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ON THE ASYMPTOTIC DISTRIBUTION OF EIGENVALUES IN AN UNBOUNDED DOMAIN

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1. Introduction

Let Ω be a domain in real space R^n with generic point $x=(x_1, \cdots, x_n)$. We denote by $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index of length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and use the notations

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \quad D_k = -\sqrt{-1}\,\partial/\partial x_k .$$

For an integer $m\geqq 0$, $H_m(\Omega)$ is to be the set of all functions whose distribution derivatives of order up to m belong to $L^2(\Omega)$ and we introduce in it the usual norm

$$||u||_m = ||u||_{m,\Omega} = (\int_\Omega \sum_{|\alpha|\leqq m} |D^\alpha u|^2\, dx)^{1/2} .$$

$\mathring{H}_m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H_m(\Omega)$.

Let B be a symmetric integro-differential sesquilinear form of order m with bounded coefficients

$$B[u, v] = \int_\Omega \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x) D^\alpha u\, \overline{D^\beta v}\, dx$$

satisfying

$$B[u, u]\geqq \delta ||u||_m^2 \qquad \text{for any } u\in\mathring{H}_m(\Omega)$$

where δ is some positive constant. Let A be the operator associated with this sesquilinear form: an element u of $\mathring{H}_m(\Omega)$ belongs to $D(A)$ and $Au=f\in L^2(\Omega)$ if $B[u, v]=(f, v)$ is vaid for any $v\in\mathring{H}_m(\Omega)$. It is well known that A is a positive definite self-adjoint operator in $L^2(\Omega)$. On the other hand, Beryer & Schecter [3] proved that the injection $\mathring{H}_m(\Omega)\subset L^2(\Omega)$ is compact if

$$\operatorname{meas}(S(x)\cap\Omega) \to 0 \qquad \text{as } |x| \to \infty \tag{1.1}$$

where $S(x)=\{y\in R^n\colon |y-x|<1\}$. Hence, when Ω satisfies (1.1), the spectrum of A consists of a sequence $\{\lambda_k\}$ of eigenvalues of finite multiplicity having $+\infty$ as the only accumulation point. For $t>0$ let $N(t)$ be the number of eigenvalues of A which do not exceed t. This paper is devoted to the investigation of the asymptotic distribution of eigenvalues of A under the assumption $2m>n$. The asymptotic distribution of eigenvalues in unbounded domains was studied by several writers. For the Laplace operator Tamura [8] and Asakura [1] obtained the asymptotic formula of the distribution. Fleckinger [4] considered a certain type of elliptic operators on domains in R^2. For the uniformly elliptic, second order, formally self-adjoint partial differential operators Hewgill [5], [6] gave upper and lower bounds for $N(t)$. In the case of order $2m$ Audrin & Pham The Lai [2] gave an upper bound for $N(t)$: under the condition $\int_\Omega \delta(x)^{-2k}\,dx<\infty$ for an integer k such that $m>n/2+k$ they established $N(t)=0(t^{(n+2k)/2m})$ where $\delta(x)=\mathrm{dist}(x, \partial\Omega)$.

In this paper we consider domains which satisfy a P_τ-condition:

$$(P_\tau)\quad \mathrm{meas}(\Omega\cap\{x\colon |x|=r\})\leqq C(1+r)^{-\tau}$$

where τ is a positive constant such that $0<\tau\leqq 1$. The conclusion of this paper is that

$$N(t)=\begin{cases} O(t^{n/2m+(n-1)(1-\tau)/2m\tau}) & \text{if } 0<\tau<1 \\ O(t^{n/2m}\log t) & \text{if } \tau=1 \end{cases} \tag{1.2}$$

as $t\to\infty$. When $\tau=1$, under some additional assumptions on Ω and the coefficients of B we shall derive the asymptotic formula:

$$N(t)\sim\int_{\Omega_t} a(x)\,dx\; t^{n/2m} \tag{1.3}$$

as $t\to\infty$ where

$$\Omega_t=\Omega\cap\{x\colon |x|\leqq t^{(n-1)/2m}\},$$
$$a(x)=(2\pi)^{-n}\,\mathrm{meas}\,\{\xi\colon \sum_{|\beta|=|\alpha|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta}<1\}.$$

The method used in this paper is different from the above papers. By this method we can estimate the eigenfunctions of A: for any positive integer k there exists a constant C_k such that

$$|\phi_j(x)|\leqq C_k\lambda_j^{n/2n+(n-1)k/4m}(1+|x|)^{-\tau k/2} \tag{1.4}$$

where $A\phi_j=\lambda_j\phi_j$, $(\phi_i, \phi_j)=\delta_{ij}$. In the proof of (1.3) we shall use the result of Tsujimoto [9].

