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ON HYPERSURFACES OF A COMPLEX GRASSMANN MANIFOLD $G_{m+n,n}(C)$

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On Kähler submanifolds of a complex projective space, J-I Hano [5] has studied complete intersections of hypersurfaces in a complex projective space and proved that if a complete intersection M of hypersurfaces is an Einstein manifold with respect to the induced metric then M is a complex projective space or a complex quadric. The purpose of this note is to investigate hypersurfaces of a complex Grassmann manifold by using Hano's method. Let $G_{m+n,n}(\boldsymbol{C})$ denote the complex Grassmann manifold of n-planes in $\boldsymbol{C}^{m+n}$. Let X be a compact complex hypersurface of $G_{m+n,n}(\boldsymbol{C})$. Then X defines a positive divisor on $G_{m+n,n}(\boldsymbol{C})$ and hence a holomorphic line bundle $\{X\}$ on $G_{m+n,n}(\boldsymbol{C})$. We denote by $c(X)$ the Chern class of the line bundle $\{X\}$. Since the second cohomology group $H^2(G_{m+n,n}(\boldsymbol{C}), \boldsymbol{Z})$ is isomorphic to $\boldsymbol{Z}$, we can write $c(X)=a(X)\cdot\sigma$, where $a(X)\in\boldsymbol{N}$ and σ is a generator of $H^2(G_{m+n,n}(\boldsymbol{C}), \boldsymbol{Z})$. We call $a(X)$ *the degree of* X. We equip an hermitian inner product on $\boldsymbol{C}^{m+n}$. The complex Grassmann manifold $G_{m+n,n}(\boldsymbol{C})$ has a Kähler metric invariant under the action of the unitary group $U(m+n)$. Moreover we may assume that $m\geqq n$. Under these notations, we have a following Theorem.

Theorem. *Let X be a compact complex hypersurface of a complex Grassmann manifold $G_{m+n,n}(\boldsymbol{C})$ and $a(X)$ the degree of X. If $a(X)\geqq r+2$, where $r=\binom{m+n}{n}-mn-1$ and $n\geqq 2$, X is not an Einstein manifold with respect to the induced metric.*

1. Preliminaries

Let $G_{m+n,n}(\boldsymbol{C})$ be the complex Grassmann manifold of n-planes in $\boldsymbol{C}^{m+n}$. An element of $G_{m+n,n}(\boldsymbol{C})$ can be given by a non-zero decomposable n-vector $\Lambda=X_1\wedge\cdots\wedge X_n\neq 0$ defined up to a constant factor. If $(e_1, \cdots, e_{m+n})$ denotes a fixed frame in $\boldsymbol{C}^{m+n}$, we can write

$$\Lambda=\sum_{\iota} p_{i_1\cdots i_n}e_{i_1}\wedge\cdots\wedge e_{i_n}\quad(1\leqq i_1, \cdots, i_n\leqq m+n) \tag{1.1}$$

where the $p_{i_1\cdots i_n}$'s are skew-symmetric in their indices. The $p_{i_1\cdots i_k}$ are called the

Plücker coordinates in $G_{m+n,n}(\boldsymbol{C})$. By considering $p_{i_1\cdots i_n}$ as the homogeneous coordinates of the complex projective space $P^\mu(\boldsymbol{C})$ of dimension $\mu=\binom{m+n}{n}-1$, we get an imbedding $j\colon G_{m+n,n}(\boldsymbol{C})\to P^\mu(\boldsymbol{C})$.

We equip an hermitian inner product in $\boldsymbol{C}^{m+n}$. Then we can define a Kähler metric on $G_{m+n,n}(\boldsymbol{C})$ which is invariant under the action of the unitary group $U(m+n)$. We also have the Fubini-Study metric on the complex projective spaec $P^\mu(\boldsymbol{C})$ induced from the hermitian inner product in the n-th exterior product $\Lambda^n\boldsymbol{C}^{m+n}$ of $\boldsymbol{C}^{m+n}$. Then the imbedding j is isometric with respect to these Kähler metrics (cf. for example [3] §8).

From now on we identify $G_{m+n,n}(\boldsymbol{C})$ with the image of the imbedding j. Let $I(V)$ denote the ideal associated to a subvariety V of $P^\mu(\boldsymbol{C})$. We recall the generators of the ideal $I(G_{m+n,n}(\boldsymbol{C}))$. Let $i_1, \cdots, i_{n-1}$ be $n-1$ distinct numbers which are chosen from a set $\{1, \cdots, m+n\}$ and let $j_0, \cdots, j_n$ be $n+1$ distinct numbers chosen from the same set. We define homogeneous polynomials $Q(i_1\cdots i_{n-1}j_0\cdots j_n)$ of degree 2 on $\boldsymbol{C}^{\mu+1}$ by

$$Q(i_1\cdots i_{n-1}j_0\cdots j_n)=\sum_{\lambda=0}^{n}(-1)^\lambda p_{i_1\cdots i_{n-1}j_\lambda}p_{j_0\cdots\hat{j}_\lambda\cdots j_n}\,. \tag{1.2}$$

Then it is known that $Q(i_1\cdots i_{n-1}\cdots j_0\cdots j_n)=0$ are the generators of the ideal $I(G_{m+n,n}(\boldsymbol{C}))$ (See [7] Chapter 7 §6 Theorem 2 and §7 Theorem 1). The relations $Q(i_1\cdots i_{n-1}j_0\cdots j_n)=0$ are called the quadratic p-relations.

Let π denote the canonical projection of $\boldsymbol{C}^{\mu+1}-(0)$ onto the complex projective space $P^\mu(\boldsymbol{C})$. The triple $(\boldsymbol{C}^{\mu+1}-(0), \pi, P^\mu(\boldsymbol{C}))$ is a principal $\boldsymbol{C}^*$-bundle over $P^\mu(\boldsymbol{C})$. Let E be the standard line bundle over $P^\mu(\boldsymbol{C})$ associated to the above principal bundle. We denote by $H^1(M, \theta^*)$ the group of all equivalent classes of holomorphic line bundles over a compact complex manifold M. On the line bundles over a Grassmann manifold $G_{m+n,n}(\boldsymbol{C})$, the following propositions are known.

Proposition 1.1. *Let H denote the dual bundle of E over $P^\mu(\boldsymbol{C})$. Then, for any integer $k>0$, the inclusion map $j\colon G_{m+n,n}(\boldsymbol{C})\to P^\mu(\boldsymbol{C})$ induces the surjective map $j^*\colon H^0(P^\mu(\boldsymbol{C}), H^k)\to H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k)$, that is, every holomorphic section of the line bundle j^*H^k is given by the restriction of a section of the line bundle H^k on $P^\mu(\boldsymbol{C})$.*

Proposition 1.2. *The inclusion map $j\colon G_{m+n,n}(\boldsymbol{C})\to P^\mu(\boldsymbol{C})$ induces the canonical isomorphism $j^*\colon H^1(P^\mu(\boldsymbol{C}), \theta^*)\to H^1(G_{m+n,n}(\boldsymbol{C}), \theta^*)$. Moreover each positive divisor X of $G_{m+n,n}(\boldsymbol{C})$ is the complete intersection of $G_{m+n,n}(\boldsymbol{C})$ and a subvariety Y of codimension 1 of $P^\mu(\boldsymbol{C})$. Furthermore, for an irreducible subvariety X of codimension 1 in $G_{m+n,n}(\boldsymbol{C})$, $I(X)=I(G_{m+n,n}(\boldsymbol{C}))+(F)$ where F is an irreducible homogeneous polynomial on $\boldsymbol{C}^{\mu+1}$.*

Proof. See [7] chapter 14 §8 Theorem 1 and [8] Theorem 3.

For a compact connected complex submanifold X of codimension 1 in $G_{m+n,n}(\boldsymbol{C})$, let $[X]$ denote the positive divisor defined by X and $c(X)$ the Chern class of the line bundle $\{X\}$ defined by $[X]$. Since $H^2(G_{m+n,n}(\boldsymbol{C}), \boldsymbol{Z}) \cong \boldsymbol{Z}$, $c(X)=a(X)\sigma$ where $a(X) \in \boldsymbol{N}$ and σ is a generator of $H^2(G_{m+n,n}(\boldsymbol{C}), \boldsymbol{Z})$. We call $a(X)$ *the degree of* X. Note that the degree of an irreducible subvariety Y of codimension 1 of $P^{\mu}(\boldsymbol{C})$ corresponding to X is given by $a(X)$.

2. The canonical line bundle

With respect to the hermitian inner product on $\boldsymbol{C}^{\mu+1}$ induced from the hermitian inner product on $\boldsymbol{C}^{m+n}$, the square of the norm $||z||$ is given by $\sum_{i_1<\cdots<i_n} |p_{i_1\cdots i_n}(z)|^2$ for an orthonormal frame $(e_1, \cdots, e_{m+n})$ of $\boldsymbol{C}^{m+n}$. The function $||z||^2$ can be regarded as a hermitian fiber metric on the standard line bundle E on $P^{\mu}(\boldsymbol{C})$. A unique connection of type (1, 0) on E is determined by the fiber metric $||z||^2$ on E and gives rise to the curvature form $-\Omega$ on $P^{\mu}(\boldsymbol{C})$. The form Ω is the associated (1, 1)-form of the Fubini-Study metric on $P^{\mu}(\boldsymbol{C})$; $\pi^*\Omega= \frac{\sqrt{-1}}{2\pi} d'd'' \log ||z||^2$.

Let K, $K(G_{m+n,n}(\boldsymbol{C}))$ and $K(X)$ be the canonical line bundle of $P^{\mu}(\boldsymbol{C})$, $G_{m+n,n}(\boldsymbol{C})$ and X respectively. The normal bundle of X in $P^{\mu}(\boldsymbol{C})$ is a holomorphic vector bundle over X whose fiber dimension is $r+1=\mu-mn+1$. We denote by N the $(r+1)$-th exterior product of the dual bundle of the normal bundle of X in $P^{\mu}(\boldsymbol{C})$. Denoting by ι the inclusion $X \subsetneq P^{\mu}(\boldsymbol{C})$, we have

$$\iota^* K = K(X) \cdot N . \tag{2.1}$$

Let $U_{i_1\cdots i_n}$ denote an open subset of $P^{\mu}(\boldsymbol{C})$ given by $\{\pi(z) \in P^{\mu}(\boldsymbol{C}) \,|\, p_{i_1\cdots i_n}(z) \neq 0\}$. The functions $u_{i_1\cdots i_n,\beta_1\cdots\beta_n} = p_{\beta_1\cdots\beta_n}/p_{i_1\cdots i_n}((\beta_1, \cdots, \beta_n) \neq (i_1, \cdots, i_n),\ \beta_1 < \cdots < \beta_n)$ form a holomorphic coordinates system on $U_{i_1\cdots i_n}$. We arrange the Plücker coordinates in the lexicographical order. Let $p_{j_1\cdots j_n}$ be the $\sigma(j_1, \cdots, j_n)$-th component of the Plücker coordinates in above order. The map $s_{i_1\cdots i_n}\colon U_{i_1\cdots i_n} \to \boldsymbol{C}^{\mu+1}-(0)$ defined by

$$s_{i_1\cdots i_n}(y) = (u_{i_1\cdots i_n,12\cdots n}(y), \cdots, \underset{\substack{\vee \\ \sigma(i_1\cdots i_n)}}{1}, \cdots, u_{i_1\cdots i_n,m+1\cdots m+n}(y)) \quad (y \in U_{i_1\cdots i_n})$$

is a holomorphic section on $U_{i_1\cdots i_n}$ of the principal $\boldsymbol{C}^*$-bundle $(\boldsymbol{C}^{\mu+1}-(0), \pi, P^{\mu}(\boldsymbol{C}))$. We put

$$g_{i_1\cdots i_n, j_1\cdots j_n} = p_{i_1\cdots i_n}/p_{j_1\cdots j_n}$$

on $U_{i_1\cdots i_n} \cap U_{j_1\cdots j_n}$. Then $(g_{i_1\cdots i_n, j_1\cdots j_n})$ is the system of transition functions of the principal bundle associated to the holomorphic local trivialization $(U_{i_1\cdots i_n}, s_{i_1\cdots i_n})$ of the bundle. Let $V_{i_1\cdots i_n}$ denote the connected open set of $G_{m+n,n}(\boldsymbol{C})$ given by

$$V_{i_1\cdots i_n} = U_{i_1\cdots i_n} \cap G_{m+n,n}(\boldsymbol{C})$$

and $W_{i_1\cdots i_n}$ the open set of X given by

$$W_{i_1\cdots i_n} = U_{i_1\cdots i_n} \cap X .$$

Now we shall consider the structure of the holomorphic line bundle N on X. Let $Q(\beta_1\cdots\beta_n i_1\cdots i_n)$ be a homogeneous polynomial of degree 2 on $\boldsymbol{C}^{\mu+1}$ defined by (1.2). It is obvious that $Q(\beta_1\cdots\beta_n i_1\cdots i_n)$ has following properties:

$$(2.2)\quad \begin{cases} 1) & Q(\beta_1\cdots\beta_n i_1\cdots i_n) \textit{ is alternating with respect to } \beta_1, \cdots, \beta_{n-1} . \\ 2) & Q(\beta_1\cdots\beta_n i_1\cdots i_n) \textit{ is alternating with respect to } \beta_n, i_1, \cdots, i_n . \\ 3) & \textit{if } \{\beta_1, \cdots, \beta_{n-1}\} \subset \{\beta_n, i_1, \cdots, i_n\},\ Q(\beta_1\cdots\beta_n i_1\cdots i_n) \equiv 0 . \end{cases}$$

Furthermore we have a following lemma which gives the relations between these polynomials.

