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ON THE CONTINUITY OF PLURISUBHARMONIC FUNCTIONS ALONG CONFORMAL DIFFUSIONS

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1. Introduction

A stochastic process $Z_t=(Z_t^1, \cdots, Z_t^n)$ taking values in C^n is called a *conformal martingale* if Z_t^α and $Z_t^\alpha Z_t^\beta$, $1\leqq\alpha, \beta\leqq n$, are continuous local martingales. When Z_t is defined only on a time interval $[0, \eta)$ for some predictable stopping time η, Z_t is said to be a conformal martingale if so is the stopped process $Z_{t\wedge\eta'}$ for any stopping time η' strictly less than η.

Let M be a complex manifold of complex dimension n. By a diffusion process $\boldsymbol{D}=(Z_t, P_z)$ on M, we mean a strong Markov process on M with continuous sample paths defined on $[0, \zeta)$, ζ being the life time. In this paper, we assume without specific mention that the diffusion $\boldsymbol{D}$ admits no killing inside M in the sense that $P_z(\tau_U<\zeta<+\infty)=P_z(\zeta<+\infty)$, $z\in U$, for any relatively compact open set $U\subset M$, where τ_U denotes the first exit time from U: $\tau_U=\inf\{t\geqq 0: Z_t\notin U\}$. We see then that, for any open set $U\subset M$, τ_U is a predictable stopping time with respect to P_z for $z\in U$.

We call a diffusion process $\boldsymbol{D}=(Z_t, P_z)$ on M a *conformal diffusion* on M if, for any holomorphic coordinate neighbourhood (U, ϕ), the C^n-valued process $\phi(Z_t)$ defined on $[0, \tau_U)$ is a conformal martingale with respect to P_z for each $z\in U$. We occasionally assume that the transition function p_t of $\boldsymbol{D}$ is absolutely continuous with respect to a volume element V on M:

$$p_t(z, \cdot)<V\,, \qquad z\in M\,. \tag{1.1}$$

We aim at proving the following theorem.

Theorem. *Let $\boldsymbol{D}=(Z_t, P_z)$ be a conformal diffusion on M satisfying the condition* (1.1). *Then, for any plurisubharmonic function u on M,*
$P_z(u(Z_t)$ is continuous in $t\in[0, \zeta)$ and finite for $t\in(0, \zeta))=1$, $z\in M$.

This is a generalization of a theorem of Doob [2] to the cases of higher complex dimension and our proof is also similar to the one given in [2] in the sense that we utilize the quasi-continuity of plurisubharmonic functions with respect to a specific capacity related to the extremal function.

As we shall see, any plurisubharmonic function u on M is $\boldsymbol{D}$-subharmonic in Dynkin's sense and consequently $u(Z_t)$ is right continuous. Therefore its continuity would follow from a Hunt's theorem on the regularity of excessive functions provided that

(1.2) every semi-polar set is polar

for the diffusion $\boldsymbol{D}$. However, it seems to be unknown whether (1.2) is fulfilled for all the conformal diffusions being considered.

Indeed, a typical conformal diffusion is a diffusion $\boldsymbol{D}$ on M whose infinitesimal generator is expressible on a local chart as

$$L=\frac{1}{2}\sum g^{\alpha\bar{\beta}}\frac{\partial^2}{\partial z^\alpha\partial\bar{z}^\beta} \tag{1.3}$$

with a continuous non-negative Hermitian tensor field g on M ([6]). If g is sufficiently smooth and non-degenerate, (1.2) is known to be correct for $\boldsymbol{D}$. But, if g is merely continuous positive or degenerate ($\boldsymbol{D}$ may still satisfy (1.1) in the latter case), we do not know to what extent (1.2) is true. (1.2) becomes true under the additional condition of the symmetrizability. But the latter condition might be false either in general in view of a Fujita's result [4] saying that there exists a manifold M where no diffusion with generator (1.3) is symmetrizable (although only smooth and non-degenerate cases are treated in [4]).

We add a remark that there are many conformal diffusions whose generators are not expressible by the usual differential operator like (1.3). It was shown in [5] that fairly general class of symmetrizable conformal diffusions can be characterized by closed positive currents of type $(n-1, n-1)$. The first two propositions of the present paper have been proven in [5] for this class of diffusions on a domain of C^n.

This work was motivated by the lectures of Professor Laurent Schwartz delivered at Kyoto University (cf. [7]). I am grateful to him for his kind guidance to the problem.