2. Main theorems

As was stated in the introduction it is assumed that $2m>n$ and we consider domains which satisfy a P_τ-condition:

$$(P_\tau) \quad \mathrm{meas}(\Omega \cap \{x\colon |x| = r\}) \leqq C(1+r)^{-\tau}.$$

Theorem 1. *Suppose that Ω satisfies (P_τ), then we have*

$$N(t) = \begin{cases} O(t^{n/2m+(n-1)(1-\tau)/2m\tau}) & \text{if } 0<\tau<1 \\ O(t^{n/2m} \log t) & \text{if } \tau = 1 \end{cases}$$

as $t \to \infty$.

Next, we consider the following assumptions:

(Q) $a_{\alpha\beta} \in \mathcal{B}^\infty(\Omega)$,

(R)–(i) $\mathrm{meas}\, \Omega_t \geqq C \log t$,

–(ii) $\lim\limits_{\varepsilon\to 0} \lim\limits_{t\to\infty} [\mathrm{meas}\,(\Omega_t \cap \{x\colon \delta(x) < \varepsilon |x|^{1/(1-n)}\})] (\log t)^{-1} = 0$,

–(iii) $\displaystyle\int_{\Omega_t \cup \{x\,:\,\delta(x) > \varepsilon |x|^{1/(1-n)}\}} \delta(x)^{-1} dx \leqq C_\varepsilon\, t^{1/2m}$,

for $\varepsilon > 0$, $t > 2$ where $\Omega_t = \Omega \cap \{x\colon |x| < t^{(n-1)/2m}\}$, $\delta(x) = \min\, \{1, \mathrm{dist}(x, \partial\Omega)\}$.

Theorem 2. *Suppose that Ω and B satisfy (P_1), (Q) and (R), then the following asymptotic formula for $N(t)$ holds as $t \to \infty$:*

$$N(t) \sim t^{n/2m} \int_{\Omega_t} a(x)\, dx$$

where

$$a(x) = (2\pi)^{-n}\, \mathrm{meas}\, \{\xi\colon \sum_{|\alpha| = |\beta| = m} a_{\alpha\beta}(x) \xi^{\alpha+\beta} < 1\}.$$

3. Some lemmas and proof of theorem 1

Lemma 3.1. *Let S be a bounded operator on the antidual $H_{-m}(\Omega)$ of $\mathring{H}_m(\Omega)$ to $\mathring{H}_m(\Omega)$. Then S has a kernel M in the following sense:*

$$(Sf)(x) = \int_\Omega M(x, y) f(y)\, dx \quad \text{for } f \in L^2(\Omega).$$

$M(x, y)$ is continuous in $\Omega \times \Omega$ and there exist a constant C such that for any $x, y \in \Omega$

$$|M(x, y)| \leqq C \|S\|_{(-m,m)}^{n^2/4m^2} \|S\|_{(-m,0)}^{n/2m-n^2/4m^2} \|S\|_{(0,m)}^{n/2m-n^2/4m^2} \|S\|_{(0,0)}^{(1-n/2m)^2}$$

where we denote by $\|S\|_{(-m,m)}$, $\|S\|_{(-m,0)}$, $\|S\|_{(0,m)}$, $\|S\|_{(0,00}$ the norms of S considered as an operator on $H_{-m}(\Omega)$ to $\mathring{H}_m(\Omega)$, on $H_{-m}(\Omega)$ to $L^2(\Omega)$, on $L^2(\Omega)$ to $\mathring{H}_m(\Omega)$, on $L^2(\Omega)$ to $L^2(\Omega)$ respectively.

Proof. We note that for any function $u \in \mathring{H}_m(\Omega)$ we can use Sobolev's inequality even if Ω does not have the cone property. Hence, the present lemma can be proved just as Lemma 3.2 of [7].

Let $K_\lambda(x, y)$ be the resolvent kernel of A. For $\lambda\in(0, \pi/2)$ we set $\Lambda=\{\lambda: \theta\leqq\arg\lambda\leqq 2\pi-\theta, |\lambda|>0\}$.