Lemma 2.1. *On* $\pi^{-1}(U_{i_1\cdots i_n})$,

(a)
$$\begin{aligned} Q(\beta_1\cdots\beta_{n-1}ki_1\cdots i_n) &= -Q(\beta_1\cdots\beta_{n-2}k\beta_{n-1}i_1\cdots i_n) \\ &\quad + \sum_{a=1}^{n}(-1)^{a+n-1}\frac{p_{\beta_{n-1}i_1\cdots\hat{i}_a\cdots i_n}}{p_{i_1\cdots i_n}}Q(i_a\beta_1\cdots\beta_{n-2}ki_1\cdots i_n) \\ &\quad + \sum_{b=1}^{n}(-1)^{b+n-1}\frac{p_{ki_1\cdots\hat{i}_b\cdots i_n}}{p_{i_1\cdots i_n}}Q(i_b\beta_1\cdots\beta_{n-1}i_1\cdots i_n) \end{aligned}$$

(b)
$$\begin{aligned} &Q(\beta_1\cdots\beta_n i_1\cdots i_{j-1}i_{j+1}\cdots i_n k) \\ &= \frac{p_{i_1\cdots\hat{i}_j\cdots i_n k}}{p_{i_1\cdots i_n}}Q(\beta_1\cdots\beta_n i_1\cdots i_n) \\ &\quad + \sum_{a\neq j}(-1)^a\frac{p_{\beta_1\cdots\beta_{n-1}i_a}}{p_{i_1\cdots i_n}}Q(\beta_n i_1\cdots\hat{i}_j\cdots\hat{i}_a\cdots i_n k i_1\cdots i_n) \\ &\quad + (-1)^n\frac{p_{\beta_n i_1\cdots\hat{i}_j\cdots i_n}}{p_{i_1\cdots i_n}}Q(\beta_1\cdots\beta_{n-1}ki_1\cdots i_n) . \end{aligned}$$

Proof. Straightforward computation.

Let $(i_1, \cdots, i_n)$ be an n-tuples such that

$$1 \leqq i_1 < i_2 < \cdots < i_n \leqq m+n$$

and let $(i_1, \cdots, i_n, s_1, \cdots, s_m)$ be the permutation of $(1, \cdots, m+n)$ such that

$$1 \leqq s_1 < \cdots < s_m \leqq m+n .$$

For a permutation $(l_1, \cdots, l_m)$ of $(1, \cdots, m)$, we introduce a linear order $\prec$ on $\{1, \cdots, m+n\}$ by $i_1 \prec i_2 \prec \cdots \prec i_n \prec s_{l_1} \prec \cdots \prec s_{l_m}$. We denote $\{\beta = (\beta_1, \cdots, \beta_n) \mid \beta_1 \prec$

$\cdots \prec \beta_n\}$ by $C(i_1 \cdots i_n, \prec)$. The associated lexicographical order on $C(i_1 \cdots i_n, \prec)$ is called an admissible order with respect to $(i_1, \cdots, i_n)$. If the linear order $\prec$ on $\{1, \cdots, m+n\}$ is given by $i_1 \prec \cdots \prec i_n \prec s_1 \prec \cdots \prec s_m$, the admissible order is called principal with respect to $\{i_1, \cdots, i_n\}$. For an admissible order with respect to $(i_1, \cdots, i_n)$, we define a subset $I(i_1 \cdots i_n, \prec)$ of $C(i_1 \cdots i_n, \prec)$ by

$$\left\{\beta = (\beta_1, \cdots, \beta_n) \,\middle|\, \begin{array}{l} \beta = (i_1, \cdots, \hat{i}_l, \cdots, i_n, s_t), \quad l = 1, \cdots, n; \\ t = 1, \cdots, m, \quad \text{or} \quad \beta = (i_1, \cdots, i_n) \end{array} \right\}.$$

Note that $I(i_1 \cdots, i_n, \prec) = I(i_1, \cdots, i_n, \prec')$ for $\prec$, $\prec'$ admissible orders, with respect to $(i_1, \cdots, i_n)$ and the number of elements in $I(i_1 \cdots i_n, \prec)$ is $mn+1$. Moreover $Q(\beta i_1 \cdots i_n) \equiv 0$ for $\beta \in I(i_1 \cdots i_n, \prec)$ by (2.2) 3).

For an admissible order $\prec$ with respect to $(i_1, \cdots, i_n)$, we define a holomorphic r-form $\tilde{q}^{\prec}_{i_1 \cdots i_n}$ on $\boldsymbol{C}^{\mu+1}$ by

$$\tilde{q}^{\prec}_{i_1 \cdots i_n} = \bigwedge_{\beta \in C(i_1 \cdots i_n, \prec) - I(i_1 \cdots i_n, \prec)} dQ(\beta i_1 \cdots i_n) \tag{2.3}$$

where we take the exterior product of $dQ(\beta i_1 \cdots i_n)$ according to the admissible order $\prec$ on $C(i_1 \cdots i_n, \prec) - I(i_1 \cdots i_n, \prec)$. If the admissible order $\prec$ is principal, we denote $\tilde{q}^{\prec}_{i_1 \cdots i_n}$ by $\tilde{q}_{i_1 \cdots i_n}$.

Lemma 2.2. *Let $\prec$, $\prec'$ be admissible orders with respect to $(i_1, \cdots, i_n)$. Then we have*

$$\tilde{q}^{\prec}_{i_1 \cdots i_n}(z) = \varepsilon(\prec, \prec') \tilde{q}^{\prec'}_{i_1 \cdots i_n}(z) \tag{2.4}$$

for $z \in \pi^{-1}(V_{i_1 \cdots i_n})$, where $\varepsilon(\prec, \prec') \in \{\pm 1\}$.

Proof. Let $\prec$ be a linear order on $\{1, \cdots, m+n\}$ given by $i_1 \prec \cdots \prec i_n \prec s_{l_1} \cdots \prec s_{l_m}$. Since the symmetric group of m elements is generated by transpositions $\{(k, k+1) \mid k = 1, \cdots, m-1\}$, we may assume that the admissible order $\prec'$ is given by a linear order

$$i_1 \prec' \cdots \prec' i_n \prec' s_{l_1} \prec' \cdots \prec' s_{l_{k-1}} \prec' s_{l_{k+1}} \prec' s_{l_k} \prec' s_{l_{k+2}} \prec' \cdots \prec' s_{l_m}.$$

Let β be an element of $C(i_1 \cdots i_n, \prec') - I(i_1 \cdots i_n, \prec')$. Then β is of the form either

1) $\beta = (\beta_1, \cdots_1 \beta_n)$, $\beta_t \neq s_{l_k}, s_{l_{k+1}}$ for any $t = 1, \cdots, n$,
2) $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_t = s_{l_k}$ for some t and $\beta_a \neq s_{l_{k+1}}$ for $a \neq t$,
3) $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_t = s_{l_{k+1}}$ for some t and $\beta_a \neq s_{l_k}$ for $a \neq t$,
4) $\beta = (\beta_1, \cdots, \beta_n)$, $\beta_t = s_{l_{k+1}}$, $\beta_{t+1} = s_{l_k}$ for some $t+1 < n$,

or 5) $\beta = (\beta_1, \cdots \beta_{n-2}, s_{l_{k+1}}, s_{l_k})$.

In the cases of 1), 2) and 3), $\beta \in C(i_1 \cdots i_n, \prec) - I(i_1 \cdots i_n, \prec)$. In the case of 4), $Q(\beta i_1 \cdots i_n) = Q(\beta_1 \cdots \beta_{t-1} s_{l_{k+}} . s_{l_k} \beta_{t+2} \cdots \beta_n i_1 \cdots i_n) = -Q(\beta_1 \cdots \beta_{t-1} s_{l_k} s_{l_{k+1}} \beta_{t+2} \cdots \beta_n i_1 \cdots$

i_n) by (2.2) 1). Note that $(\beta_1\cdots\beta_{t-1}s_{l_k}s_{l_{k+1}}\beta_{t+2}\cdots\beta_n)\in C(i_1\cdots i_n, \multimap)-I(i_1\cdots i_n, \multimap)$. In the case of 5), we have

$$\begin{aligned}
Q(\beta_1\cdots\beta_{n-2}s_{l_{k+1}}s_{l_k}i_1\cdots i_n) &= -Q(\beta_1\cdots\beta_{n-2}s_{l_k}s_{l_{k+1}}i_1\cdots i_n)\\
&+\sum_{a=1}^{n}(-1)^{a+n-1}\frac{p_{s(l(k+1))i_1\cdots\hat{i}_a\cdots i_n}}{p_{i_1\cdots i_n}}Q(i_a\beta_1\cdots\beta_{n-2}s_{l_k}i_1\cdots i_n)\\
&+\sum_{b=1}^{n}(-1)^{b+n-1}\frac{p_{s(l(k))i_1\cdots\hat{i}_b\cdots i_n}}{p_{i_1\cdots i_n}}Q(i_b\beta_1\cdots\beta_{n-2}s_{l_{n+1}}i_1\cdots i_n)
\end{aligned}$$

by Lemma 2.1 (a). Note that $(\beta_1, \cdots, \beta_{n-2}, s_{l_k}, s_{l_{k+1}})\in C(i_1\cdots, i_n, \cdots)-I(i_1\cdots i_n, \multimap)$, $i_a\multimap s_{l_k}$ and $i_b\multimap s_{l_{k+1}}$. By (2.2) 1), $Q(i_a\beta_1\cdots\beta_{n-2}s_{l_k}i_1\cdots i_n)=Q(\beta_1'\cdots\beta_{n-1}'s_{l_k}i_1\cdots i_n)$ where $\beta_1', \cdots, \beta_{n-1}'$ is a permutation of $i_a, \beta_1, \cdots, \beta_{n-2}$ such that $\beta_1'\multimap\cdots\multimap\beta_{n-1}'\multimap s_{l_k}$. If $Q(i_a\beta_1\cdots\beta_{n-2}s_{l_k}i_1\cdots i_n)\not\equiv 0$, then $(\beta_1', \cdots, \beta_{n-1}'s_{l_k})\in C(i_1, \cdots, i_n, \cdots)-I(i_1\cdots i_n, \multimap)$ and $\beta=(\beta_1', \cdots, \beta_{n-1}', s_{l_k})$ is of the form of the case 2). Similarly, $Q(i_b\beta_1\cdots\beta_{n-2}s_{l_{k+1}}i_1\cdots i_n)=\pm Q(\beta_1'\cdots\beta_{n-1}'s_{l_{k+1}}i_1\cdots i_n)$ where $\beta_1', \cdots, \beta_{n-1}'$ is a permutation of $i_b, \beta_1, \cdots, \beta_{n-2}$ such that $\beta_1'\multimap\cdots\multimap\beta_{n-1}'\multimap s_{l_{k+1}}$. If $Q(i_b\beta_1\cdots\beta_{n-2}s_{l_{k+1}}i_1\cdots i_n)\not\equiv 0$, then $(\beta_1', \cdots, \beta_{n-1}', s_{l_{k+1}})\in C(i_1\cdots i_n, \multimap)-I(i_1\cdots i_n, \multimap)$ and $\beta=(\beta_1', \cdots, \beta_{n-1}', s_{l_{k+1}})$ is of the form of the case 3). Now we get our claim by taking differential.

q.e.d.