2. $\boldsymbol{D}$-subharmonicity of plurisubharmonic functions

A function u on an open set $E\subset M$ taking values in $[-\infty, +\infty)$ is said to be *plurisubharmonic* on E if, on each holomorphic coordinate neighbourhood $U\subset E$, u is locally integrable, $\sum \frac{\partial^2 u}{\partial z^\alpha\partial\bar{z}^\beta}\xi^\alpha\bar{\xi}^\beta$ is a non-negative distribution for any $\xi=(\xi^1, \cdots, \xi^n)\in C^n$ and $\operatorname{ess}\limsup_{z'\to z} u(z')=u(z)$, $z\in U$. Those properties are intrinsic because they are preserved under holomorphic transformations. Any plurisubharmonic function is upper semicontinuous.

Let $\boldsymbol{D}=(Z_t, P_z)$ be a conformal diffusion on M with transition function p_t.

A Borel function u on M taking values in $[0, +\infty]$ is called *p_t-excessive* if $p_t u(Z) \uparrow u(z)$ as $t \downarrow 0$ for each $z \in M$. A Borel function u on an open set $E \subset M$ taking values in $[-\infty, +\infty)$ is called ***D**-subharmonic* on E if u is $\boldsymbol{D}$-finely upper semicontinuous, locally bounded from above and, for any open set V with compact closure $\bar{V} \subset E$,

$$u(z) \leqq E_z(u(Z_{\tau_V}); \tau_V < \zeta), \qquad z \in V. \tag{2.1}$$

Any $\boldsymbol{D}$-subharmonic function is $\boldsymbol{D}$-finely continuous and hence right continuous along Z_t in $t \in [0, \zeta)$ P_z-almost surely ([3]). The negative of a $\boldsymbol{D}$-subharmonic function is said to be $\boldsymbol{D}$-superharmonic. A Dynkin's theorem [3; Theorem 12.4] says that a non-negative Borel function on M is p_t-excessive if and only if it is $\boldsymbol{D}$-superharmonic.

Proposition 1. *Any plurisubharmonic function on M is $\boldsymbol{D}$-subharmonic. The negative of a non-positive plurisubharmonic function on M is p_t-excessive.*

Proof. Let u be a plurisubharmonic function on M. u is then locally bounded from above on each coordinate neighbourhood. Besides the subharmonicity for the diffusion process is a local property according to a Sur's theorem ([3; Theorem 12.11]). Hence we may only prove the $\boldsymbol{D}$-subharmonicity of u by assuming that M is a bounded domain $D \subset C^n$ and u is non-positive.

Take any open set V with compact closure in D and denote by τ the first exit time of Z_t from V. Since (Z_t, P_z) is a conformal martingale, we see, by virtue of Schwartz [6; Proposition (5.10)], that $(u(Z_{t\wedge\tau}), P_z)$ is a generalized submartingale for $z \in V$, and consequently,

$$-\infty \leqq u(z) \leqq E_z(u(Z_{t\wedge\tau})), \qquad z \in V.$$

The right hand side is not greater than $E_z(u(Z_{t\wedge\tau}); \tau < \zeta)$ and we get the inequality (2.1) by letting $t \to +\infty$. Since u is upper semicontinuous, we conclude that u is $\boldsymbol{D}$-subharmonic.

In the remainder of this section, we only consider a bounded domain D of C^n. For $E \subset D$, the extremal function u_E^* is defined by $u_E(z) = \sup\{v(z)\colon v$ plurisubharmonic on D, $-1 \leqq v \leqq 0$ on D, $v = -1$ on $E\}$, $u_E^*(z) = \overline{\lim}_{z' \to z} u_E(z')$, $z \in D$. We further introduce a set function C_* by

$$C_*(E) = -\int_D u_E^*(z)dV(z) \quad \left(= -\int_D u_E(z)dV(z)\right) \tag{2.2}$$

where V denotes the Lebesgue measure on D. C_* is known to be a Choquet capacity ([1; Proposition 8.4]). Moreover $C_*(N) = 0$ if and only if N is pluripolar, namely, there exists a plurisubharmonic function v on D with $N \subset v^{-1}(-\infty)$.

Let $\boldsymbol{D} = (Z_t, P_z)$ be a conformal diffusion on D. Denote by σ_E the hitting time of a set $E\subset D$ after $0+$: $\sigma_E=\inf\{t>0\colon Z_t\in E\}$. We let $\sigma_E=+\infty$ if the event in the braces is empty.