Lemma 3.2. *There exist constants C, d such that*

$$|K_\lambda(x, y)|\leqq C|\lambda|^{n/2m-1}e^{-d|\lambda|^{1/2m}|x-y|} \tag{3.1}$$

for $x, y\in\Omega$, $\lambda\in\Lambda$.

Proof. Using Lemma 3.1, the present lemma can be proved just as Lemma 5.1 of [10].

Next, we consider the iterated kernels of $K_\lambda(x, y)$:

$$K_\lambda^{(k)}(x, y)=\int_\Omega K_\lambda^{(k-1)}(x, z)\, K_\lambda(z, y)\, dz\,,$$
$$K_\lambda^{(0)}(x, y)=K_\lambda(x, y)\,.$$

We note that $K_\lambda^{(k)}$ is the kernel of $(A-\lambda)^{-(k+1)}$.

Lemma 3.3. *For any positive integer k there exists a constant C_k such that*

$$|K_\lambda^{(k)}(x, y)|\leqq C_k|\lambda|^{n/2m+(n-1)k/2m-k-1}(1+|x|)^{-\tau k/2}(1+|y|)^{-\tau k/2} \tag{3.2}$$

for any $x, y\in\Omega$, $\lambda\in\Lambda$.

Proof. We prove (3.2) by induction on k.

In the case of $k=1$. Using (3.1) and Schwartz's inequality, we have

$$|K_\lambda^{(1)}(x, y)|\leqq C|\lambda|^{2n/2m-2}\Big(\int_\Omega e^{-d|\lambda|^{1/2m}|x-z|}dz\Big)^{1/2}\times\Big(\int_\Omega e^{-d|\lambda|^{1/2m}|z-y|}dz\Big)^{1/2}. \tag{3.3}$$

In proving (3.2) we may assume that $|x|$, $|y|>2$. We set

$$\Omega_{1,x}=\Omega\cap\{z: |z-x|>|x|^{1/2}\}\,,$$
$$\Omega_{2,x}=\Omega\cap\{z: |z-x|<|x|^{1/2}\}\,.$$

Then we have

$$\int_\Omega e^{-d|x-z||\lambda|^{1/2m}}dz$$
$$=\int_{\Omega_{1,n}}+\int_{\Omega_{2,n}}=I_1+I_2\,.$$

Introducing polar coordinates, we have for any positive integer N

$$I_1\leqq C\int_{|x-z|>|x|^{1/2}}e^{-d|\lambda|^{1/2m}|x-z|}dz$$

$$
\begin{aligned}
&= C \int_{r>|x|^{1/2}} e^{-d|\lambda|^{1/2m} r} r^{n-1} dr \\
&\leqq C_N \int_{r>|x|^{1/2}} (|\lambda|^{1/2m} r)^{-N} r^{n-1} dr \\
&\leqq C_N |\lambda|^{-N/2m} |x|^{(n-N)/2} . \qquad (3.4)
\end{aligned}
$$

We set $\omega_r = \text{meas}(\Omega \cap \{z: |z|=r\})$. From (P_τ) we have that $\omega_r \geqq C(1+r)^{-\tau}$. Hence, introducting polar coordinates, we have

$$
\begin{aligned}
I_2 &\leqq C \int_{|x|-|x|^{1/2}}^{|x|+|x|^{1/2}} e^{-d|\lambda|^{1/2m} |r-|x||} \omega_r dr \\
&\leqq C(1+|x|)^{-\tau} |\lambda|^{-1/2m} \int_{|\lambda|^{1/2m}(|x|-|x|^{1/2})}^{|\lambda|^{1/2m}(|x|+|x|^{1/2})} e^{-d|r-|\lambda|^{1/2m}|x||} dr \\
&\leqq C|\lambda|^{-1/2m}(1+|x|)^{-\tau} . \qquad (3.5)
\end{aligned}
$$

From (3.4) and (3.5) we have

$$
\int_\Omega e^{-d|\lambda|^{1/2m}|x-z|} dz \leqq C|\lambda|^{-1/2m}(1+|x|)^{-\tau} . \qquad (3.6)
$$

Hence, from (3.3) we have (3.2) for $k=1$.