Let $(i_1\cdots\hat{i}_j\cdots i_n i_j s_1\cdots s_m)$ be a permutation of $(1\cdots m+n)$. We define a linear order $\lhd$ on $\{1, \cdots, m+n\}$ by $i_1\lhd\cdots\lhd\hat{i}_j\lhd\cdots\lhd i_n\lhd i_j\lhd s_1\lhd\cdots\lhd s_m$. We define a set $C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)$ by $\{\beta=(\beta_1\cdots\beta_n)\,|\,\beta_1\lhd\cdots\lhd\beta_n\}$ and a subset $I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)$ of $C(i_1\cdots i_k\cdots i_n i_j, \lhd)$ by

$$\left\{\beta\in C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)\;\middle|\;\begin{array}{l}\beta=(i_1\cdots\hat{i}_j\cdots i_l\cdots i_n i_j s_t)\\ \text{or }\ \beta=(i_1\cdots\hat{i}_j\cdots i_n s_t)\\ \quad t=1, \cdots, m;\ l=1, \cdots, \hat{j}, \cdots, n\\ \text{or }\ \beta=(i_1\cdots\hat{i}_j\cdots i_n i_j)\end{array}\right\}.$$

Lemma 2.3. $\displaystyle\bigwedge_{\beta\in C(i_1\cdots i_n, \lhd)-I(i_1\cdots i_n, \lhd)} dQ(\beta i_1\cdots i_n)=\varepsilon(i_1\cdots i_n, i_1\cdots\hat{i}_j\cdots i_n i_j)$ $\displaystyle\times\bigwedge_{\gamma\in C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)-I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)} dQ(\gamma i_1\cdots\hat{i}_j\cdots i_n i_j)$ *on* $\pi^{-1}(V_{i_1\cdots i_n})$, *where* $\varepsilon(i_1\cdots i_n, i_1\cdots\hat{i}_j\cdots i_n i_j)\in\{\pm 1\}$ *and the exterior product is taken according to the lexicographical order induced from the linear order* $\lhd$.

Proof. Note that there is a natural bijection between $C(i_1\cdots i_n, \multimap)-I(i_1\cdots i_n, \multimap)$ and $C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)-I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)$. We denote this map by

$$f\colon\ C(i_1\cdots i_n, \multimap)-I(i_1\cdots i_n, \multimap)\rightarrow C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)-I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)\,.$$

Then, for $\beta\in C(i_1\cdots i_n, \multimap)-I(i_1\cdots i_n, \multimap)$, $Q(\beta i_1\cdots i_n)$ and $Q(f(\beta)i_1\cdots\hat{i}_j\cdots i_n i_j)$ coincide up to sign by (2.2) 1) and 2).

q.e.d.

Let $(i_1\cdots\hat{i}_j\cdots i_n s_k i_j s_1\cdots\hat{s}_k\cdots s_m)$ be a permutation of $(1, \cdots, m+n)$. We define a linear order $\prec$ on $\{1, \cdots, m+n\}$ by

$$i_1\prec\cdots\prec i_j\cdots\prec i_n\prec s_k\prec i_j\prec s_1\prec\cdots\prec\hat{s}_k\prec\cdots\prec s_m\,.$$

We define a set $C(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec)$ by $\{\beta=(\beta_1\cdots\beta_n)\,|\,\beta_1\prec\cdots\prec\beta_n\}$ and a subset $I(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec)$ by

$$\left\{\beta\in C(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec) = \left|\begin{array}{l} \beta=(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n s_k s_t)\,, \\ \beta=(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n s_k i_j)\,, \\ \beta=(i_1\cdots\hat{i}_j\cdots i_n s_i)\,, \\ \beta=(i_1\cdots\hat{i}_j\cdots i_n i_j) \quad \text{or} \\ \beta=(i_1\cdots\hat{i}_j\cdots i_n s_k) \\ t=1, \cdots, \hat{k}, \cdots m\,, \\ l=1, \cdots, \hat{j}, \cdots, m\,, \end{array}\right.\right\}.$$

Lemma 2.4. *For* $l=1, \cdots, \hat{j}, \cdots, n$, $t=1, \cdots, \hat{k}, \cdots, m$, $Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n i_j s_t i_1\cdots\hat{i}_j\cdots i_n s_k)=(-1)^{j+n}Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n s_t s_k i_1\cdots i_n)=(-1)^{j+n+1}Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n s_k s_t i_1\cdots i_n)$.

Proof. The first part is nothing but Lemmas 2.1 (b). Noting that only three terms of Q are non trivial in our case, we get the second part by the definition. q.e.d.

Now we define a linear order $\lhd'$ on $\{1, \cdots, m+n\}$ by $i_1\lhd'\cdots\lhd'\hat{i}_j\lhd'\cdots\lhd' i_n\lhd' i_j\lhd' s_k\lhd' s_1\lhd'\cdots\lhd'\hat{s}_k\lhd'\cdots\lhd' s_m$.

We define a set $C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd')$ by $\{\beta=(\beta_1\cdots\beta_n)\,|\,\beta_1\lhd'\cdots\lhd'\beta_n\}$ and a subset $I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd')$ of $C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd')$ by $I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd)$. We put

$$V(i_1\cdots\hat{i}_j\cdots i_n i_j, \prec) = C(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec)-I(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec)$$

and

$$V(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd') = C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd')-I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd')\,.$$

Let $\tilde{h}=\{(1, \cdots, m+n\}, \prec)\to\{(1, \cdots, m+n\}, \lhd')$ be an order preserving bijection defined by

$$\begin{cases} \tilde{h}(i)=i \quad \text{for} \quad i\neq i_j, s_k \\ \tilde{h}(i_j)=s_k \\ \tilde{h}(s_k)=i_j\,. \end{cases}$$

Then $\tilde{h}$ induces order preserving bijections

$$h\colon\ C(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec)\to C(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd')$$

and

$$h\colon I(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec) \to I(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd').$$

Hence, we have an order preserving bijection

$$h\colon V(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec) \to V(i_1\cdots\hat{i}_j\cdots i_n i_j, \lhd').$$

Proposition 2.5. *On* $\pi^{-1}(V_{i_1\cdots i_n})$,

$$\bigwedge_{\gamma\in V(i_1\cdots\hat{i}_j\cdots i_n s_k,\prec)} dQ(\beta i_1\cdots\hat{i}_j\cdots i_n s_k)$$

$$=\varepsilon(i_1\cdots\hat{i}_j\cdots i_n s_k,\, i_1\cdots\hat{i}_j\cdots i_n i_j)\left(\frac{p_{i_1\cdots\hat{i}_j\cdots i_n s_k}}{p_{i_1\cdots i_n}}\right)^t \bigwedge_{\gamma\in V(i_1\cdots\hat{i}_j\cdots i_n i_j,\lhd')} dQ(\gamma i_1\cdots\hat{i}_j\cdots i_n i_j)$$

where $\varepsilon(i_1\cdots\hat{i}_j\cdots i_n s_k, i_1\cdots\hat{i}_j\cdots i_n i_j)$ *is constant and valued in* $\{\pm 1\}$, *and* $t=r-(n-1)(m-1)$.

Proof. By Lemma 2.4, we have

$$Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n i_j s_t i_1\cdots\hat{i}_j\cdots i_n s_k) = \pm Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n s_k s_t i_1\cdots\hat{i}_j\cdots i_n i_j)$$

for $l=1,\cdots,\hat{j},\cdots,n$, $t=1,\cdots,\hat{k},\cdots,m$. In other words, for

$$\beta=(i_1\cdots\hat{i}_j\cdots i_l\cdots i_n i_j s_t)\ (l=1,\cdots,\hat{j},\cdots,n;\ t=1,\cdots,\hat{k},\cdots,m)$$
$$Q(\beta i_1\cdots\hat{i}_j\cdots i_n s_k) = \pm Q(h(\beta) i_1\cdots\hat{i}_j\cdots i_n i_j).$$

We put

$$S(i_1\cdots\hat{i}_j\cdots i_n s_k) = \left\{\beta\in V(i_1\cdots\hat{i}_j\cdots i_n s_k, \prec)\ \middle|\ \begin{array}{l}\beta=(i_1\cdots\hat{i}_j\cdots i_l\cdots i_n i_j s_t)\\ l=1,\cdots,\hat{j},\cdots,n;\ t=1,\cdots,\hat{k},\cdots,m\end{array}\right\}$$

and

$$S(i_1\cdots\hat{i}_j\cdots i_n i_j) = \left\{\beta\in V(i_1\cdots\hat{i}_j\cdots i_n s_k, \lhd')\ \middle|\ \begin{array}{l}\beta=(i_1\cdots\hat{i}_j\cdots\hat{i}_l\cdots i_n s_k s_t)\\ l=1,\cdots,j,\cdots,n;\ t=1,\cdots,\hat{k},\cdots,m\end{array}\right\}.$$

Obviously $h(S(i_1\cdots\hat{i}_j\cdots i_n s_k))=S(i_1\cdots\hat{i}_j\cdots i_n i_j)$. Now we claim that on $\pi^{-1}(U_{i_1\cdots i_n})$

$$\text{(2.5)}\qquad Q(\beta i_1\cdots\hat{i}_j\cdots i_n s_k) = \pm\frac{p_{i_1\cdots\hat{i}_j\cdots i_n s_k}}{p_{i_1\cdots i_n}} Q(h(\beta) i_1\cdots\hat{i}_j\cdots i_n i_j)$$
$$+\sum_{\gamma'\underset{\neq}{\lhd} h(\beta)} P_\gamma\left(\cdots\frac{p_{\lambda_1\cdots\lambda_n}}{p_{i_1\cdots i_n}}\cdots\right) Q(\gamma i_1\cdots\hat{i}_j\cdots i_n i_j),$$

where $P_\gamma\left(\cdots\frac{p_{\lambda_1\cdot\lambda_n}}{p_{i_1\cdots i_n}}\cdots\right)$ denotes a polynomial of $\frac{p_{\lambda_1\cdots\lambda_n}}{p_{i_1\cdots i_n}}$, for each

$\beta \in V(i_1 \cdots \hat{i}_j \cdots i_n s_k, \prec) - S(i_1 \cdots \hat{i}_j \cdots i_n s_k)$.

Case 1. $\beta = (i_1 \cdots \hat{i}_j \cdots i_{l-1} \hat{i}_l i_{\alpha_1} \cdots i_{\mu_t} s_k i_j s_{\mu_1} \cdots s_{\mu_q})$ where $l = 1, \cdots, \hat{j}, \cdots, n-1$, $l < \alpha_a$ $(a = 1, \cdots, t)$, $\mu_b \neq k$ $(b = 1, \cdots, q)$.

By Lemma 2.1 (b) and (2.2) 1) 2),

$$
\begin{aligned}
(2.6)\quad & Q(i_1 \cdots \hat{i}_j \cdots i_{l-1} \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_q} i_1 \cdots \hat{i}_j \cdots i_n s_k) \\
&= (-1)^{n-j+1} \frac{p_{i_1 \cdots \hat{i}_j \cdots i_n s_k}}{p_{i_1 \cdots i_n}} Q(i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} i_j s_k s_{\mu_1} \cdots s_{\mu_q} i_1 \cdots \hat{i}_j \cdots i_n i_j) \\
&+ \sum_{a \neq j} (-1)^{a+n-j} \frac{p_{i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-1}} i_a}}{p_{i_1 \cdots i_1}} Q(s_{\mu_q} i_1 \cdots \hat{i}_j \cdots \hat{i}_a \cdots i_n s_k i_1 \cdots \hat{i}_j \cdots i_n i_j) \\
&+ (-1)^j \frac{p_{s_{\mu_q} i_1 \cdots \hat{i}_j \cdots i_n}}{p_{i_1 \cdots i_n}} Q(i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-1}} s_k i_1 \cdots \hat{i}_j \cdots i_n i_j)\,.
\end{aligned}
$$

Note that $p_{i_1 \cdots \hat{i}_j \cdots i_l \cdots i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-1}} i_a} \neq 0$ if and only if $a \geqq l$ and $a \neq \alpha_1, \cdots, \alpha_t$. By (2.2) 1) and Lemma 2.4, we also have

$$Q(s_{\mu_q} i_1 \cdots \hat{i}_j \cdots \hat{i}_a \cdots i_n s_k i_1 \cdots \hat{i}_j \cdots i_n i_j) = \pm Q(i_1 \cdots \hat{i}_j \cdots \hat{i}_a \cdots i_n s_k s_{\mu_q} i_1 \cdots \hat{i}_j \cdots i_n i_j)\,.$$

Put

$$\gamma = (i_1 \cdots \hat{i}_j \cdots \hat{i}_a \cdots i_n s_k s_{\mu_q})\,.$$

Then

$$\gamma \underset{\neq}{\trianglelefteq}' (i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} i_j s_k s_{\mu_1} \cdots s_{\mu_q})$$

for $a \geqq l$.