Proposition 2. *For any Borel set $E\subset D$,*

$$\int_D P_z(\sigma_E<\zeta)dV(z)\leqq C_*(E).$$

Proof. By Choquet's lemma, there is a non-decreasing sequence of plurisubharmonic functions v_k such that $-1\leqq v_k\leqq 0$, $v_k=-1$ on E and $u_E^*(z)=\overline{\lim}_{z'\to z} v_0(z')$ for $v_0=\lim_{k\to\infty} v_k$. By Proposition 1, $\{-v_k(Z_t), P_z\}$ is supermartingale and $-v_k(z)\geqq -E_z(v_k(Z_{\sigma_K})\colon \sigma_K<\zeta)\geqq P_z(\sigma_K<\zeta)$ for any compact set $K\subset E$ and $z\in D$, on account of the optional sampling theorem. Letting $k\to\infty$ and integrating by dV, we have

$$\int_D P_z(\sigma_K<\zeta)dV(z)\leqq C_*(E)$$

since $v_0=u_E^*$ V-a.e. Taking then an increasing sequence of compact sets $K_m\subset E$ such that $\sigma_{K_m}\downarrow\sigma_E$ as $m\to\infty$, P_V-a.e., we get the desired inequality.

Corollary 1

(i) *If $N\subset D$ is pluripolar, then there exists a Borel set $N'\supset N$ and*

(2.3) $$P_z(\sigma_{N'}<\zeta)=0 \qquad V\text{-}a.e.\quad z\in D.$$

(ii) *If $O_k\subset D$ are decreasing open sets such that $\lim_{k\to\infty} C_*(O_k)=0$, then*

(2.4) $$P_z(\lim_{k\to\infty}\sigma_{O_k}<\zeta)=0 \qquad V\text{-}a.e.\quad z\in D.$$

Proof. (ii) is a stronger assertion than (i). (ii) is immediate from Proposition 2.

We denote by θ_s the usual shift operator defined by $Z_t(\theta_s\omega)=Z_{s+t}(\omega)$. In particular we have

(2.5) $$s+\sigma_E\circ\theta_s(\omega)=\inf\{t>s\colon Z_t(\omega)\in E\},\qquad s\geqq 0.$$

Corollary 2. *Suppose that the transition function p_t of $\boldsymbol{D}$ satisfies the absolute continuity condition* (1.1).

(i) *If $N\subset D$ is pluripolar, then N is $\boldsymbol{D}$-polar: there exists a Borel set $N'\supset N$ and* (2.3) *holds for every $z\in D$.*

(ii) *If $O_k\subset D$ are decreasing open sets such that $\lim_{k\to\infty} C_*(O_k)=0$, then*

(2.6) $$P_z(\lim_{k\to\infty}(s+\sigma_{O_k}\circ\theta_s)<\zeta,\ s<\zeta)=0$$

for every $s>0$ and $z\in D$.

Proof. (i) Denote by $f(z)$ the left hand side of (2.3). Then $f(z)=$

$\lim_{s\downarrow 0} p_s f(z)=0$, $z\in D$, by the assumption and Corollary 1 (i). (ii) The left hand side of (2.6) equals $p_s f(z)$ for the function f defined by the left hand side of (2.4).

3. $C_\sharp$-quasi-continuity of plurisubharmonic functions

We continue to consider a bounded domain $D\subset C^n$ and the capacity $C_\sharp$ defined by (2.2).

Proposition 3. *Suppose that the domain D is strongly pseudo-convex. Any plurisubharmonic function u on D is then $C_\sharp$-quasi-continuous. More specifically, for any $\varepsilon>0$, there exists an open set $O\subset D$ with $C_\sharp(O)<\varepsilon$ such that u is finite valued and continuous on $D-O$ with respect to the relative topology.*

Proof. We deduce this from several results of Bedford-Taylor [1]. First, according to [1; Theorem 3.5], any plurisubharmonic function on a bounded domain D is quasi-continuous in the above sense but with respect to another capacity which we shall denote by C_{BT}. C_{BT} admits the expression

$$C_{BT}(O) = \int_D (dd^c u_O^*)^n, \tag{3.1}$$

for open set O with compact closure $\bar{O}\subset D$. Therefore it suffices to show the implication

$$C_{BT}(O_k) \underset{k\to\infty}{\longrightarrow} 0 \Rightarrow C_\sharp(O_k) \underset{k\to\infty}{\longrightarrow} 0 \tag{3.2}$$

for decreasing sequence of open sets $O_k\subset D$.

A function on D is quasi-continuous relative to a capacity if and only if it is so on each open set E with compact closure $\bar{E}\subset D$. Hence, in proving (3.2), we may assume that O_1 has compact closure $\bar{O}_1\subset D$. Set $v=\lim_{k\to\infty} u_{O_k}^*$, $v^*(z)=\lim_{z'\to z} v(z')$, and assume now the strong pseudo-convexity of D. We then easily see that $v^*(z)\to 0$ as $z\to\partial D$. Moreover by the continuity of the Bedford-Taylor measures [1; Proposition 5.2], $(dd^c u_{O_k}^*)^n\to(dd^c v^*)^n$, $k\to\infty$. Hence we get, from (3.1) and the hyposesis in (3.2), $\int_D (dd^c v^*)^n=0$ and consequently $(dd^c v^*)^n$ is the zero measure. We can finally use a comparison theorem [1, Corollary 4.4] to obtain $v^*=0$ and $v=0$ V-a.e. We arrive at the conclusion in (3.2): $\lim_{k\to\infty} C_\sharp(O_k)=\int_D v(z)dV(z)=0$.