Assuming now that (3.2) holds for k, we shall prove it for $k+1$. From (3.1) and the induction assumption, we have

$$
\begin{aligned}
|K_\lambda^{(k+1)}(x, y)| \leqq C|\lambda|^{2n/2m+(n-1)k/2m-k-2}(1+|x|)^{-\tau/2} \\
\times \int_\Omega (1+|z|)^{-\tau k/2} e^{-d|\lambda|^{1/2m}|z-y|} dz .
\end{aligned}
$$

By the same way as the proof of (3.6), we have

$$
\int_\Omega (1+|z|)^{-\tau k/2} e^{-d|\lambda|^{1/2m}|z-y|} dz \leqq C(1+|y|)^{-\tau-\tau k/2} |\lambda|^{-1/2m} .
$$

Hence we have

$$
\begin{aligned}
|K_\lambda^{(k+1)}(x, y)| \leqq C|\lambda|^{n/2m+(n-1)(k+1)/2m-k-2}(1+|x|^{-\tau k/2} \\
\times(1+|y|^{-\tau-\tau k/2}) . \qquad (3.7)
\end{aligned}
$$

Using $K_\lambda^{(k+1)}(x, y) = \int_\Omega K_\lambda(x, z) K_\lambda^{(k)}(z, y)\, dz$, analogously we get

$$
\begin{aligned}
|K_\lambda^{(k+1)}(x, y)| \leqq C|\lambda|^{n/2m+(n-1)(k+1)/2m-k-2}(1+|x|)^{-\tau-\tau k/2} \\
\times(1+|y|)^{-\tau k/2} . \qquad (3.8)
\end{aligned}
$$

From (3.7) and (3.8) we have (3.2) for $k+1$. This completes the induction and establishes (3.2).

Let $\{E_s\}$ be the spectral resolution of A: $A = \int_0^\infty t\, dE_t$ and $e(x, y; t)$ be the spectral function of A, that is, the kernel of E_t. It is well known that

$$\int_0^\infty (t-\lambda)^{-k-1}\, dE_t = (A-\lambda)^{-k-1}, \tag{3.9}$$

$$\int_0^\infty (t-\lambda)^{-k-1}\, de(x, y;\ t) = K_\lambda^{(k)}(x, y)\,. \tag{3.10}$$

Lemma 3.4. *For any non negative integer k there exists a constant C_k such that for any $x\in\Omega$*

$$|e(x, x;\ t)| \leqq C_k\, t^{n/2m+(n-1)k/2m}(1+|x|)^{-\tau k}\,. \tag{3.11}$$

Proof. Since $de(x, x;\, t)$ is a positive measure, we have

$$\begin{aligned}\int_0^t (s+t)^{-k-2} de(x, x;\, s) &\leqq \int_0^\infty (s+t)^{-k-1} de(x, x;\, s)\\ &= K_{-t}^{(k)}(x, x)\,. \end{aligned}\tag{3.12}$$

Noting that $e(x, x;\, t)\leqq C\, t^{k+1}\int_0^t (s+t)^{-k-1} de(x, x;\, s)$, from (3.1), (3.2) and (3.12) we have the present lemma.

REMARK. Noting that $e(x, x;\, t)=\sum_{\lambda_j\leqq t}|\phi_j(x)|^2$, from (3.11) we get (1.4).

Proof of Theorem 1. We set $\Omega_1=\Omega\cap\{x\colon |x|<t^{(n-1)/2m\tau}\}$, $\Omega_2=\Omega\cap\{x\colon |x| >t^{(n-1)/2m\tau}\}$. Then we have for $t>2$

$$\begin{aligned}\int_{\Omega_1} e(x, x;\ t)\, dx &\leqq Ct^{n/2m}\int_{\Omega_1} dx\\ &\leqq Ct^{n/2m}\int_0^{t^{(n-1)/2m\tau}} (1+r)^{-\tau} dr\\ &\leqq \begin{cases} C\ t^{n/2m+(n-1)(1-\tau)/2m\tau} & \text{if } 0<\tau<1\,,\\ C\ t^{n/2m}\log t & \text{if } \tau=1\,.\end{cases}\end{aligned}$$

Using (3.11), we have for $k>1/\tau-1$

$$\begin{aligned}\int_{\Omega_2} e(x, x;\ t)\, dx &\leqq C_k\ t^{n/2m+(n-1)k/2m}\int_{\Omega_2} (1+|x|)^{-\tau k} dx\\ &\leqq C_i\ t^{n/2m+(n-1)k/2m}\int_{t^{(n-1)/2m\tau}}^\infty (1+r)^{-\tau-\tau k} dr\\ &\leqq C_k\ t^{n/2m+(n-1)(1-\tau)/2m\tau}\,.\end{aligned}$$

Hence, noting that $N(t)=\int_\Omega e(x, x;\ t)\, dx$, we get Theorem 1.