By Lemma 2.1 (a) and (2.2) 2),

$$
\begin{aligned}
(2.7)\quad & Q(i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-1}} s_k i_1 \cdots \hat{i}_j \cdots i_n i_j) \\
&= -Q(i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-2}} s_k s_{\mu_{q-1}} i_1 \cdots \hat{i}_j \cdots i_n i_j) \\
&+ \sum_a (-1)^{a+n-1} \frac{p_{s_{\mu_{q-1}} i_1 \cdots \hat{i}_a \cdots i_n}}{p_{i_1 \cdots i_n}} Q(i_a i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-2}} s_k i_1 \cdots \hat{i}_j \cdots i_n i_j) \\
&+ \sum_b (-1)^{b+n-1} \frac{p_{s_k i_1 \cdots \hat{i}_b \cdots i_n}}{p_{i_1 \cdots i_n}} Q(i_b i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j s_{\mu_1} \cdots s_{\mu_{q-1}} i_1 \cdots \hat{i}_j \cdots i_n i_j)
\end{aligned}
$$

Note that

$$Q(i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k \hat{i}_j s_k i_1 \cdots \hat{i}_j \cdots i_n i_j) \equiv 0 \quad (t+l = n-1)\,,$$

$$Q(i_b i_1 \cdots \hat{i}_j \cdots \hat{i}_l i_{\alpha_1} \cdots i_{\alpha_t} s_k i_j i_1 \cdots \hat{i}_j \cdots i_n i_j) \equiv 0 \quad (t+l = n-1)$$

and

$$Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}s_k i_j s_{\mu_1}\cdots s_{\mu_{q-2}}s_k s_{\mu_{q-1}}i_1\cdots\hat{i}_j\cdots i_n i_j)\equiv 0$$

if $q\geqq 2$. Thus the first term in the right hand side of (2.7) is identically zero. Obviously

$$\begin{aligned}&Q(i_b i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}s_k i_j s_{\mu_1}\cdots s_{\mu_{q-1}}i_1\cdots\hat{i}_j\cdots i_n i_j)\,.\\ &\quad = -Q(i_b i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}i_j s_k s_{\mu_1}\cdots s_{\mu_{q-1}}i_1\cdots\hat{i}_j\cdots i_n i_j)\end{aligned}$$

by (2.2) 1). Inductively we get

$$\begin{aligned}&Q(i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}s_k i_j s_{\mu_1}\cdots s_{\mu_{q-1}}s_k i_1\cdots\hat{i}_j\cdots i_n i_j)\\ &\quad =\sum_{\gamma \underset{\neq}{\triangleleft'} h(\beta)} P_\gamma\Big(\cdots,\frac{p_{\lambda_1\cdots\lambda_n}}{p_{i_1\cdots i_n}},\cdots\Big)Q(\gamma i_1\cdots\hat{i}_j\cdots i_n i_j)\end{aligned}\tag{2.8}$$

for some polynomial functions P_γ. Hence we get our claim (2.5) in this case. By the same way, we can show our claim in the following cases:

Case 2. $\beta=(i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}s_k s_{\mu_1}\cdots s_{\mu_q})$
$l=1,\cdots,\hat{j},\cdots,n-1,\ t\geqq 0,\ q\geqq 2,\ l<\alpha_a\neq j$
$(a=1,\cdots,t)\ \mu_b\neq k\ (b=1,\cdots,q)$.

Case 3. $\beta=(i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}i_j s_{\mu_1}\cdots s_{\mu_q})$
$l=1,\cdots,\hat{j},\cdots,n-1,\ t\geqq 0,\ q\geqq 2,\ l<\alpha_a\neq j\quad (a=1,\cdots,t)$
$\mu_b\neq k\ (b=1,\cdots,q)$.

Case 4. $\beta=(i_1\cdots\hat{i}_j\cdots\hat{i}_l i_{\alpha_1}\cdots i_{\alpha_t}s_{\mu_1}\cdots s_{\mu_q})$
$l=1,\cdots,\hat{j},\cdots,n,\ t\geqq 0,\ q\geqq 2,\ l<\alpha_a\neq j\quad (a=1,\cdots,t)$
$\mu_b\neq k\ (b=1,\cdots,q)$.

Hence, on $\pi^{-1}(V_{i_1\cdots i_n})$, we have

$$dQ(\beta i_1\cdots\hat{i}_j\cdots i_n s_k)=\pm dQ(h(\beta)i_1\cdots\hat{i}_j\cdots i_n i_j)$$

for $\beta\in S(i_1\cdots\hat{i}_j\cdots i_n s_k)$ and

$$\begin{aligned}dQ(\beta i_1\cdots\hat{i}_j\cdots i_n s_k)&=\pm\frac{p_{i_1\cdots\hat{i}_j\cdots i_n s_k}}{p_{i_1\cdots i_n}}dQ(h(\beta)i_1\cdots\hat{i}_j\cdots i_n i_j)\\ &\quad+\sum_{\gamma \underset{\neq}{\triangleleft'} h(\beta)} P_\gamma\Big(\cdots,\frac{p_{\lambda_1\cdots\lambda_n}}{p_{i_1\cdots i_n}},\cdots\Big)dQ(\gamma i_1\cdots\hat{i}_j\cdots i_n i_j)\end{aligned}$$

for $\beta\in V(i_1\cdots\hat{i}_j\cdots i_n s_k,\prec)-S(i_1\cdots\hat{i}_j\cdots i_n s_k)$.

Since h is order preserving and the number of elements in $S(i_1\cdots\hat{i}_j\cdots i_n s_k)$ is $(n-1)(m-1)$, we get Proposition 2.5. q.e.d.

Proposition 2.6. *For n-tuples* $(i_1,\cdots,i_n)$, $(j_1,\cdots,j_n)$ $(1\leqq i_1<\cdots<i_n\leqq m+n$,

$1 \leqq j_1 < \cdots < j_n \leqq m+n$),

$$\tilde{q}_{j_1 \cdots j_n} = \varepsilon(j_1 \cdots j_n, i_1 \cdots i_n) \left(\frac{p_{j_1 \cdots j_n}}{p_{i_1 \cdots i_n}} \right)^t \tilde{q}_{i_1 \cdots i_n}$$

on $\pi^{-1}(V_{i_1 \cdots i_n})$, *where* $\varepsilon(j_1 \cdots j_n, i_1 \cdots i_n)$ *is constant and valued in* $\{\pm 1\}$.

Proof. It is enough to see that for n-tuples $(i_1, \cdots, i_n)$ and $(i_1 \cdots \hat{i}_j \cdots i_l s_k i_{l+1} \cdots i_n)$ $(1 \leqq i_1 < \cdots < \hat{i}_j < i_l < s_k < i_{l+1} < \cdots < i_n \leqq m+n)$

$$\tilde{q}_{i_1 \cdots \hat{i}_j \cdots i_l s_k i_{l+1} \cdots i_n} = \varepsilon(i_1 \cdots \hat{i}_j \cdots i_l s_k i_{l+1} \cdots i_n, i_1 \cdots i_n) \left(\frac{p_{i_1 \cdots \hat{i}_j \cdots i_l s_k i_{l+1} \cdots i_n}}{p_{i_1 \cdots i_n}} \right)^t \tilde{q}_{i_1 \cdots i_n} \tag{2.9}$$

on $\pi^{-1}(V_{i_1 \cdots i_n})$.

By Lemma 2.2, 2.3 and Proposition 2.5, the equality (2.9) holds on

$$\pi^{-1}(V_{i_1 \cdots i_n}) \cap \pi^{-1}(V_{i_1 \cdots \hat{i}_j \cdots i_l s_k i_{l+1} \cdots i_n}).$$

Since $\tilde{q}_{i_1 \cdots \hat{i}_j i_l s_k i_{l+1} \cdots i_n}$ and $\tilde{q}_{i_1 \cdots i_n}$ are holomorphic forms on $C^{\mu+1}$, the equality (2.9) holds on $\pi^{-1}(V_{i_1 \cdots i_n})$. q.e.d.

Lemma 2.7. *For n-tuples* $(i_1 \cdots i_n)$, $(j_1 \cdots j_n)$, $(k_1 \cdots k_n)$, $\varepsilon(i_1 \cdots i_n, j_1 \cdots j_n)\varepsilon(j_1 \cdots j_n, k_1 \cdots k_n)\varepsilon(k_1 \cdots k_n, i_1 \cdots i_n) = 1$ *on* $V_{i_1 \cdots i_n} \cap V_{j_1 \cdots j_n} \cap V_{k_1 \cdots k_n}$.

Proof. Since

$$\tilde{q}_{i_1 \cdots i_n}(z) = (p_{i_1 \cdots i_n}(z))^r \bigwedge_{\beta \in C(i_1 \cdots i_n, \theta) - I(i_1 \cdots i_n, \theta)} (dp_\beta)_z + \text{other terms},$$

$$\tilde{q}_{i_1 \cdots i_n}(z) \neq 0 \quad \text{for} \quad z \in \pi^{-1}(V_{i_1 \cdots i_n}).$$

By Proposition 2.6, we get

$$\varepsilon(i_1 \cdots i_n, j_1 \cdots j_n)\varepsilon(j_1 \cdots j_n, k_1 \cdots k_n)\varepsilon(k_1 \cdots k_n, i_1 \cdots i_n) = 1$$

on $\pi^{-1}(V_{i_1 \cdots i_n}) \cap \pi^{-1}(V_{j_1 \cdots j_n}) \cap \pi^{-1}(V_{k_1 \cdots k_n})$. Since $\varepsilon(i_1 \cdots i_n, j_1 \cdots j_n)$ is constant, we get our claim. q.e.d.

Lemma 2.8 (Principle of monodromy). *Let G be an abelian group and M a simply connected manifold. Let* $\mathfrak{U} = \{U_\alpha\}_\alpha$ *be an open covering of M such that each* U_α *is connected. Then* $H^1(\mathfrak{U}, G) = (0)$.

Proof. See Weil [12] Chap. 5 Lemma 1.

Applying Lemma 2.8, for the complex Grassmann manifold $G_{m+n,n}(\boldsymbol{C})$ and the system of transition functions $\{\varepsilon(i_1 \cdots i_n, j_1 \cdots j_n)\}$, we get a system of constant functions $\{\delta(i_1 \cdots i_n)\}$ $(\delta(i_1 \cdots i_n)\colon V_{i_1 \cdots i_n} \to \{\pm 1\})$ such that $\varepsilon(i_1 \cdots i_n, j_1 \cdots j_n) = \delta(j_1 \cdots j_n)^{-1}\delta(i_1 \cdots i_n)$. We put $q_{i_1 \cdots i_n} = \delta(i_1 \cdots i_n)\tilde{q}_{i_1 \cdots i_n}$. Then, by Proposition 2.6, we have

$$(2.10) \qquad q_{j_1\cdots j_n} = \left(\frac{p_{j_1\cdots j_n}}{p_{i_1\cdots i_n}}\right)^t q_{i_1\cdots i_n} \quad \text{on} \quad \pi^{-1}(V_{i_1\cdots i_n}) .$$

By Proposition 1.2, a compact complex hypersurface X of $G_{m+n,n}(\boldsymbol{C})$ is the complete intersection of $G_{m+n,n}(\boldsymbol{C})$ and an irreducible subvariety Y of codimension 1 in $P^{\mu}(\boldsymbol{C})$. Let (F) denote the homogeneous ideal associated to Y. Note that the degree of homogeneous polynomial F on $\boldsymbol{C}^{\mu+1}$ is the degree of X and $W_{i_1\cdots i_n}=\{\pi(z)\in V_{i_1\cdots i_n} \mid F(z)=0\}$.

Lemma 2.9. *On* $\pi^{-1}(W_{i_1\cdots i_n})$, $q_{i_1\cdots i_n}\wedge dF\neq 0$.