4. Proof of Theorem

The right continuity of $u(Z_t)$ at $t=0$

$$P_z(\lim_{t\downarrow 0} u(Z_t) = u(z)) = 1, \qquad z\in M, \tag{4.1}$$

follows from Proposition 1.

For a stopping time $\eta \in [0, +\infty]$, let us consider the event $\Lambda_\eta = \{u(Z_t)$ is finite and continuous for $t \in (0, \eta)\}$. We aim at proving

$$(4.2) \qquad P_z(\Lambda_\zeta) = 1\,, \qquad z \in M\,.$$

We assume the condition (1.1). Choose a system of holomorphic coordinate neighbourhoods (U_α, ϕ_α) of M such that $\phi_\alpha(U_\alpha)$ is a strongly pseudo-convex bounded domain of C^n. Proposition 3 and Corollary 2 (ii) to Proposition 2 are applicable to the function $u|_{U_\alpha}$ and to the part $\boldsymbol{D}_\alpha$ of $\boldsymbol{D}$ on U_α respectively. In view of (2.5), we then readily see that $P_z(u(Z_t)$ is finite and continuous for $t \in (s, \tau_{U_\alpha})$, $s' < \tau_{U_\alpha}) = P_z(s' < \tau_{U_\alpha})$, $0 < s \leqq s'$, $z \in U_\alpha$. By letting $s \downarrow 0$ and then $s' \downarrow 0$, we get

$$(4.3) \qquad P_z(\Lambda_{\tau_{U_\alpha}}) = 1\,, \qquad z \in U_\alpha\,.$$

We now use a Sur's method. Take two members, say, U_0 and U_1 from the chart system and let V be an arbitrary open set with $\bar{V} \subset U_0 \cup U_1$. By [3; Lemma 12.6], we can find open sets V_0 and V_1 such that $V = V_0 \cup V_1$, $\bar{V}_0 \subset U_0$, $\bar{V}_1 \subset U_1$ and $\overline{V_0 \cap (M - V_1)} \cap \overline{V_1 \cap (M - V_0)} = \phi$. Denote by τ^0 and τ^1 the exit time from V_0 and V_1 respectively, and let $\gamma_0 = 0$, $\gamma_{k+1} = \gamma_k + \tau^{\varepsilon_k} \circ \theta_{\gamma_k}$, $k \geqq 1$, where $\varepsilon_k = k \bmod 2$. By virtue of [3; Lemma 12.4], it holds then that

$$(4.4) \qquad \gamma_k = \tau_V \qquad \text{from some } k \text{ on.}$$

Together with the event Λ_η for the stopping time η, we also consider the event $\tilde{\Lambda}_\eta = \{u(Z_t)$ is finite continuous at each $t \in (0, \eta)$ and also at $t = \eta$ if $\eta < +\infty\}$. In view of (4.3), we have

$$(4.5) \qquad P_z(\tilde{\Lambda}_{\tau^i}) = 1\,, \qquad z \in V_i\,,\ i = 0, 1\,.$$

Since (4.5) is trivially true for $z \in D - V_i$, we obtain from the strong Markov property and (4.5),
$P_z(\tilde{\Lambda}_{\gamma_2}) = P_z(u(Z_t)$ is finite continuous for $t > 0$, $\tau^0 = +\infty) + E_z(u(Z_t)$ is finite continuous for $t \in (0, \tau^0]$, $\tau^0 < +\infty\,;\ P_{z_{\tau^0}}(\tilde{\Lambda}_{\tau^1})) = P_z(\tilde{\Lambda}_{\tau^0}) = 1$, $z \in V$. By induction and (4.4), we get

$$(4.6) \qquad P_z(\tilde{\Lambda}_{\tau_V}) = 1\,, \qquad z \in V\,.$$

By letting $V \downarrow U_0 \cup U_1$, we are led from (4.6) to

$$(4.7) \qquad P_z(\Lambda_{\tau_G}) = 1\,, \qquad z \in G\,,$$

for $G = U_0 \cup U_1$. Repeating the same argument, (4.7) can be seen to be true for the union G of finite number of U_α's. Now (4.7) holds for any relatively compact open set $G \subset M$. We finally let $G \uparrow M$ to get the desired identity (4.2).

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