4. Proof of theorem 2

From the assumption (Q) and Lemma 3.2 we see that A satisfies the assumption of the main theorem of [9]. Hence, we have

$$|e(x, x; t)-a(x)t^{n/2m}| \leqq C\delta(x)^{-1}t^{(n-1)/2m}. \tag{4.1}$$

We note that

$$\begin{aligned} &|N(t)-t^{n/2m}\int_{\Omega_t} a(x)dx| \\ \leqq &\int_{\Omega_1} |e(x,x; t)-a(x)t^{n/2m}|dx+\int_{\Omega_2} |e(x, x; t)|dx \\ = &I_1+I_2. \end{aligned}$$

From the proof of Theorem 1 we have

$$I_2 \leqq C\, t^{n/2m}. \tag{4.2}$$

We set for sufficiently small ε $\Omega^{\varepsilon}_{1,1}=\Omega_1 \cap \{x\in\Omega: \delta(x)<\varepsilon|x|^{1/(1-n)}\}$, $\Omega^{\varepsilon}_{1,2}=\Omega_1 \cap \{x\in\Omega: \delta(x)>\varepsilon|x|^{1/(1-n)}\}$. Then we have

$$I_1 = \int_{\Omega^{\varepsilon}_{1,1}} + \int_{\Omega^{\varepsilon}_{1,2}} = I_{1,1}(\varepsilon, t)+I_{1,2}(\varepsilon, t).$$

From the assumption (R)–(ii) we have

$$\lim_{\varepsilon\to 0} \lim_{t\to\infty} I_{1,1}(\varepsilon, t)\, (t^{n/2m} \log t)^{-1} = 0. \tag{4.3}$$

Moreover from (4.1) and the assumption (R)–(iii) we have

$$I_{1,2}(\varepsilon, t) \leqq C_{\varepsilon}\, t^{n/2m}. \tag{4.4}$$

From the assumption (R)–(i) we see that there exists a constant C such that for $t>2$

$$t^{n/2m}\int_{\Omega_t} a(x)\, dx \geqq C\, t^{n/2m} \log t. \tag{4.5}$$

Hence, from (4.2), (4.3), (4.4) and (4.5) we get Theorem 2.

Bibliography

[1] F. Asakura: *The asymptotic distribution of eigenvalues for the Laplacian in semi-infinite domains*, J. Math. Kyoto Univ. **19** (1979), 583–599.

[2] Jean-Marie Audrin & Pham The Lai: *Comportement asymptotique des valeurs propres d'un probleme de Dirichlet elliptique autoadjoint sur un ouvert non borne,* C.R. Acad. Sci. Paris **276** (1973), Serie A 1197–1200.

[3] M.S. Beryeı & M. Schecter: *Embedding theorems and quasi-linear elliptic boundary value problems for unbounded domains,* Trans. Amer. Math. Soc. **172** (1972), 261–278.

[4] J. Fleckinger: *Asymptotic distribution of eigenvalues of elliptic operators on unbounded domains,* Ordinary and Partial Differential Equations, Proceedings,

Dundee, Scotland, (1980).

[5] D. Hewgill: *On the eigenvalues of a second order elliptic operator in an unbounded domain,* Pacific J. Math. **51** (1974), 467–476.

[6] D. Hewgill: *On the growth of the eigenvalues of an elliptic operator in a quasi-bounded domain,* Arch. Rational Mech. Anal. **56** (1974), 367–371.

[7] K. Maruo & H. Tanabe: *On the asymptotic distribution of eigenvalues of operators associated with strongly elliptic sesqui-linear forms,* Osaka J. Math. **8** (1971), 323–345.

[8] H. Tamura: *The asymptotic distribution of eigenvalues of the Laplace operator in an unbounded domain,* Nagoya Math. J. **60** (1976), 7–33.

[9] J. Tsujimoto: *On the asymptotic behavior of spectral functions of elliptic operators,* Japan. J. Math. **8** (1982), 177–210.

[10] J. Tsujimoto: *On the remainder estimates of asymptotic formulas for eigenvalues of operators associated with strongly elliptic sesquilinear forms,* J. Math. Soc. Japan **33** (1981), 557–569.

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