Proof. Suppose that there is a point $z_0\in\pi^{-1}(W_{i_1\cdots i_n})$ such that $(q_{i_1\cdots i_n}\wedge dF)_{z_0}=0$. Since $\pi^{-1}(X)$ is a complex submanifold of $\boldsymbol{C}^{\mu+1}-(0)$, there are an open neighborhood U of z_0 in $\boldsymbol{C}^{\mu+1}-(0)$ and holomorphic functions $f_j(j=1, \cdots, r+1)$ such that $U\cap\pi^{-1}(X)=\{z\in U \mid f_j(z)=0, j=1, \cdots, r+1\}$ and $(df_j)_z(j=1, \cdots, r+1)$ are linearly independent for $z\in U\cap\pi^{-1}(X)$. By the Nullstellensatz for prime ideals ([4] chap. 2A Theorem 7),

$$f_j = \sum_{\alpha} q_{j\alpha}Q_{\alpha}+h_jF$$

where $q_{j\alpha}$, h_j are holomorphic functions on U and Q_{α} are generators of the ideal $I(G_{m+n,n}(\boldsymbol{C}))$. Thus we have

$$(df_j)_{z_0} = \sum_{\alpha} q_{j\alpha}(z_0)(dQ_{\alpha})_{z_0}+h_j(z_0)(dF)_{z_0} .$$

By Lemma 2.1 a) and b) and (2.2), we see that for each Q_{α}

$$(dQ_{\alpha})_{z_0} = \sum_{\gamma\in V(i_1\cdots i_n,\theta)} C_{\alpha}(\gamma)(dQ(\gamma i_1\cdots i_n))_{z_0}$$

for some $C_{\alpha}(\gamma)\in\boldsymbol{C}$. Hence, $\bigwedge_{j=1}^{r+1}(df_j)_{z_0}=c(q_{i_1\cdots i_n}\wedge dF)_{z_0}$ for some $c\in\boldsymbol{C}$ and hence $\bigwedge_{j=1}^{r+1}(df_j)_{z_0}=0$. This is a contradiction. q.e.d.

We define a local holomorphic section $t_{i_1\cdots i_n}$ of the line bundle N on $W_{i_1\cdots i_n}$ by

$$(2.11) \qquad t_{i_1\cdots i_n}(x) = (s_{i_1\cdots i_n}^{*}(q_{i_1\cdots i_n}\wedge dF))_x$$

for $x\in W_{i_1\cdots i_n}$.

Lemma 2.10. *The system of transition functions associated to the local trivialization* $(W_{i_1\cdots i_n}, t_{i_1\cdots i_n})$ *of the line bundle* N *is* $(\iota^*g_{i_1\cdots i_n,j_1\cdots j_n}^{2r+a-t})$, *where* a *is the degree of* X. *In particular,* $N=\iota^*E^{2r+a-t}$.

Proof. By Lemma 2.9, we have $t_{i_1\cdots i_n}(x)\neq 0$ for any $x\in W_{i_1\cdots i_n}$. Since $Q(\beta i_1\cdots i_n)$ are of degree 2 and F is of degree a,

$$t_{j_1\cdots j_n}(x)=\left(\frac{p_{i_1\cdots i_n}}{p_{j_1\cdots j_n}}(x)\right)^{-t+2r+a} t_{i_1\cdots i_n}(x)$$
$$=(\iota^* g_{i_1\cdots i_n,j_1\cdots j_n}(x))^{2r+a-t} t_{i_1\cdots i_n}(x)$$

on $W_{i_1\cdots i_n}\cap W_{j_1\cdots j_n}$, by (2.10). q.e.d.

The canonical line bundle K of $P^{\mu}(\boldsymbol{C})$, the holomorphic line bundle of co-vectors of bi-degree $(\mu, 0)$ on $P^{\mu}(\boldsymbol{C})$, is isomorphic to $E^{\mu+1}$. By (2.1) and Lemma 2.10,

$$K(X)=\iota^* E^{m+n-a}, \tag{2.12}$$

since $t=r-(n-1)(m-1)$.

REMARK. Let j: $G_{m+n,n}(\boldsymbol{C})\to P^{\mu}(\boldsymbol{C})$ be the inclusion. Then $K(G_{m+n,n}(\boldsymbol{C}))=j^*E^{m+n}$ ([1] §16). Let X be a compact complex submanifold of codimension 1 in $G_{m+n,n}(\boldsymbol{C})$ and ι_0: $X\to G_{m+n,n}(\boldsymbol{C})$ the inclusion. Then $K(X)=(j\circ\iota_0)^*E^{m+n-a}$, by considering the normal bundle $N(X, G_{m+n,n}(\boldsymbol{C}))$ of X in $G_{m+n,n}(\boldsymbol{C})$ and by Proposition 1.2.

The first Chern class of X, which is the Chern class of the dual bundle $K(X)^*$ of $K(X)$, is the cohomology class containing the form $(m+n-a)\omega$, where $\omega=\iota^*\Omega$ is the Kähler form on X associated to the induced Kähler metric on X. We shall determine a local section $k_{i_1\cdots i_n}$ of $K(X)^*$ on each $W_{i_1\cdots i_n}$ so that the system of transition functions associated to the local trivialization $(W_{i_1\cdots i_n}, k_{i_1\cdots i_n})$ is $(\iota^* g_{i_1\cdots i_n,j_1\cdots j_n}{}^{a-(m+n)})$. We put

$$l_{i_1\cdots i_n}=(-1)^{\sigma(i_1\cdots i_n)-1}\bigwedge_{(\alpha_1\cdots\alpha_n)\neq(i_1\cdots i_n)}\partial/\partial u_{i_1\cdots i_n,\alpha_1\cdots\alpha_n} \tag{2.13}$$

on $U_{i_1\cdots i_n}$, where we take the exterior product of $\partial/\partial u_{i_1\cdots i_n,\alpha_1\cdots\alpha_n}$ according to the natural lexicographical order. Then $(U_{i_1\cdots i_n}, l_{i_1\cdots i_n})$ is the local trivialization of the holomorphic line bundle K on $P^{\mu}(\boldsymbol{C})$ and the system of transition functions is $(g_{i_1\cdots i_n,j_1\cdots j_n}{}^{\mu+1})$.

Lemma 2.11. *Let $k_{i_1\cdots i_n}$ be a local holomorphic section of $K(X)^*$ on $W_{i_1\cdots i_n}$ defined by*

$$k_{i_1\cdots i_n}(x)=l_{i_1\cdots i_n}(x)\llcorner t_{i_1\cdots i_n}(x) \tag{2.14}$$

for $x\in W_{i_1\cdots i_n}$, where $\llcorner$ denotes the right interior multiplication. Then the system of transition functions associated to the local trivialization $(W_{i_1\cdots i_n}, k_{i_1\cdots i_n})$ of $K(X)^*$ is $(\iota^* g_{i_1\cdots i_n,j_1\cdots j_n}{}^{a-(m+n)})$.

Proof. By (2.1) and Lemma 2.10, $(k_{i_1\cdots i_n}, W_{i_1\cdots i_n})$ is a local trivialization of $K(X)^*$ and the system of transition functions is $(\iota^* g_{i_1\cdots i_n,j_1\cdots j_n})^{-(\mu+1)+2r+a-t}$. Since $-(\mu+1)+2r+a-t=a-(m+n)$, we get our claim. q.e.d.

3. The relation between volumes

Let C_n denote the set $\{(i_1, \cdots, i_n) \mid 1 \leqq i_1 < \cdots < i_n \leqq m+n\}$. For an element $i=(i_1, \cdots, i_n) \in C_n$, we put

$$q_i = \sum H^i_{\lambda_1 \cdots \lambda_r} dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_r} \tag{3.1}$$

where the summation runs over all $(\lambda_1, \cdots, \lambda_r) \in \underbrace{C_n \times \cdots \times C_n}_{r}$ such that $\lambda_1 < \cdots < \lambda_r$ with respect to the lexicographical order $<$ on C_n. Note that $H^i_{\lambda_1 \cdots \lambda_r}$ are homogeneous polynomials of degree r.

Proposition 3.1. *There exist homogeneous polynomials $H_{\lambda_1 \cdots \lambda_r}$ of degree $(n-1)(m-1)$ on $C^{\mu+1}$ such that*

$$H^i_{\lambda_1 \cdots \lambda_r} = p_i^t H_{\lambda_1 \cdots \lambda_r} \quad \text{on} \quad \pi^{-1}(V_i) \quad \text{for each} \quad i \in C_n\,. \tag{3.2}$$

Proof. By (2.10), we have

$$H^i_{\lambda_1 \cdots \lambda_r} = \left(\frac{p_i}{p_j}\right)^t H^j_{\lambda_1 \cdots \lambda_r} \tag{3.3}$$

on $\pi^{-1}(V_i \cap V_j)$ for each $(\lambda_1, \cdots, \lambda_r)$. Thus we get

$$\frac{H^i_{\lambda_1 \cdots \lambda_r}}{p_i^r} = \left(\frac{p_j}{p_i}\right)^{(n-1)(m-1)} \frac{H^j_{\lambda_1 \cdots \lambda_r}}{p_j^r} \tag{3.4}$$

On $V_i \cap V_j$. Hence, $\{H^i_{\lambda_1 \cdots \lambda_r}/p_i^r\}_{i \in C_n}$ define a holomorphic section of the line bundle $j^*H^{(n-1)(m-1)}$. Note that a holomorphic section of line bundle $H^{(n-1)(m-1)}$ on $P^\mu(C)$ is nothing but a homogeneous polynomial of degree $(n-1)(m-1)$ on $C^{\mu+1}$. By Proposition 1.1, there is a homogeneous polynomial $H_{\lambda_1 \cdots \lambda_r}$ of degree $(n-1)(m-1)$ on $C^{\mu+1}$ such that

$$\frac{H_{\lambda_1 \cdots \lambda_r}}{p_i^{(n-1)(m-1)}} = \frac{H^i_{\lambda_1 \cdots \lambda_r}}{p_i^r} \quad \text{on} \quad V_i\,.$$

Thus we get (3.2). q.e.d.

Now we have

$$q_i = p_i^t \sum H_{\lambda_1 \cdots \lambda_r} dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_r} \tag{3.5}$$

on $\pi^{-1}(V_i)$ for each $i \in C_n$, and hence

$$q_i \wedge dF = p_i^t \sum G_{\lambda_1 \cdots \lambda_{r+1}} dp_{\lambda_1} \wedge \cdots \wedge dp_{\lambda_{r+1}} \tag{3.6}$$

on $\pi^{-1}(W_i)$, where $G_{\lambda_1 \cdots \lambda_{r+1}} (\lambda_1 < \cdots < \lambda_{r+1})$ are homogeneous polynomials of degree $(n-1)(m-1)+(a-1)$.

For homogeneous polynomials $P_1, \cdots, P_s$ on $C^{\mu+1}$, we put

$$dP_1\wedge\cdots\wedge dP_s=\sum P_{\lambda_1\cdots\lambda_s}dp_{\lambda_1}\wedge\cdots\wedge dp_{\lambda_s}$$

where the summation runs over all $(\lambda_1,\cdots,\lambda_s)\in\underbrace{C_n\times\cdots\times C_n}_{s}$ such that $\lambda_1<\cdots<\lambda_s$ with respect to the lexicographical order $<$ on C_n, and we define

$$\|dP_1\wedge\cdots\wedge dP_s\|^2(z)=\sum|P_{\lambda_1\cdots\lambda_s}(z)|^2 \tag{3.7}$$

for $z\in \boldsymbol{C}^{\mu+1}$. Then we have

$$\|q_i\wedge dF\|^2(z)=|p_i(z)|^{2t}\sum|G_{\lambda_1\cdots\lambda_{r+1}}(z)|^2 \tag{3.8}$$

for $z\in\pi^{-1}(W_i)$.

Now we can define a C^∞-function φ: $X\to\boldsymbol{R}$ by

$$\varphi(x)=\frac{\|q_i\wedge dF\|^2(z)}{|p_i(z)|^{2t}\|z\|^{2((n-1)(m-1)+(a-1))}} \tag{3.9}$$

where $z\in\pi^{-1}(x)$.

Note that $\varphi(x)=(\sum|G_{\lambda_1\cdots\lambda_{r+1}}(z)|^2)/\|z\|^{2((n-1)(m-1)+(a-1))}$ for $z\in\pi^{-1}(x)$, $x\in X$.

Since the dual bundle $K(X)^*$ of the canonical line bundle $K(X)$ is the line bundle of $(mn-1)$ vectors of bi-degree $(mn-1, 0)$, the set of hermitian fiber metrics on $K(X)^*$ and the set of positive volume elements on X are canonically in one to one correspondence. Let $\mathfrak{v}$ denote the volume element on X corresponding to the fiber metric $\iota^*\|z\|^{2(a-(m+n))}$ on $K(X)^*$. Then the curvature form of the connection determined by the fiber metric $\iota^*\|z\|^{2(a-(n+n))}$ is $(m+n-a)\omega$, where $\omega=\iota^*\Omega$ is the Kähler form of the induced metric on X.

Now the relation between two volume elements ω^{mn-1} and $\mathfrak{v}$ is given by the following Proposition.

