extension of the mapping D^α: $C^m(G)\to C^0(G)$.

Let $C^m(\bar{G})$ be the space of all functions u from G into $\boldsymbol{R}$ which are uniformly continuous together with derivatives $D^\alpha(u)$ for $|\alpha|\leq m$.

In the space $C^m(\bar{G})$ we define a norm putting

$$|u|_m=\sum_{|x|<n}\sup_{x\in G}|D^\alpha u(x)|\,.$$

Let $C^{m+\mu}(\bar{G})$, $0<\mu<1$, be the Hölder space with the norm

$$|u|_{m+\mu}=|u|_m+\sum_{|\alpha|=m}\sup\frac{|D^\alpha u(x)-D^\alpha u(y)|}{|x-y|^\mu}:\ x,y\in G,\ x\neq y\,.$$

We have $C^{m+\mu}(\bar{G})\subset C^m(\bar{G})$. Note the following (see [5]).

(5.2) The embedding i: $C^{m+\mu}(\bar{G})\to C^m(\bar{G})$, given by $i(u)=u$ is a completely continuous mapping.

From the Sobolev embedding theorem (see [5]) we obtain the following:

Proposition 5.1. *Let $p>n$. Then, for $\mu=1-n/p$, the mapping j: $H_{m,p}(G)\to C^{m-1+\mu}(\bar{G})$ given as follows: $j(\tilde{u})=u$, $u\in C^{m-1+\mu}(\bar{G})$ and $u(x)=\tilde{u}(x)$ a.e. on G, is well defined and it is a continuous mapping.*

Let A_p: $H_{m,p}(G)\to L^p(G)$ be an elliptic operator given by

$$A_p(u)(x)=\sum_{|\alpha|\leq m}a_\alpha(x)\,D^\alpha u(x)\,,$$

where

$$a_\alpha(\cdot)\in\bigcap_{m=0}^{\infty}C^m(\bar{G})=C^\infty(\bar{G})$$

and let B_j: $C^{m-1}(\bar{G})\to C^0(\bar{G})$; $j=1,2,\cdots,k$ be a differential boundary operator given by

$$B_j(u)(x)=\sum_{|\alpha|\leq m_j}b_\alpha^j(x)\,D^\alpha u(x)\,,$$

where $m_j<m$, $b_\alpha^j(\cdot)\in C^\infty(\bar{G})$ for $j=1,2,\cdots,k$ and $|\alpha|\leq m_j$.

For a multivalued mapping f: $\bar{G}\times R\times R\to 2^R$ we formulate the following boundary value problem:

(5.3) $$\begin{cases}u\in C^{m-1}(\bar{G})\\ A_p(u)(x)\in f(x,u(x,D^\beta u(x))\text{ a.e. on }G,\ |\beta|\langle m,p\rangle n\\ B_j(u)(x)=0\quad\text{for}\quad x\in\partial G,\ j=1,2,\cdots,m/2\,.\end{cases}$$

DEFINITION 5.1. We say that a multivalued mapping $f: \bar{G}\times R\times R\to 2^R$ satisfies the Caratheodory conditions if:

(C_1) for each pair $(u, v)\in \boldsymbol{R}^2$, the multivalued mapping $f(\cdot, u, v)$ is measurable, i.e., for every open set $U\subset \boldsymbol{R}$, the set $f^{-1}(U, u, v)=\{x\in \bar{G}: f(x, u, v)\cap U\neq\Phi\}$ is Lebesque measurable;

(C_2) for each $x\in\bar{G}$, the multivalued mapping $f(x, \cdot, \cdot)$ is u.s.c.

Theorem 5.1. *Suppose that the multivalued mapping f stated in the boundary value problem* (5.3) *satisfies*:

(i) *the Caratheodory conditions* C_1, C_2;

(ii) *for each* $(x, u, v)\in\bar{G}\times\boldsymbol{R}\times\boldsymbol{R}$, $f(x, u, v)$ *is a convex set*;

(iii) $|f(x, u, v)|\leqq g(x)(1+|u|+|v|)^\rho$, *for each* $(x, u, v)\in\bar{G}\times R\times R$ *and some* $g\in L^p(G)$ *and some* ρ, $\rho<1$.