Proposition 3.2. *Let φ be a C^∞-function on X defined by* (3.9). *Then*

$$\omega^{mn-1}=\frac{(mn-1)!}{(2\pi)^{mn-1}}\varphi\mathfrak{v}\quad on\quad X. \tag{3.10}$$

We need several lemmas to prove Proposition 3.2. Note that the norm defined by (3.7) does not depend on the choice of unitary cartesian coordinates on $\boldsymbol{C}^{\mu+1}$. That is, for a unitary matrix $A\in U(\mu+1)$ and homogeneous polynomials P_j, we put $P_j'(w)=P_j(A^{-1}w)$ for $w\in\boldsymbol{C}^{\mu+1}$. Then

$$\|dp_1\wedge\cdots\wedge dp_s\|^2(z)=\|dp_1'\wedge\cdots\wedge dp_s'\|^2(w) \tag{3.11}$$

for $w=Az$, $z\in\boldsymbol{C}^{\mu+1}$.

In order to prove Proposition 3.2, it suffices to verify (3.10) at an arbitrary point $x_0\in X$. Fix a point $x_0\in X$ and let z_0 denote an element of $\boldsymbol{C}^{\mu+1}$ such that

$||z_0||=1$ and $\pi(z_0)=x_0$. For an element $A\in U(\mu+1)$, let p'_i denote $p'_i=\sum_j A^j_i p_j$, where $A=(A^j_i)$, and put $w=(\cdots, p'_i, \cdots)$. For a homogeneous polynomial P of degree k on $C^{\mu+1}$, put $P'(w)=P(A^{-1}w)$, $P'_{i_0}(w)=P'(w)/(p'_{i_0})^k$, where $i_0=(1,\cdots,n)\in C_n$ and put $u'_{i_0,\lambda}(x)=p'_\lambda(z)/p'_{i_0}(z)(z\in\pi^{-1}(x))$. $\lambda\in C_n$, $(\lambda\neq i_0)$.

Lemma 3.3. *If $x_0\in W_i$ $(i\in C_n)$, there is an element $A\in U(\mu+1)$ such that $p'_{i_0}(z_0)=1$, $p'_j(z_0)=0$ for $j\in C_n$, $j\neq i_0$ and $(dQ'(\beta, i)_{i_0})_{x_0}$ $(\beta\in C(i, \dashv)-I(i, \dashv))$ (where the order is principal with respect to i), $(dF'_{i_0})_{x_0}$ are linear combination of*

$$(du'_{i_0,\lambda})_{x_0}\quad (\lambda\in C(i_0, <)-I(i_0, <),\quad (du'_{i_0,12\cdots n-1n+1})_{x_0}\,.$$

Proof. By a routine computation of linear algebra.

Now we put $p_j=\sum_j B^k_j p'_k$ and $C^\lambda_\nu=(\partial u_{i,\lambda}/\partial u'_{i_0,\nu})(x_0)$.

Lemma 3.4.

$$C^\lambda_\nu = (B^{i_0}_i)^{-2}(B^\lambda_\nu B^{i_0}_i - B^\nu_i B^{i_0}_\lambda) \tag{3.12}$$

for $\lambda\neq i$, $\nu\neq i_0$, λ, $\nu\in C_n$

Proof. Straightforward computation.

Let $J(i_0, <)$ denote $I(i_0, <)-\{i_0, (12\cdots n-1n+1)\}$. We put $J(i_0, <)=\{\nu_1, \cdots, \nu_{mn-1}\}$ with $\nu_k<\nu_{k+1}$ $(k=1, \cdots, mn-2)$, $C(i, \dashv)-I(i, \dashv)=\{\beta_1, \cdots, \beta_r\}$ with $\beta_l \dashv \beta_{l+1}$ $(l=1, \cdots, r-1)$ and $C(i_0, <)-I(i_0, <)=\{\lambda_1, \cdots, \lambda_r\}$ with $\lambda_s<\lambda_{s+1}$ $(s=1, \cdots, r-1)$.

Lemma 3.5. *Let k_i be the holomorphic section of $K(X)^*$ on W_i defined in Lemma 2.12. Then, at $x_0\in W_i$,*

$$\begin{aligned} k_i(x_0) &= (-1)^{\sigma(i)-1}\cdot\delta(i)\cdot[\det(C^\lambda_\nu)]^{-1} \\ &\times\left(\frac{p'_{i_0}}{p_i}(x_0)\right)^{2r+a}\cdot\frac{\partial(Q'(\beta_1, i)_{i_0}, \cdots, Q'(\beta_r, i)_{i_0}, F'_{i_0})}{\partial(u'_{i_0,12\cdot n-1n+1}, u'_{i_0\,\lambda_1}, \cdots, u'_{i_0,\lambda_r})}(x_0) \\ &\times(\partial/\partial u'_{i_0,\nu_1}\wedge\cdots\wedge\partial/\partial u'_{i_0,\nu_{mn-1}})_{x_0}\,. \end{aligned} \tag{3.13}$$

Proof. For a homogeneous polynomial P of degree k on $C^{\mu-1}$, put $P_i=P/(p_i)^k$ on U_i. By the definition,

$$t_i(x_0)=\delta(i)s_1^*(dQ(\beta_1, i)\wedge\cdots\wedge dQ(\beta_r, i)\wedge dF)_{x_0}\,.$$

Thus

$$\begin{aligned} t_i(x_0) &= \delta(i)(dQ(\beta_1, i)_i\wedge\cdots\wedge dQ(\beta_r, i)_i\wedge dF_i)_{x_0} \\ &= \delta(i)(p'_{i_0}/p_i)(x_0)^{2r+a}(dQ'(\beta_1, i)_{i_0}\wedge\cdots\wedge dQ'(\beta_r, i)_{i_0}\wedge dF'_{i_0})_{x_0} \end{aligned}$$

On the other hand, we have

$$\det(C^\lambda_\nu)\Big(\bigwedge^{\prec}_{\alpha\in C(\iota,\prec)-\{\iota\}} \partial/\partial u_{\iota,\alpha}\Big)_{x_0} = \Big(\bigwedge^{<}_{\beta\in C(\iota_0,<)-\{\iota_0\}} \partial/\partial u'_{\iota_0,\beta}\Big)_{x_0}$$

By the definition of k_ι,

$$k_i(x_0) = (-1)^{\sigma(\iota)-1}\delta(i)\cdot[\det(C^\lambda_\nu)]^{-1}(p'_{\iota_0}/p_i)(x_0)^{2r+a}$$
$$\times\Big(\bigwedge^{<}_{\beta\in C(\iota_0,<)-\{\iota_0\}} \partial/\partial u'_{\iota_0,\beta}\Big)_{x_0} \llcorner (dQ'(\beta_1, i)_{\iota_0}\wedge\cdots\wedge dQ'(\beta_r, i)_{\iota_0}\wedge dF'_{\iota_0})_{x_0}$$

By Lemma 3.3, we get (3.13). q.e.d.

Now the local expression of the volume element $\mathfrak{v}$ at x_0 is given by the following Lemma.

Lemma 3.6.

$$\text{(3.14)}\qquad \mathfrak{v}_{x_0} = (\sqrt{-1})^{(mn-1)^2}|\det(C^\lambda_\nu)|^2\cdot|(p'_{\iota_0}/p_\iota)(x_0)|^{-2(m+n+2r)}$$
$$\times\left|\frac{\partial(Q'(\beta_1, i)_{\iota_0}, \cdots, Q'(\beta_r, i)_{\iota_0}, F'_{\iota_0})}{\partial(u'_{\iota_0,12\cdots n-1n+1}, u'_{\iota_0,\lambda_1}, \cdots, u'_{\iota_0,\lambda_r})}(x_0)\right|^{-2}(d\eta'\wedge d\bar{\eta}')_{x_0}$$

where $(d\eta')_{x_0}=(du'_{\iota_0,\nu_1}\wedge\cdots\wedge du'_{\iota_0,\nu_{mn-1}})_{x_0}$

Proof. By the definition, $\mathfrak{v}$ is the volume element on X corresponding to the fiber metric $\iota^*||z||^{2(a-(m+n))}$ on $K(X)^*$. Note that

$$1+\sum_{\substack{\alpha\neq\iota\\ \alpha\in C_n}}|(p_\alpha/p_i)(x_0)|^2 = |(p'_{\iota_0}/p_i)(x_0)|^2.$$

Put

$$T_\iota(x_0) = (-1)^{\sigma(\iota)-1}\delta(i)\cdot[\det(C^\lambda_\nu)]^{-1}$$
$$\times(p'_{\iota_0}/p_i)(x_0)^{2r+a}\cdot\frac{\partial(Q'(\beta_1, i)_{\iota_0}, \cdots, Q'(\beta_r, i)_{\iota_0}, F'_{\iota_0})}{\partial(u'_{\iota_0,12\cdots n-1n+1}, u'_{\iota_0,\lambda_1}, \cdots, u'_{\iota_0\lambda_r})}(x_0)$$

Then $\mathfrak{v}_{x_0}$ is given by

$$\frac{1}{|T_\iota(x_0)|^2}|(p'_{\iota_0}/p_i)(x_0)|^{2(a-(m+n))}(d\eta'\wedge d\bar{\eta}')_{x_0}.$$

Hence

$$\mathfrak{v}_{x_0} = (\sqrt{-1})^{(mn-1)^2}|\det(C^\lambda_\nu)|^2\cdot|(p'_{\iota_0}/p_\iota)(x_0)|^{2(-(m+n)-2r)}$$
$$\times\left|\frac{\partial(Q'(\beta_\iota, i)_{\iota_0}, \cdots, Q'(\beta_r, i)_{\iota_0}, F'_{\iota_0})}{\partial(u'_{\iota_0,12\cdots n-1n+1}, u'_{\iota_0,\lambda_1}, \cdots, u'_{1_0,\lambda_r})}(x_0)\right|^{-2}(d\eta'\wedge d\bar{\eta}')_{x_0}$$

q.e.d.

Lemma 3.7. *At* $x_0\in W_i$,

$$\text{(3.15)}\qquad \varphi(x_0) = |(p'_{\iota_0}/p_\iota)(x_0)|^{2t}\cdot\left|\frac{\partial(Q'(\beta_1, i)_{\iota_0}, \cdots, Q'(\beta_r, i)_{\iota_0}, F'_{\iota_0})}{\partial(u'_{\iota_0,12\cdots n-1n+1}, u'_{\iota_0,\lambda_1}, \cdots, u'_{\iota_0,\lambda_r})}(x_0)\right|^2.$$

Proof. Fix $c\in \boldsymbol{C}^*$ so that $||cs_i(x_0)||^2=1$. Then $|c|^2\cdot(1+\sum_{\substack{\alpha\in C_n\\ \alpha\neq i}}|(p_\alpha/p_i)(x_0)|^2)=1$ and $|c|^2=|(p'_{i_0}/p_i)(x_0)|^{-2}$. Note that

$$\varphi(x_0)=\frac{||q_i\wedge dF||^2(cs_i(x_0))}{|c|^{2t}||cs_i(x_0)||^{2((n-1)(m-1)+(a-1))}}=\frac{||q'_i\wedge dF'||^2(1,0,\cdots,0)}{|c|^{2t}}\quad \text{by (3.11).}$$

Since

$$\frac{\partial Q'(\beta_k,i)}{\partial p'_{i_0}}(1,0,\cdots,0)=0\quad \text{for}\quad k=1,\cdots,r,\ \frac{\partial F'}{\partial p'_{i_0}}(1,0,\cdots,0)=0$$

and

$$\frac{\partial Q'(\beta_k,i)}{\partial p'_j}(1,0,\cdots,0)=\frac{\partial Q'(\beta_k,i)_{i_0}}{\partial u'_{i_0,j}}(x_0),$$

$$\frac{\partial F'}{\partial p'_j}(1,0,\cdots,0)=\frac{\partial F'_{i_0}}{\partial u'_{i_0,j}}(x_0)\quad \text{for}\quad j\in C_n,\ j\neq i_0,$$

$$\begin{aligned}&||q'_i\wedge dF||^2(1,0,\cdots,0)\\ &=||dQ'(\beta_1,i)\wedge\cdots\wedge dQ'(\beta_r,i)\wedge dF'||^2(1,0,\cdots,0)\\ &=\left|\frac{\partial(Q'(\beta_1,i)_{i_0},\cdots,Q'(\beta_r,i)_{i_0},F'_{i_0})}{\partial(u'_{i_0,12\cdots n-1n+1},u'_{i_0},\cdots,u'_{i_0,\lambda_r})}(x_0)\right|^2\end{aligned}$$

by Lemma 3.3. q.e.d.

By Lemma 3.3, the Kähler form ω of the induced metric on X is given by

$$\omega_{x_0}=\frac{\sqrt{-1}}{2\pi}\Big(\sum_{\nu\in J(i_0,<)}du'_{i_0,\nu}\wedge d\bar{u}'_{i_0,\nu}\Big)_0\quad \text{at}\quad x_0\in X.$$

Hence,

$$\omega_{x_0}^{mn-1}=\frac{(\sqrt{-1}^{(mn-1)^2})(mn-1)!}{(2\pi)^{mn-1}}(d\eta'\wedge d\bar{\eta}')_{x_0}. \tag{3.16}$$

Lemma 3.8.

$$|\det(C^\lambda_\nu)|^2=|(p'_{i_0}/p_i)(x_0)|^{2(\mu+1)} \tag{3.17}$$

Proof. Put $D^\lambda_\nu=B^\nu_\lambda B^{i_0}_i-B^\mu_i B^{i_0}_\lambda$ for $\lambda\neq i$, $\nu\neq i_0$, $\lambda,\nu\in C_n$. Note that

$$|\det(D^\lambda_\nu)|^2=\det(D^\lambda_\nu)\cdot\det{}^t(\bar{D}^\lambda_\nu)=\det\Big(\big(\sum_{\alpha\neq i_0}D^\lambda_\alpha\bar{D}^\tau_\alpha\big)_{\lambda,\tau\neq i}\Big),$$

and that

$$\begin{aligned}\sum_{\alpha\neq i_0}D^\lambda_\alpha\bar{D}^\tau_\alpha&=\sum_{\alpha\neq i_0}(B^\alpha_\lambda B^{i_0}_i-B^\alpha_i B^{i_0}_\lambda)\overline{(B^\alpha_\tau B^{i_0}_i-B^\alpha_i B^{i_0}_\tau)}\\ &=\sum_{\alpha\in C_n}(B^\alpha_\lambda B^{i_0}_i-B^\alpha_i B^{i_0}_\lambda)\overline{(B^\alpha_\tau B^{i_0}_i-B^\alpha_i B^{i_0}_\tau)}\\ &=\delta_{\lambda\tau}|B^{i_0}_i|^2+B^{i_0}_\lambda\overline{B^{i_0}_\tau},\end{aligned}$$

since $\sum_{\alpha \in C_n} B^{\alpha}_{\lambda} \bar{B}^{\alpha}_{\tau} = \delta_{\lambda\tau}$.

Thus

$$
\begin{aligned}
|\det (D^{\lambda}_{\nu})|^2 &= \det (\delta_{\lambda\tau} |B_i^{i_0}|^2 + B_{\lambda}^{i_0} \bar{B}_{\tau}^{i_0}) \\
&= |B_i^{i_0}|^{2\mu} \det (\delta_{\lambda\tau} + (B_{\lambda}^{i_0} \bar{B}_{\tau}^{i_0} / |B_{\tau}^{i_0}|^2)) \\
&= |B_i^{i_0}|^{2\mu} (1 + \sum_{\lambda \neq i_0} |B_{\lambda}^{i_0} / B_i^{i_0}|^2) \\
&= |B_i^{i_0}|^{2(\mu-1)} .
\end{aligned}
$$

Now

$$
\begin{aligned}
|\det (C^{\lambda}_{\nu})|^2 &= |B_i^{i_0}|^{-2\times 2\mu} |\det (D^{\lambda}_{\nu})|^2 \\
&= |B_i^{i_0}|^{-2\times 2\mu} \times |B_i^{i_0}|^{2(\mu-1)} = |B_i^{i_0}|^{-2(\mu+1)}
\end{aligned}
$$

Since $B_i^{i_0} = (p_i / p'_{i_0})(x_0)$, we get our claim. q.e.d.

Proof of Proposition 3.2.

By Lemma 3.6, Lemma 3.7 and Lemma 3.8, we have

$$\varphi(x_0) \mathfrak{v}_{x_0} = (\sqrt{-1})^{(mn-1)^2} |(p'_{i_0}/p_i)(x_0)|^{2(-m-n-2r+\mu+1+t)} (d\eta' \wedge d\bar{\eta}')_{x_0} .$$

Since

$$
\begin{aligned}
r - t &= (m-1)(n-1) = mn - (m+n) + 1 , \\
\mu + 1 + t - 2r - m - n &= \mu + 1 - r - (m+n) - mn + m + n - 1 \\
&= \mu + 1 - r - mn - 1 = 0 .
\end{aligned}
$$

Hence

$$\varphi(x_0) \mathfrak{v}_{x_0} = (\sqrt{-1})^{(mn-1)^2} (d\eta' \wedge d\bar{\eta}')_{x_0} .$$

Now our claim follows from (3.16).

Corollary of Proposition 3.2 (*cf.* Hano [5] Corollary of Proposition 2). *Let g_0 denote the Kähler metric on X induced from the Fubini-Study metric on $P^{\mu}(\boldsymbol{C})$. Then (X, g_0) is an Einstein manifold if and only if φ is a constant function on X.*

Proof. The Ricci form of the Kähler metric g_0 on X is the curvature form of the connection of type (1.0) on the holomorphic line bundle $K(X)^*$ determined by the volume element ω^{mn-1}. Suppose that g_0 is Einstein, that is, the Ricci form is a constant multiple of the Kähler form ω. Then the Ricci form is harmonic. On the other hand, the volume element $\mathfrak{v}$ determines the curvature form $(m+n-a)\omega$, which is also harmonic. Since the Ricci form and $(m+n-a)\omega$ are both curvature form of the bundle $K(X)^*$, they are cohomologous. Thus the Ricci form must be $(m+n-a)\omega$. Since ω^{mn-1} and $\mathfrak{v}$ define the same curvature form, $d'd'' \log \varphi = 0$, and hence $\log \varphi$ is a harmonic function on X. This implies that φ is a constant function. Conversely, if φ is a constant function, then the metric g_0 is Einstein. q.e.d.

4. The dual map and Veronese map

In this section we recall the dual map and Veronese map due to Hano [5].

Let $\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$ denote the $(r+1)$-th exterior product of the dual space of the vector space $\boldsymbol{C}^{\mu+1}$. We identify the tangent space of $\boldsymbol{C}^{\mu+1}$ at a point with $\boldsymbol{C}^{\mu+1}$ itself. We regard $(q_i\wedge dF)_z$ as an element in $\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$. Let $(\zeta_{\lambda_1\cdots\lambda_{r+1}})$ be the standard base of $\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$. Then

$$(q_i\wedge dF)_z=(p_i(z))^t\sum G_{\lambda_1\cdots\lambda_{r+1}}(z)\zeta_{\lambda_1\cdots\lambda_{r+1}}\quad\text{for}\quad z\in\pi^{-1}(W_i)\,.$$

Now we define a map $G\colon \boldsymbol{C}^{\mu+1}\to\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$ by

$$G(z)=\sum G_{\lambda_1\cdots\lambda_{r+1}}(z)\zeta_{\lambda_1\cdots\lambda_{r+1}}\,. \tag{4.1}$$

We denote by $P^e(\boldsymbol{C})$ the complex projective space associated to the complex vector space $\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$, where $e+1=\binom{\mu+1}{r+1}$. Since the map $G\colon \boldsymbol{C}^{\mu+1}\to\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$ is a polynomial map of degree $(n-1)(m-1)+(a-1)$ and $G(z)\neq 0$ for $z\in\pi^{-1}(X)$, it induces a holomorphic map $g\colon X\to P^e(\boldsymbol{C})$. We call g the dual map of X in $P^\mu(\boldsymbol{C})$. Let $||w||$ be the norm of an element w in $\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$ induced from the hermitian inner product on $\boldsymbol{C}^{l+1}$. Let Ω' be the Fubini-Study form on $P^e(\boldsymbol{C})$ determined from $||w||^2$.

Proposition 4.1 (cf. [5] Proposition 3). *The induced metric g_0 on X is Einstein if and only if the reciprocal image of the Fubin-Study metric on $P^e(\boldsymbol{C})$ under the dual map g is $(n-1)(m-1)+(a-1)$ times of the induced metric*; $g^*\Omega'=((n-1)(m-1)+(a-1))\omega$.

Proof. Since the degree of G is $(n-1)(m-1)+(a-1)$, the reciprocal image of the standard line bundle E' over $P^e(\boldsymbol{C})$ under the map g is $\iota^*E^{(n-1)(m-1)+(a-1)}$ where E denotes the standard line bundle over $P^\mu(\boldsymbol{C})$. We regard $||w||^2$ as the fiber metric on E' over $P^e(\boldsymbol{C})$. Its reciprocal image under g is the restriction of $\sum|G_{\lambda_1\cdots\lambda_{r+1}}(z)|^2$ to $\pi^{-1}(X)$ and is a fiber metric on $\iota^*E^{(n-1)(m-1)+(a-1)}$. Then

$$\pi^*g^*\Omega'=\frac{\sqrt{-1}}{2\pi}d'd''\log\left(\sum|G_{\lambda_1\cdots\lambda_{r+1}}(z)|^2\right).$$

Now our claim follows from Corollary of Proposition 3.2. q.e.d.

Let S_k be the vector space of homogeneous polynomials on $\boldsymbol{C}^{\mu+1}$ of degree k and S_k^* the dual space of S_k. We denote by $P^d(\boldsymbol{C})$ the complex projective space associated to S_k^*, where $d+1=\dim S_k$. Each point $z\in\boldsymbol{C}^{\mu+1}$defines a linear function $\Psi(z)$ on S_k given by $\Psi(z)(P)=P(z)$ for $P\in S_k$. We denote by ψ the map $z\mapsto\Psi(z)$. The polynomial map Ψ induces an injective holomorphic map

$$\psi\colon P^{\mu}(\boldsymbol{C}) \to P^{d}(\boldsymbol{C}) \tag{4.2}$$

if $k \geqq 1$. The map ψ is called the Veronese map of degree k.

For simplicity we denote the Plücker coordinate $(\cdots, p_{\iota}, \cdots)$ by $(z_0, \cdots, z_{\mu})$. With respect to the hermitian inner product on S_k induced from the one on $\boldsymbol{C}^{\mu+1}$, the set of all monomials

$$z_0^{\nu_0}\cdots z_{\mu}^{\nu_{\mu}}/(\nu_0!\cdots\nu_{\mu}!)^{1/2},\ \nu_0+\cdots+\nu_{\mu}=k \tag{4.3}$$

is a unitary base of S_k. Moreover

$$|z_0^{\nu_0}\cdots z_{\mu}^{\nu_{\mu}}/(\nu_0!\cdots\nu_{\mu}!)^{1/2}|^2 = ||z||^{2k}/k!\,. \tag{4.4}$$

Obviously the reciprocal image of the standard line bundle over $P^d(\boldsymbol{C})$ under the map ψ is E^k. By (4.4), if Ω'' denotes the Fubini-Study form on $P^d(\boldsymbol{C})$, then $\psi^*\Omega''=k\Omega$. That is, the Veronese map ψ is homothetic and the ratio of the metrics is the degree k of the map ψ.

Now we specify k to be $(n-1)(m-1)+(a-1)$, and define a linear map $L\colon S^*_{(n-1)(m-1)+(a-1)}\to\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$ so that $L\circ\psi=G$ on the cone $\pi^{-1}(X)$. Let $(\xi_{\nu_0\cdots\nu_{\mu}})$ be the dual base of the unitary base of $S_{(n-1)(m-1)+(a-1)}$ chosen above. Since $G_{\lambda_1\cdots\lambda_{r+1}}$ is of degree $(n-1)(m-1)+(a-1)$,

$$G_{\lambda_1\cdot\lambda_{r+1}} = \sum_{\nu_0\cdots\nu_{\mu}} a(\lambda_1\cdots\lambda_{r-1};\ \nu_0\cdots\nu_{\mu})(z_0^{\nu_0}\cdots z_{\mu}^{\nu_{\mu}}/(\nu_0!\cdots\nu_{\mu}!)^{1/2})\,. \tag{4.5}$$

Using these coefficients, a linear map L is defined by

$$L(\xi_{\nu_0\cdots\nu_{\mu}}) = \sum a(\lambda_1\cdots\lambda_{r+1};\ \nu_0\cdots\nu_{\mu})\zeta_{\lambda_1\cdots\lambda_{r+1}}\,. \tag{4.6}$$

By the way L is defined, it is clear that

$$(L\circ\psi)(z) = G(z) \quad \text{for} \quad z\in\pi^{-1}(X)\,.$$

Consider the rational map $l\colon P^d(\boldsymbol{C})\to P^e(\boldsymbol{C})$ induced from the linear map $L\colon S^*_{(n-1)(m-1)+(a-1)}\to\bigwedge^{r+1}(\boldsymbol{C}^{\mu+1})^*$. The map l is holomorphic at a point $x\in P^d(\boldsymbol{C})$ if the image under L at a point of $S^*_{(n-1)(m-1)+(a-1)}$ lying over x is not zero. Since $||q_{\iota}\wedge dF||^2$ vanishes nowhere on $\pi^{-1}(W_{\iota})$, L does not vanishes at each point on the image of $\pi^{-1}(X)$ under ψ. Therefore l is holomorphic on $\psi(X)$.