Moreover, suppose that $R(A_p)=L^p(G)$ *and* $j(\ker A_p)\subset C^\infty(G)$, *where* A_p *is the elliptic operator stated above and that the system*

$$\begin{cases} A_p(u)(x)=0 \\ B_j(u)(x)=0 \quad \textit{for} \quad x\in\partial G,\ j=1, 2, \cdots, m/2 \end{cases}$$

admit only a finite number of linearly independent solutions. Then, the problem (5.3) *admits a solution.*

Proof. Let us put $X_1=C^{m-1}(\bar{G})$; $X_2=L^p(G)$, $p>n$ and

$$X=\{u\in C^{m-1}(\bar{G}): u\in H_{m,p}(G), B_j(u)_{/\partial G}=0, j=1, \cdots, m/2\}$$

Now, let us specify the following:

$$N: X_1\to 2^{X_2}$$
$$N(u)=\{v\in L^p(G); v(x)\in f(x, u(x), D^\beta u(x)) \text{ a.e. on } G\}$$

and

$$L: D(L)=X\to X_2; \quad L(u)=A_p(u)$$

So, the problem (5.3) is equivalent to the equation

$$L(u)\in N(u).$$

The mapping L defined above is a linear Fredholm mapping with ind $L\geq 0$.

Let P be a projection in $\langle X_1, |\cdot|_{m-1}\rangle$ such that $R(P)=\ker L$. By the Banach theorem, the mapping $(L_{/X_0})^{-1}$: $X_2\to\langle X, \|\cdot\|_{m,p}\rangle$ is continuous, where $X_1=\ker L\oplus X_0$ with X_0 a closed vector subspace of $\langle X_1, |\cdot|_{m-1}\rangle$. In virtue of (5.2) and Proposition 5.1, we see that the mapping K_p is a completely continuous mapping from X_2 into $\langle X_1, |\cdot|_{m-1}\rangle$, where K_p is the mapping stated in the following commutative diagram:

$$\begin{tikzcd}
X_2 = L^p(G) \arrow[r, "(L_{/X_0})^{-1}"] \arrow[ddr, "K_p = i_{/X}\cdot j/X.\,(L_{/X_0})^{-1}"'] & \langle X, \|\cdot\|_{m,p}\rangle \arrow[d, "j_{/X}"] \\
& \langle X, |\cdot|_{m-1+\mu}\rangle \arrow[d, "i_{/X}"] \\
& \langle X, |\cdot|_{m-1}\rangle
\end{tikzcd}$$

Let $T: X_1 \to C(\bar{G}, \boldsymbol{R}^2)$ be a linear continous mapping given by $T(u)=(u, D^\beta u)$ for every $u \in X$, and let $S: C(\bar{G}; \boldsymbol{R}^2) \to 2^{X_2}$ be the multivalued mapping defined by

$$S(u) = \{Z \in L^p(G);\ Z(x) \in f(x, u(x), v(x)) \text{ a.e. on } G\}$$

Since by condition C_3, $N = S \circ T$ maps a bounded set into a bounded set, the multivalued mapping K_pN is compact since K_p is a compact linear mapping. Furthermore, by conditions (i) and (ii) we have that K_pN is u.s.c. and for each $u \in X$, $K_pN(u)$ is a convex set. The closdness of $K_pN(u)$ for each $u \in X$ follows by the upper semicontinuity and the compactness of K_pN.
Now, by the surjectivity of L, the projector Q stated in the Definition 3.3 is the null operator. Then,

$$K_p(I-Q)\, N = K_pN\,.$$

Still by the surjectivity of L, coker $L = \{0\}$ and then,

$$\tilde{M}_\Psi = K_pN\,,$$

where $\tilde{M}_\Psi$ is the multivalued mapping stated in the Theorem 3.1. Thus, if u is a solution of $L(u) \in N(u)$, then $u \in K_pN(u)$, and so, by condition (iii) we have

$$|u|_{m-1} \leqq D(1+|u|_{m-1})^\rho\,, \quad \rho<1\,,$$

where C is a positive constant. This implies that there exists a positive constant $\bar{C}$ such that if u is a solution of $L(u) \in N(u)$ then,

$$|u|_{m-1} \leqq \bar{C}\,.$$

So, if we take $\delta > \bar{C}$, then, for each $u \in \partial B(O, \delta)$, we have

$$u \notin K_pN(u)$$

Let M be the multivalued mapping defined by

$$M(\lambda, u) = \lambda K_pN(u), \quad \lambda \in [0, 1], \quad u \in X_1$$

It is easily seen that if $u \in \partial B(0, \delta)$ then we have $u \notin M(\lambda_0, u)$ for each $\lambda_0 \in [0, 1]$.

So, by the homotopy property of topological degree, we have

$$d(I-K_pN, B(0, \delta), 0) = d(I, B(0, \delta), 0) .$$

Now, it is well known that $d(I, B(0, \delta), 0)=1$ and so, by the existence property of coincidence degree, we have that the equation $L(u)\in N(u)$ admit a solution, ie, the boundary value problem (5.3) admit a solution. QED.

REMARK. Compare our application with the application 5 of [6] and note that while in [6] the mapping S is considered injective, in our case this is exchanged by the more general condition dim ker $S<\infty$. Another point to be remarked is that our hypothesis (iii) of Theorem 5.1 is more general that the condition (C_3') stated in Theorem 5.6 of [6].

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