Proposition 4.2. *Let be ψ the Veronese map of degree $(n-1)(m-1)+(a-1)$ of $P^{\mu}(\boldsymbol{C})$ into $P^d(\boldsymbol{C})$ and let g be the dual map of X into $P^e(\boldsymbol{C})$. Then there is a projective transformation l of $P^d(\boldsymbol{C})$ into $P^e(\boldsymbol{C})$ which is holomorphic on $\psi(X)$ and satisfies the equality $(l\circ\psi)(x)=g(x)$ for $x\in X$. Moreover the induced metric on X is Einstein if and only if the restriction of l to $\psi(X)$ is everywhere locally isometric with respect to the induced metric on $\psi(X)$ and the Fubini-Study metric on $P^e(\boldsymbol{C})$.*

Proof. By Proposition 4.1 and the above observation (cf. [5] Proposition 4).

Now we have the following Lemma due to Hano ([5] Lemma 7).

Lemma 4.3. *Let Φ be a linear map of C^{s+1} into C^{t+1} and ϕ the induced projective transformaticn of $P^s(C)$ into $P^t(C)$. Let U be a connected algebraic submanifold in $P^s(C)$ which is not contained in any hyperplane in $P^s(C)$. We equip on U the metric induced from a Funibi-Study metric on $P^s(C)$, and on $P^t(C)$ a Fubini-Study metric. Suppose that the restriction of ϕ to U is holomorphic and locally iscmetric everywhere, then Φ is a constant multiple of an isometry, and particularly Φ is injective.*

Now we have the following necessary condition from Lemma 4.3.

Proposition 4.4 (cf. [5] Hano §8). *Let X be a hypersurface of $G_{m+n,n}(C)$ of degree a. If the induced is metric on X Einstein, then*

$$\dim\left(S_{(n-1)(m-1)+(a-1)}/I_{(n-1)(m-1)+(a-1)}\right) \leqq e+1 = \binom{\mu+1}{r+1}, \tag{4.8}$$

where $I_{(n-1)(m-1)+(a-1)} = S_{(n-1)(m-1)+(a-1)} \cap I(X)$.

Proof. For $P \in S_{(n-1)(m-1)+(a-1)}$, the equation $\langle \xi, P \rangle = 0$, $\xi \in S^*_{(n-1)(m-1)+(a-1)}$, defines a hyperplane in $P^d(C)$. By the definition of the Vcronese map ψ, a homogeneous polynomial P in $S_{(n-1)(m-1)+(a-1)}$ defines a hyperplane containing $\psi(X)$ if and only if P belongs to $I_{(n-1)(m-1)+(a-1)}$. Thus, the minimal linear variety $P^{d'}(C)$ containing $\psi(X)$ is the intersection of these hyperplanes each of which is associated to a polynomial in $I_{(n-1)(m-1)+(a-1)}$. Its dimension d' is given by $\dim\left(S_{(n-1)(m-1)+(a-1)}/I_{(n-1)(m-1)+(a-1)}\right)-1$. Let $C^{d'+1}$ be the subspace in $S^*_{(n-1)(m-1)+(a-1)}$ perpendicular to the subspace $I_{(n-1)(m-1)+(a-1)}$. Let L' be the restriction to $C^{d'+1}$ of the linear map $L\colon S^*_{(n-1)(m-1)+(a-1)} \to \bigwedge^{r+1}(C^{\mu+1})^*$, and let l' be the restriction to $P^{d'}(C)$ of projective transformation l. Now the connected algebraic submanifold $\psi(X)$ in $P^{d'}(C)$ is not contained in any hyperplane of $P^{d'}(C)$. By Proposition 4.2, the restriction to $\psi(X)$ of l' is everywhere locally isometric. Applying Lemma 4.3, to $\psi(X)$ in $P^{d'}(C)$, we see that the linear map

$$L'\colon C^{d'+1} \to \bigwedge^{r+1}(C^{\mu+1})^*$$

is injective, and hence we get (4.8). q.e.d.

5. Proof of Theorem

Let J denote the ideal $I(G_{m+n,n}(C))$ of homogeneous polynomials S on $C^{\mu+1}$.

Lemma 5.1. *Let J_k denote $J \cap S_k$. Then*

$$\dim(S_k/J_k) = \prod_{i=1}^{n} \prod_{j=n+1}^{m+n} \frac{k+j-i}{j-i}$$

Proof. By Proposition 1.1, the inclusion $j: G_{m+n,n}(\boldsymbol{C}) \to P^{\mu}(\boldsymbol{C})$ induces a surjective linear map

$$j^*: H^0(P^{\mu}(\boldsymbol{C}), H^k) \to H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k).$$

Noting that $H^0(P^{\mu}(\boldsymbol{C}), H^k)$ is the space of homogeneous polynomials S_k of degree k,

$$\begin{aligned} \operatorname{Ker} j^* &= \{P \in S_k \mid P(z) = 0 \quad \text{for any} \quad z \in \pi^{-1}(G_{m+n,n}(\boldsymbol{C}))\} \\ &= J \cap S_k. \end{aligned}$$

Hence, $\dim (S_k/J_k) = \dim H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k)$.

On the other hand, by a Theorem of Borel-Weil [2] and the dimension formula of Weyl [10], we have

$$\dim H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k) = \prod_{i=1}^{n} \prod_{j=n+1}^{m+n} \frac{k+j-i}{j-i}.$$

q.e.d.

Lemma 5.2. *Let I_k denote $I(X) \cap S_k$. Then*

$$\dim (S_k/I_k) = \dim (S_k/J_k) - \dim (S_{k-a}/J_{k-a})$$

if $k \geqq a$, where a is the degree of X.

Proof. Let $[X]$ denote the non-singular divisor defined by X and $\{X\}$ the holomorphic line bundle on $G_{m+n,n}(\boldsymbol{C})$ defined by $[X]$. Then there is an exact sequence

$$0 \to j^*H^{k-a} \to j^*H^k \to \widehat{\iota^*H^k} \to 0 \tag{5.1}$$

of holomorphic sheaves on $G_{m+n,n}(\boldsymbol{C})$. (cf. [6])

Then (5.1) induces the following exact sequence of cohomologies

$$\begin{aligned} 0 &\to H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^{k-a}) \to H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k) \\ &\to H^0(X, \iota^*H^k) \to H^1(G_{m+n,n}(\boldsymbol{C}), j^*H^{k-a}) \to \cdots. \end{aligned} \tag{5.2}$$

Since $H^1(G_{m+n,n}(\boldsymbol{C}), j^*H^{k-a}) = 0$ if $k \geqq a$, by a theorem of Bott [2],

$$\dim H^0(X, \iota^*H^k) = \dim H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k) - \dim H^0(G_{m+n,n}(\boldsymbol{C}), j^{**}H^{k-a}).$$

On the other hand, $j^*: H^0(P^{\mu}(\boldsymbol{C}), H^k) \to H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k)$ is surjective, and hence $\iota^*: H^0(P^{\mu}(\boldsymbol{C}), H^k) \to H^0(X, \iota^*H^k)$ is surjective if $k \geqq a$. Noting that $\operatorname{Ker} \iota^* = I(X) \cap S_k$, we have

$$\begin{aligned} \dim (S_k/I_k) &= \dim H^0(X, \iota^*H^k) \\ &= \dim H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^k) - \dim H^0(G_{m+n,n}(\boldsymbol{C}), j^*H^{k-a}) \\ &= \dim (S_k/J_k) - \dim (S_{k-a}/J_{k-a}). \end{aligned}$$

q.e.d.

Proof of Theorem. Put $k=(n-1)(m-1)+(a-1)$. If $n\geqq 2$ and $m\geqq n$, then $k\geqq a$. Thus, by Lemma 5.2,

$$\begin{aligned}\dim(S_{(n-1)(m-1)+(a-1)}/I_{(n-1)(m-1)+(a-1)})&\\ =\dim(S_{(n-1)(m-1)+(a-1)}/J_{(n-1)(m-1)+(a-1)})&\\ -\dim(S_{(n-1)(m-1)-1}/J_{(n-1)(m-1)-1})&.\end{aligned}$$

By Lemma 5.1, we see that $\dim(S_k/I_k)$ is increasing in k. Hence, it is enough to prove the following inequality (5.3) by Proposition 4.4;

$$\dim(S_{\mu-(m+n)+2}/I_{\mu-(m+n)+2})>\binom{\mu+1}{mn}. \tag{5.3}$$

By Lemma 5.1, we have

$$\dim(S_k/J_k)=\frac{(k+1)(k+2)^2\cdots(k+n)^n\cdots(k+m)^n(k+m-1)^{n-1}\cdots(k+m+n-1)}{1\cdot 2^2\cdots n^n\cdots m^n\cdot(m+1)^{n-1}\cdots(m+n-1)}.$$

Thus

$$\begin{aligned}&\dim(S_{\mu-(m+n)+2}/I_{\mu-(m+n)+1})-\binom{\mu+1}{mn}\\ &=\frac{(\mu+1)\mu^2(\mu-1)^3\cdots(\mu-n+2)^n\cdots(\mu-m+2)^n(\mu-m+1)^{n-1}\cdots(\mu-m-n+3)}{1\cdot 2^2\cdot 3^3\cdots n^n\cdots m^n\cdot(m+1)^{n-1}\cdots(m+n-1)}\\ &\quad-\frac{(mn-m-n+1)(mn-m-n+2)^2\cdots(mn-m-1)^{n-1}(mn-m)^n\cdots(mn-n)^n}{1\cdot 2^2\cdots n^n\cdots m^n(m+1)^{n-1}\cdots(m+n-1)}\\ &\qquad\times(mn-n+1)^{n-1}\cdots(mn-1)\\ &\quad-\frac{(\mu+1)\mu(\mu-1)\cdots(\mu+2-mn)}{1\cdot 2\cdot 3\cdots(mn)}\\ &>\frac{1}{(mn)!}\{(\mu+1)\mu^2(\mu-1)^3\cdots(\mu-n+2)^n\cdots(\mu-n+2)^n(\mu-m+1)^{n-1}\cdots\\ &\qquad\times(\mu-m-n+3)-(\mu+1)\mu(\mu-1)\cdots(\mu+2-mn)\\ &\quad-(mn-m-n+1)(mn-m-n)^2\cdots(mn-m)^n\cdots(mn-n)^n(mn-n+1)^{n-1}\cdots(mn-1)\}\end{aligned}$$

Now we have

$$\begin{aligned}&(\mu+1)\mu^2(\mu-1)^3\cdots(\mu-n+2)^n\cdots(\mu-m+2)^n(\mu-m+1)^{n-1}\cdots(\mu-m-n+3)\\ &\qquad-(\mu+1)\mu(\mu-1)\cdots(\mu+2-mn)\\ &\quad=(\mu+1)\mu(\mu-1)\cdots(\mu-m-n+3)\{\mu(\mu-1)^2\cdots(\mu-n+2)^{n-1}\cdots(\mu-m+2)^{n-1}\cdots\\ &\qquad\quad\times(\mu-m+1)^{n-2}\cdots(\mu-m-n+4)-(\mu-m-n+2)\cdots(\mu-mn+2)\}\\ &\quad>(\mu+1)\mu(\mu-1)\cdots(\mu-mn+3)(mn-m-n+2).\end{aligned}$$

On the other hand,

$$(\mu-mn+3)-(mn-n-m+2)=\binom{m+n}{n}-2mn+m+n>0.$$

Thus we have

$$\begin{aligned}&(\mu+1)\mu(\mu-1)\cdots(\mu-mn+3)(mn-u-n+2)\\&\quad-(mn-1)\cdots(mn-m-n+1)^{n-1}(mn-n)^{n}\cdots(mn-m)^{n}(mn-m-1)^{n-1}\cdots\\&\quad\times(mn-m-n+1)>(\mu+1)\mu\cdots(\mu-mn+3)(mn-m-n+2)-(2mn-m-m)\cdots\\&\quad\times(mn-m-n+2)(mn-m-n+1)>0\,.\end{aligned}$$

Hence, we get (5.3). q.e.d.

REMARK. In the case of $G_{5,2}(\boldsymbol{C})$, we can see that if the degree $a(X)$ of X satisfies $a(X)\geqq 3$ a hypersurface X is not an Einstein manifold with respect to the induced metric by the same way. But we do not know the cases when $a(X)=1$, 2.

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